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EXISTENCE AND STABILITY FOR NONLINEAR VARIABLE-COEFFICIENT DIFFERENTIAL EQUATIONS WITH NON-SINGULAR KERNELS

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Abstract. In this work, we study an initial value problem for variable-coefficient differential equation with nonsingular kernel. Firstly, we obtain existence and uniqueness of solutions for the above problem. Secondly, we obtain a stability result within the Ulam-Hyers sense of the given problem. As a special case, we obtain the corresponding conclusions with constant coefficients. An example is provided to verify the theoretical results.

Keywords: Caputo-Fabrizio derivative, variable-coefficient, Ulam-Hyers stability, constant-coefficient.

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1 Introduction

Compared to traditional integer-order equations, fractional differential equations(FDEs), by virtue of the nonlocality of their operators, can accurately characterize complex processes with memory effects and hereditary properties([1, 2]). Therefore, they have become important research tools in fields such as anomalous diffusion, viscoelastic mechanics, soft matter physics, biological system modeling, and control theory([3]).

In 2015, Caputo and Fabrizio proposed a fractional derivative as a modification of the traditional Caputo derivative, which is now known as the Caputo-Fabrizio (CF) derivative([4]). It replaces the power-law kernel in the classical definition with an exponential function. This derivative is extensively applied in modeling across diverse practical domains, including biomedicine, physics

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and engineering, economics and finance, as well as ecology. Research on equations containing the CF derivative has yielded some results([5, 6, 7, 8, 10, 9, 11, 12]), however, to the best of the authors' knowledge, there is still a great deal of research that has not been addressed. For instance, research on the initial value problem (IVP) for variable-coefficient differential equations involving the CF derivative is scarce.

When the coefficients in the equation are time-varying functions, they can more realistically describe phenomena where memory properties and time-varying characteristics coexist in practical scenarios, such as the dynamic evolution of cells in biophysics or the diffusion of pollutants in environmental science.

Ulam-Hyers (UH) stability serves as a tool for analyzing the error bound between approximate and exact solutions. This theory was introduced by Ulam and Hyers in 1940 ([13, 14]) and has now been widely applied in the study of various types of FDEs. However, its application to variable-coefficient differential equations with the CF derivative remains rare.

In this paper, we study the following initial value problem (IVP) for a variable-coefficient differential equation with nonsingular kernel

$$\begin{cases} {}^{CF}D^{\alpha}_{0+}x(t) + \lambda(t)\,{}^{CF}D^{\beta}_{0+}x(t) = \displaystyle\int_0^t f(s,x(s))\,\mathrm{d}s,\ t \in J := [0,T], & (1.1) \\ x(0) = x_0, & (1.2) \end{cases}$$

where $\alpha, \beta \in (0,1)$ and $\lambda(t) \in C^1(J,\mathbb{R})$, $\lambda(t) \neq 0$, $(1-\alpha)\lambda(t) \neq 1-\beta$, $f(t,x(t))$ will be specified later.

By overcoming the effect of the coefficient $\lambda(t)$, we obtain the existence and uniqueness result for the solution to (1.1)-(1.2), and the UH stability of (1.1). As a special case, we obtain the corresponding results for constant coefficients.

This paper is divided into six sections. Section 2 presents the foundational concepts and properties for later use. In section 3, we obtain the existence and uniqueness of solutions to (1.1)-(1.2). In section 4, we study the UH stability of (1.1). In section 5, we obtain the results for the constant-coefficient case. In section 6, we present the application of the theoretical result.

2 Preliminaries

Throughout this paper, the notation $C(J,\mathbb{R})$ is used for the space of continuous functions on J with the norm $\|x\|_{\mathcal{C}} = \max_{t\in J} |x(t)|$, and $C^1(J,\mathbb{R})$ denotes the space of continuously differential functions on J with the norm $\|x\|_{C^1} = \|x\|_{\mathcal{C}} + \|x'\|_{\mathcal{C}}$.

Definition 2.1 ([6]) *Let* $0 < \eta \leq 1$. *The fractional integral of Caputo-Fabrizio type of a function* φ *is defined by*

$$({}^{CF}I^{\eta}_{0+}\varphi)(t) = (1-\eta)\varphi(t) + \eta \int_0^t \varphi(s)\,\mathrm{d}s, \quad t \geq 0.$$

Clearly, ${}^{CF}I^{1}_{0+}\varphi(t) = I^{1}_{0+}\varphi(t) = \int_0^t \varphi(s)\,\mathrm{d}s.$

Definition 2.2 ([6]) *Let* $0 < \eta < 1$*. The generalized fractional Caputo-Fabrizio derivative of a function* $\varphi : [0, +\infty) \to \mathbb{R}$ *is defined by*

$$\begin{aligned}({}^{CF}D_{0+}^{\eta}\varphi)(t) &= \frac{1}{1-\eta}\int_0^t \varphi'(s)\exp\Big(-\frac{\eta(t-s)}{1-\eta}\Big)\,\mathrm{d}s \\ &= \frac{1}{1-\eta}[\varphi(t) - \exp\Big(-\frac{\eta t}{1-\eta}\Big)\varphi(0)] - \frac{\eta}{(1-\eta)^2}\int_0^t \varphi(s)\exp\Big(-\frac{\eta(t-s)}{1-\eta}\Big)\,\mathrm{d}s.\end{aligned}$$

Remark 2.3 *From Definition 2.2, for* $\varphi \in C^1(J, \mathbb{R})$*, we have*

$$\frac{d}{dt}\,{}^{CF}D_{0+}^{\eta}\varphi(t) = \frac{1}{1-\eta}\varphi'(t) - \frac{\eta}{1-\eta}\,{}^{CF}D_{0+}^{\eta}\varphi(t), \quad 0 < \eta < 1.$$

Lemma 2.4 ([10]) *Let* $0 < \eta < 1$ *and* $\varphi \in C^1(J, \mathbb{R})$*. Then*

$${}^{CF}I_{0+}^{\eta}\,{}^{CF}D_{0+}^{\eta}\varphi(t) = \varphi(t) - \varphi(0).$$

Clearly, $I_{0+}^1[\frac{d}{dt}\varphi(t)] = \varphi(t) - \varphi(0)$.

Lemma 2.5 *Let* $0 < \eta < 1$ *and* $\varphi \in C^1(J, \mathbb{R})$*. Then*

$${}^{CF}D_{0+}^{\eta}\,{}^{CF}I_{0+}^{\eta}\varphi(t) = \varphi(t) - e^{-\frac{\eta t}{1-\eta}}\varphi(0).$$

Proof. From Definitions 2.1 and 2.2, one has

$$\begin{aligned}{}^{CF}D_{0+}^{\eta}\,{}^{CF}I_{0+}^{\eta}\varphi(t) &= \frac{1}{1-\eta}\int_0^t \big[(1-\eta)\varphi'(s) + \eta\varphi(s)\big]\exp(-\frac{\eta}{1-\eta}(t-s))\,\mathrm{d}s \\ &= (1-\eta)\,{}^{CF}D_{0+}^{\eta}\varphi(t) + \int_0^t \varphi(s)\,\mathrm{d}\big(\exp(-\frac{\eta}{1-\eta}(t-s))\big) \\ &= \varphi(t) - e^{-\frac{\eta t}{1-\eta}}\varphi(0).\end{aligned}$$

□

Lemma 2.6 *Let* $\varphi \in C(J, \mathbb{R})$*,* $0 < \xi, \eta < 1$*. Then*

$$\begin{aligned}{}^{CF}I_{0+}^{\xi}\,{}^{CF}I_{0+}^{\eta}\varphi(t) &= {}^{CF}I_{0+}^{\eta}\,{}^{CF}I_{0+}^{\xi}\varphi(t), \\ {}^{CF}I_{0+}^{\xi}I_{0+}^{1}\varphi(t) &= I_{0+}^{1}\,{}^{CF}I_{0+}^{\xi}\varphi(t).\end{aligned}$$

Moreover,

$${}^{CF}I_{0+}^{\xi}\,{}^{CF}I_{0+}^{\eta}\varphi(t) = (1-\eta)(1-\xi)\varphi(t) + (\eta + \xi - 2\eta\xi)I_{0+}^{1}\varphi(t) + \eta\xi I_{0+}^{2}\varphi(t).$$

Proof. From Definition 2.1, we have

$$\begin{aligned}{}^{CF}I_{0+}^{\xi}\,{}^{CF}I_{0+}^{\eta}\varphi(t) &= {}^{CF}I_{0+}^{\xi}\Big[(1-\eta)\varphi(t) + \eta I_{0+}^{1}\varphi(t)\Big] \\ &= (1-\xi)\Big[(1-\eta)\varphi(t) + \eta I_{0+}^{1}\varphi(t)\Big] + \xi I_{0+}^{1}\Big[(1-\eta)\varphi(t) + \eta I_{0+}^{1}\varphi(t)\Big] \\ &= (1-\eta)\,{}^{CF}I_{0+}^{\xi}\varphi(t) + \eta I_{0+}^{1}\,{}^{CF}I_{0+}^{\xi}\varphi(t) \;=\; {}^{CF}I_{0+}^{\eta}\,{}^{CF}I_{0+}^{\xi}\varphi(t).\end{aligned}$$

Likewise, the second relation follows. From Definition 2.1, we can see that

$$\begin{aligned} {}^{CF}I^{\xi}_{0^+}\,{}^{CF}I^{\eta}_{0^+}\varphi(t) &= (1-\xi)\,{}^{CF}I^{\eta}_{0^+}\varphi(t)+\xi\, I^{1}_{0^+}\,{}^{CF}I^{\eta}_{0^+}\varphi(t)\\ &= (1-\xi)\big[(1-\eta)\varphi(t)+\eta\, I^{1}_{0^+}\varphi(t)\big]+\xi\big[(1-\eta)\, I^{1}_{0^+}\varphi(t)+\eta\, I^{2}_{0^+}\varphi(t)\big]. \end{aligned}$$

Then the conclusion can be proved to hold. □

Theorem 2.7 (Gronwall-Bellman Inequality [15]) *Let $\psi(t)$ and $\theta_2(t)$ be non-negative continuous functions defined on the interval $[a,b]$, and let $\theta_1(t)$ be a non-negative and nondecreasing function. If $\psi(t)$ satisfies the following integral inequality for all $t\in[a,b]$:*

$$\psi(t)\le\theta_1(t)+\int_a^t\theta_2(s)\psi(s)\,\mathrm{d}s,$$

then it follows that

$$\psi(t)\le\theta_1(t)\exp\left(\int_a^t\theta_2(s)ds\right),\quad t\in[a,b].$$

3 Existence and uniqueness of solutions

In this section, we present the existence and uniqueness of solution for IVP (1.1)-(1.2).

Theorem 3.1 *If for any $x(t)\in C(J,\mathbb{R})$, $f(t,x(t))\in C(J,\mathbb{R})$, then $x\in C^1(J,\mathbb{R})$ satisfies (1.1)-(1.2) if and only if $x(t)$ satisfies the following equation*

$$\begin{aligned} x(t) &= -a_1(t)\int_0^t\frac{x(s)}{\lambda(s)}\,\mathrm{d}s+a_2(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s\right]\\ &\quad+a(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t f(s,x(s))\,\mathrm{d}s\right]+H(t), \end{aligned}\tag{3.1}$$

where $a_1(t)$, $a_2(t)$, $a(t)$, $H(t)$ are defined by

$$a(t) = \frac{(1-\alpha)\lambda(t)}{1-\beta+(1-\alpha)\lambda(t)};\tag{3.2}$$

$$a_1(t) = \frac{\beta}{1-\alpha}a(t),\quad a_2(t)=\frac{\alpha}{(1-\alpha)^2}a(t),\tag{3.3}$$

$$H(t) = a(t)x_0+\frac{a(t)}{1-\alpha}\,{}^{CF}I^{\beta}_{0^+}\left[\frac{e^{-\frac{\alpha}{1-\alpha}t}}{\lambda(t)}\right]x_0.\tag{3.4}$$

.

Proof. Let $x(t)$ satisfy (1.1)-(1.2). By Definition 2.2, we have

$$\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x'(s)ds+\lambda(t)\ {}^{CF}D^{\beta}_{0^+}x(t)=\int_0^t f(s,x(s))\,\mathrm{d}s.$$

By calculation, we obtain

$$\frac{1}{1-\alpha}(x(t)-\frac{1}{1-\alpha}x_0e^{-\frac{\alpha}{1-\alpha}t}-\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s+\lambda(t)\ {}^{CF}D^{\beta}_{0^+}x(t) \\ =\int_0^t f(s,x(s))\,\mathrm{d}s.$$

This can be rewritten as

$${}^{CF}D^{\beta}_{0^+}x(t)=-\frac{1}{1-\alpha}\frac{x(t)}{\lambda(t)}+\frac{1}{1-\alpha}\frac{x_0}{\lambda(t)}e^{-\frac{\alpha}{1-\alpha}t} \\ +\frac{\alpha}{(1-\alpha)^2}\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)ds+\frac{1}{\lambda(t)}\int_0^t f(s,x(s))\,\mathrm{d}s. \tag{3.5}$$

Applying the operator ${}^{CF}I^{\beta}_{0^+}$ to both sides of the above equation and taking Lemma 2.4 into account, we have

$$x(t)=\left\{1+\frac{1}{1-\alpha}\,{}^{CF}I^{\beta}_{0^+}\left[\frac{e^{-\frac{\alpha}{1-\alpha}t}}{\lambda(t)}\right]\right\}x_0-\frac{1-\beta}{1-\alpha}\frac{x(t)}{\lambda(t)}-\frac{\beta}{1-\alpha}\int_0^t\frac{x(s)}{\lambda(s)}\,\mathrm{d}s \\ +\frac{\alpha}{(1-\alpha)^2}\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s\right]+{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t f(s,x(s))\,\mathrm{d}s\right], \tag{3.6}$$

which implies

$$x(t)=-a_1(t)\int_0^t\frac{x(s)}{\lambda(s)}\,\mathrm{d}s+a_2(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)ds\right] \\ +a(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t f(s,x(s))\,\mathrm{d}s\right]+H(t),$$

where $H(t)$ is defined by (3.4).

Then we conclude that (3.1) holds. On the other hand, if $x(t)$ satisfies (3.1), then it satisfies (3.6). Applying the operator ${}^{CF}D^{\beta}_{0^+}$ to both sides of (3.6) and taking Lemma 2.5 into account, we have

$${}^{CF}D^{\beta}_{0^+}x(t)=-\frac{1}{1-\alpha}\,{}^{CF}D^{\beta}_{0^+}\,{}^{CF}I^{\beta}_{0^+}\frac{x(t)}{\lambda(t)}+\frac{x_0}{1-\alpha}\,{}^{CF}D^{\beta}_{0^+}\,{}^{CF}I^{\beta}_{0^+}\left[\frac{e^{-\frac{\alpha}{1-\alpha}t}}{\lambda(t)}\right] \\ +\frac{\alpha}{(1-\alpha)^2}\,{}^{CF}D^{\beta}_{0^+}\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s\right]+{}^{CF}D^{\beta}_{0^+}\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\,I^1_{0^+}f(t,x(t))\right] \\ =-\frac{1}{1-\alpha}\frac{x(t)}{\lambda(t)}+\frac{1}{1-\alpha}\frac{x_0}{\lambda(0)}e^{-\frac{\beta t}{1-\beta}}+\frac{x_0}{1-\alpha}\frac{e^{-\frac{\alpha}{1-\alpha}t}}{\lambda(t)}-\frac{x_0e^{-\frac{\beta t}{1-\beta}}}{(1-\alpha)\lambda(0)} \\ +\frac{\alpha}{(1-\alpha)^2}\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s+\frac{1}{\lambda(t)}\,I^1_{0^+}f(t,x(t)).$$

This is (3.5), so we can prove that $x(t)$ satisfies (1.1). Obviously, $x(0)=x_0$, the theorem is proved. □

Next, we study the existence and uniqueness of solutions for IVP (1.1)-(1.2).

Theorem 3.2 *Let* $f(t,\, x(t)) \in C(J, \mathbb{R})$ *for any* $x(t) \in C(J, \mathbb{R})$. *If there exists a function* $l_f(t) \in C(J, \mathbb{R}^+)$ *such that*

$$|f(t, x_1) - f(t, x_2)| \le l_f(t)|x_1 - x_2|, \quad \textit{for} \quad x_1, x_2 \in \mathbb{R}, \tag{3.7}$$

then (1.1)-(1.2) *has a unique solution* $x(t) \in C^1(J, \mathbb{R})$.

For convenience, we set

$$\begin{aligned} &\|a\|_{\mathcal{C}} := a^*, \quad \|a_i\|_{\mathcal{C}} := a_i^*,\, i = 1, 2, \quad m_\lambda := \min_{t\in[0,T]} |\lambda(t)|, \\ &\Lambda = \frac{1}{m_\lambda}\Big[a_1^* + (1 - \beta + \frac{\beta T}{2})(a_2^* + a^*\|l_f\|_{\mathcal{C}})\Big]. \end{aligned}$$

Proof. We define an operator $\mathcal{T} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as follows:

$$\begin{aligned} (\mathcal{T}x)(t) \;&=\; -a_1(t) I^1_{0^+}\frac{x(s)}{\lambda(s)} + a_2(t)\, {}^{CF}I^\beta_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x(s)\,\mathrm{d}s\right] \\ &\quad + a(t)\, {}^{CF}I^\beta_{0^+}\left[\frac{1}{\lambda(t)}\, I^1_{0^+} f(t, x(t))\right] + H(t). \end{aligned}$$

Clearly, $\mathcal{T}$ is well-defined and a fixed point of $\mathcal{T}$ is a solution of (3.1).

For $x_1(t), x_2(t) \in C(J, \mathbb{R})$, one obtains

$$\Big|\int_0^t \frac{x_1(s)}{\lambda(s)}\,\mathrm{d}s - \int_0^t \frac{x_2(s)}{\lambda(s)}\,\mathrm{d}s\Big| \le \frac{t}{m_\lambda}\|x_1 - x_2\|_{\mathcal{C}}\;,$$

$$\begin{aligned} &\Big|\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x_1(s)\,\mathrm{d}s - \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x_2(s)\,\mathrm{d}s\Big| \\ &\qquad\qquad \le \int_0^t |x_1(s) - x_2(s)| e^{-\frac{\alpha(t-s)}{1-\alpha}}\,\mathrm{d}s \le t\,\|x_1 - x_2\|_{\mathcal{C}}\;, \end{aligned}$$

$$|I^1_{0^+} f(t, x_1(t)) - I^1_{0^+} f(t, x_2(t))| \le \|l_f\|_{\mathcal{C}}\, t\, \|x_1 - x_2\|_{\mathcal{C}}\;.$$

Furthermore, we get

$$\begin{aligned} &\Big|{}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x_1(s)\,\mathrm{d}s\Big] - {}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x_2(s)\,\mathrm{d}s\Big]\Big| \\ &\qquad\qquad \le \frac{t(1 - \beta + \frac{\beta T}{2})}{m_\lambda}\|x_1 - x_2\|_{\mathcal{C}} \end{aligned}$$

and

$$\Big|{}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\, I^1_{0^+} f(t, x_1(t))\Big] - {}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\, I^1_{0^+} f(t, x_2(t))\Big]\Big| \le \frac{t(1 - \beta + \frac{\beta T}{2})\|l_f\|_{\mathcal{C}}}{m_\lambda}\|x_1 - x_2\|_{\mathcal{C}}.$$

Thus we have

$$|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)| \le \frac{t}{m_\lambda}\Big[a_1^* + (1 - \beta + \frac{\beta T}{2})(a_2^* + a^*\|l_f\|_{\mathcal{C}})\Big]\|x_1 - x_2\|_{\mathcal{C}} = \Lambda t\|x_1 - x_2\|_{\mathcal{C}}.$$

Since

$$\begin{aligned}
\Big|\int_0^t \frac{(\mathcal{T}x_1)(s)}{\lambda(s)}\,\mathrm{d}s - \int_0^t \frac{(\mathcal{T}x_2)(s)}{\lambda(s)}\,\mathrm{d}s\Big| &\le \frac{\Lambda}{m_\lambda}\frac{t^2}{2}\|x_1-x_2\|_{\mathcal{C}}\,,\\
\Big|\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(\mathcal{T}x_1)(s)\,\mathrm{d}s - \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(\mathcal{T}x_2)(s)\,\mathrm{d}s\Big| & \\
\le \int_0^t |(\mathcal{T}x_1)(s)-(\mathcal{T}x_2)(s)| e^{-\frac{\alpha(t-s)}{1-\alpha}}\,\mathrm{d}s &\le \frac{\Lambda t^2}{2}\|x_1-x_2\|_{\mathcal{C}}\,,\\
|I^1_{0^+}f(t,(\mathcal{T}x_1)(t)) - I^1_{0^+}f(t,(\mathcal{T}x_2)(t))| &\le \|l_f\|_{\mathcal{C}}\frac{\Lambda t^2}{2}\|x_1-x_2\|_{\mathcal{C}}\,,
\end{aligned}$$

we have

$$\begin{aligned}
&\Big|{}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(\mathcal{T}x_1)(s)\,\mathrm{d}s\Big] - {}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(\mathcal{T}x_2)(s)\,\mathrm{d}s\Big]\Big|\\
&\qquad \le \frac{\Lambda(1-\beta+\frac{\beta T}{3})}{m_\lambda}\frac{t^2}{2}\|x_1-x_2\|_{\mathcal{C}} \le \frac{\Lambda(1-\beta+\frac{\beta T}{2})}{m_\lambda}\frac{t^2}{2}\|x_1-x_2\|_{\mathcal{C}}
\end{aligned}$$

and

$$\begin{aligned}
&\Big|{}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\,I^1_{0^+}f(t,(\mathcal{T}x_1)(t))\Big] - {}^{CF}I^\beta_{0^+}\Big[\frac{1}{\lambda(t)}\,I^1_{0^+}f(t,(\mathcal{T}x_2)(t))\Big]\Big|\\
&\qquad\qquad \le \frac{\Lambda\,(1-\beta+\frac{\beta T}{2})\,\|l_f\|_{\mathcal{C}}}{m_\lambda}\,\frac{t^2}{2}\,\|x_1-x_2\|_{\mathcal{C}}
\end{aligned}$$

Then, we have

$$\big|(\mathcal{T}^2x_1)(t)-(\mathcal{T}^2x_2)(t)\big| \le \frac{(\Lambda t)^2}{2}\|x_1-x_2\|_{\mathcal{C}}\,.$$

By induction, we deduce that

$$\Big|(\mathcal{T}^kx_1)(t)-(\mathcal{T}^kx_2)(t)\Big| \le \frac{(\Lambda t)^k}{k!}\|x_1-x_2\|_{\mathcal{C}}\,.$$

According to the generalized Banach contraction principle, $\mathcal{T}$ has a unique fixed point $x(t) \in C(J,\mathbb{R})$, which is precisely the unique solution of (1.1)–(1.2). □

4 Stability result

Definition 4.1 *Let $f(t,\,x(t)) \in C(J,\mathbb{R})$ for any $x(t) \in C(J,\mathbb{R})$. Equation* (1.1) *is UH stable if there is a constant $C_x > 0$ such that for any $\varepsilon > 0$ and for $\hat{u}(t) \in C^1(J,\mathbb{R})$ satisfying*

$$|\,{}^{CF}D^\alpha_{0^+}\hat{x}(t) + \lambda(t)\,{}^{CF}D^\beta_{0^+}\hat{x}(t) -\, I^1_{0^+}f(t,\,\hat{x}(t))| \le \varepsilon, \quad t \in J, \tag{4.1}$$

there exists a solution $x(t)$ to equation (1.1) *with $|x(t)-\hat{x}(t)| \le C_x\varepsilon$.*

Remark 4.2 *The function $\hat{x}(t) \in C^1(J,\mathbb{R})$ is a solution of* (4.1) *if and only if there exists a function $\sigma(t) \in C^1(J,\mathbb{R})$ (depending on $\hat{x}(t)$) such that (i) $|\sigma(t)| \le \varepsilon, \quad t \in J$;*
(ii) for $t \in J$, ${}^{CF}D^\alpha_{0^+}\hat{x}(t) + \lambda(t)\,{}^{CF}D^\beta_{0^+}\hat{x}(t) -\, I^1_{0^+}f(t,\,\hat{x}(t)) = \sigma(t)$.

Theorem 4.3 *Let $f(t,\, x(t)) \in C(J,\mathbb{R})$ for any $x(t) \in C(J,\mathbb{R})$. Then* (1.1) *is UH stable.*

Proof. Let $\hat{x}(t)$ be a solution to (4.1) and $\hat{x}(0) = x_0$, denote

$$\sigma(t) = {}^{CF}D^{\alpha}_{0^+}\hat{x}(t) + \lambda(t)\,{}^{CF}D^{\beta}_{0^+}\hat{x}(t) - \, I^1_{0^+} f(t,\, \hat{x}(t)),$$

then $|\sigma(t)| \le \varepsilon$ and $\sigma(0) = 0$. From Theorem 3.1, $\hat{x}(t)$ satisfies

$$\begin{aligned} \hat{x}(t) &= -a_1(t) I^1_{0^+}\frac{\hat{x}(t)}{\lambda(t)} + a_2(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}\hat{x}(s)\,\mathrm{d}s\right] \\ &\quad + a(t)\,{}^{CF}I^{\beta}_{0^+}\left\{\frac{1}{\lambda(t)}\, I^1_{0^+}[f(t,\hat{x}(t))] + \frac{\sigma(t)}{\lambda(t)}]\right\} + H(t). \end{aligned}$$

We denote by $x(t) \in C^1(J,\mathbb{R})$ the unique solution of IVP (1.1) with (1.2) and

$$\begin{aligned} x(t) &= -a_1(t) I^1_{0^+}\frac{x(t)}{\lambda(t)} + a_2(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)ds\right] \\ &\quad + a(t)\,{}^{CF}I^{\beta}_{0^+}\left[\frac{1}{\lambda(t)}\, I^1_{0^+} f(t,x(t))\right] + H(t). \end{aligned}$$

Since

$$\begin{aligned} \Big|\int_0^t \frac{x(s)}{\lambda(s)}\,\mathrm{d}s - \int_0^t \frac{\hat{x}(s)}{\lambda(s)}\,\mathrm{d}s\Big| &\le \frac{1}{m_\lambda}\int_0^t |x(s)-\hat{x}(s)|\,\mathrm{d}s, \\ \Big|\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s - \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}\hat{x}(s)\,\mathrm{d}s\Big| &\le \int_0^t |x(s)-\hat{x}(s)|\,\mathrm{d}s, \\ \big|I^1_{0^+} f(t,x(t)) - I^1_{0^+} f(t,\hat{x}(t))\big| &\le \|l_f\|_{\mathcal{C}}\int_0^t |x(s)-\hat{x}(s)|\,\mathrm{d}s, \end{aligned}$$

we get

$$\Big|\,{}^{CF}I^{\beta}_{0^+}\Big[\frac{1}{\lambda(t)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}\big(x(s)-\hat{x}(s)\big)\,\mathrm{d}s\Big]\Big| \le \frac{1-\beta+\beta T}{m_\lambda}\int_0^t |x(s)-\hat{x}(s)|\,\mathrm{d}s,$$

and

$$\begin{aligned} &\Big|\,{}^{CF}I^{\beta}_{0^+}\Big[\frac{1}{\lambda(t)}\, I^1_{0^+} f(t,x(t))\Big] - \,{}^{CF}I^{\beta}_{0^+}\Big[\frac{1}{\lambda(t)}\, I^1_{0^+} f(t,\hat{x}(t))\Big]\Big| \\ &\qquad\qquad \le \frac{(1-\beta+\beta T)\|l_f\|_{\mathcal{C}}}{m_\lambda}\int_0^t |x(s)-\hat{x}(s)|\,\mathrm{d}s. \end{aligned}$$

We deduce

$$|x(t)-\hat{x}(t)| \;\le\; \widetilde{\Lambda}\int_0^t |x(s)-\hat{x}(s)|\,\mathrm{d}s,$$

where

$$\widetilde{\Lambda} \;=\; \frac{1}{m_\lambda}\Big[a_1^* + (1-\beta+\beta T)(a_2^* + a^*\|l_f\|_{\mathcal{C}})\Big].$$

By Theorem 2.7, we conclude that

$$|x(t)-\hat{x}(t)| \;\le\; C\varepsilon,$$

where $C := \frac{a^*(1-\beta+\beta T)e^{\tilde{\Lambda}T}}{m_\lambda}$. From Definition 4.1, equation (1.1) is UH stable. □

5 The case of constant coefficients

We first recall Theorem 3.1 in [16], $\alpha = 1$, as follows.

Lemma 5.1 *For any $\mu \in \mathbb{R}$, we have the following results:*

(i) For any $h \in C(J, \mathbb{R})$, the series $\sum_{k=0}^{\infty} (-\mu)^k I_{0+}^k h(t)$ is convergent and the sum is given by

$$\sum_{k=0}^{\infty} (-\mu)^k I_{0+}^k h(t) = h(t) - \mu \int_0^t e^{-\mu(t-s)} h(s) \, \mathrm{d}s, \qquad t \in J.$$

(ii) The operator $I + \mu I_{0+}^1 : C[J, \mathbb{R}] \to C[J, \mathbb{R}]$ is invertible and continuous, and satisfies

$$\left(I + \mu I_{0+}^1\right)^{-1} h(t) = \sum_{k=0}^{\infty} (-\mu)^k I_{0+}^k h(t), \qquad t \in J.$$

When $\lambda(t) \equiv \lambda \neq 0$ is a constant, $a(t),\ a_1(t),\ a_2(t)$ in (3.2)–(3.3) can be rewritten as

$$a = \frac{(1-\alpha)\lambda}{1-\beta+(1-\alpha)\lambda}, \quad a_1 = \frac{\lambda\beta}{1-\beta+(1-\alpha)\lambda}, \quad a_2 = \frac{\alpha}{1-\alpha} \frac{\lambda}{1-\beta+(1-\alpha)\lambda}. \tag{5.1}$$

Theorem 5.2 *Let $f(t,\, x(t)) \in C(J, \mathbb{R})$ for any $x(t) \in C(J, \mathbb{R})$. If* (3.7) *holds, then the IVP*

$$\begin{cases} {}^{CF}D_{0+}^{\alpha} x(t) + \lambda \, {}^{CF}D_{0+}^{\beta} x(t) = \int_0^t f(s, x(s)) \, \mathrm{d}s, \ t \in J := [0, T], & (5.2) \\ x(0) = 0, & (5.3) \end{cases}$$

has a unique solution $x(t) \in C^1(J, \mathbb{R})$ satisfying the integral equation

$$x(t) = \frac{1}{D} \, {}^{CF}I_{0+}^{\beta} \, {}^{CF}I_{0+}^{\alpha} \int_0^t e^{-\mu(t-s)} f(s, x(s)) \, \mathrm{d}s,$$

where $D = 1 - \beta + (1-\alpha)\lambda \neq 0$ and $\mu := \frac{\beta + \lambda\alpha}{D}$.

Proof. By Theorem 3.1, $\lambda(t) \equiv \lambda$, we can deduce that (5.2)–(5.3) is equivalent to the following integral equation

$$x(t) \ = \ -\tilde{a}_1 \int_0^t x(s) \, \mathrm{d}s + \tilde{a}_2 \, {}^{CF}I_{0+}^{\beta} \Big[\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} x(s) \, \mathrm{d}s \Big] + \tilde{a} \, {}^{CF}I_{0+}^{\beta} \Big[I_{0+}^1 f(t, x(t)) \Big],$$

where

$$\tilde{a} = \frac{a}{\lambda} = \frac{1-\alpha}{D}, \quad \tilde{a}_1 = \frac{a_1}{\lambda} = \frac{\beta}{D}, \quad \tilde{a}_2 = \frac{a_2}{\lambda} = \frac{\alpha}{D(1-\alpha)}. \tag{5.4}$$

Applying ${}^{CF}I_{0+}^{\alpha}$ to both sides of the above equation and combining with

$$-\tilde{a}_1 \, {}^{CF}I_{0+}^{\alpha} \int_0^t x(s) \, \mathrm{d}s \ = \ -\tilde{a}_1 \Big[(1-\alpha) \int_0^t x(s) \, \mathrm{d}s + \alpha \int_0^t \int_0^s x(\tau) \, \mathrm{d}\tau ds \Big]$$

and

$$\begin{aligned}
{}^{CF}I^{\alpha}_{0^+}\Big[\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s\Big] &= (1-\alpha)\int_0^t x(s)e^{-\frac{\alpha(t-s)}{1-\alpha}}\,\mathrm{d}s+\alpha\int_0^t\int_0^s x(\tau)e^{-\frac{\alpha(s-\tau)}{1-\alpha}}\,\mathrm{d}\tau\,\mathrm{d}s\\
&= (1-\alpha)\int_0^t x(s)\,\mathrm{d}s,
\end{aligned}$$

we find

$$\begin{aligned}
{}^{CF}I^{\alpha}_{0^+}x(t) &= -\tilde{a}_1\Big[(1-\alpha)\int_0^t x(s)\,\mathrm{d}s+\alpha\int_0^t\int_0^s x(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big]\\
&\quad+\tilde{a}_2\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\Big[\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}x(s)\,\mathrm{d}s\Big]+\tilde{a}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\Big[\,I^1_{0^+}f(t,x(t))\Big]\\
&= -\tilde{a}_1\Big[(1-\alpha)\int_0^t x(s)\,\mathrm{d}s+\alpha\int_0^t\int_0^s x(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big]\\
&\quad+\tilde{a}_2(1-\alpha)\,{}^{CF}I^{\beta}_{0^+}\int_0^t x(s)\,\mathrm{d}s+\tilde{a}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\Big[\,I^1_{0^+}f(t,x(t))\Big]\\
&= -\tilde{a}_1\Big[(1-\alpha)\int_0^t x(s)\,\mathrm{d}s+\alpha\int_0^t\int_0^s x(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big]\\
&\quad+\tilde{a}_2(1-\alpha)(1-\beta)\int_0^t x(s)\,\mathrm{d}s+\tilde{a}_2(1-\alpha)\beta\int_0^t\int_0^s x(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad+\tilde{a}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\Big[\,I^1_{0^+}f(t,x(t))\Big]\\
&= \frac{\alpha-\beta}{D}\int_0^t x(s)\,\mathrm{d}s+\tilde{a}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\Big[\,I^1_{0^+}f(t,x(t))\Big].
\end{aligned}$$

Therefore

$$(I+\frac{\beta+\lambda\alpha}{D}I^1_{0^+})x(t) = \frac{1}{D}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\Big[\,I^1_{0^+}f(t,x(t))\Big]. \tag{5.5}$$

By Lemma 5.1, we obtain

$$\begin{aligned}
x(t) &= \frac{1}{D}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\sum_{k=0}^{\infty}(-\mu)^k I^{k+1}_{0^+}f(t,x(t))\\
&= \frac{1}{D}\,{}^{CF}I^{\beta}_{0^+}\,{}^{CF}I^{\alpha}_{0^+}\int_0^t e^{-\mu(t-s)}f(s,x(s))\,\mathrm{d}s.
\end{aligned}$$

□

6 Applications

Example 6.1 *We consider the following IVP*

$$\begin{cases}
{}^{CF}D^{\frac{1}{2}}_{0^+}x(t)+e^t\,{}^{CF}D^{\frac{1}{3}}_{0^+}x(t)=\displaystyle\int_0^t\frac{s^2\,x(s)}{1+x^2(s)}\,\mathrm{d}s,\ t\in[0,1], & (6.1)\\
x(0)=2. & (6.2)
\end{cases}$$

For

$$\alpha=\frac{1}{2},\ \ \beta=\frac{1}{3},\ x_0=2,\ T=1,\ \lambda(t)=e^t,\ \ f(t,x(t))=\frac{t^2\,x(t)}{1+x^2(t)},$$

we can rewrite (6.1)–(6.2) *as IVP* (1.1)–(1.2)*. Clearly, for* $x \in C(J,\mathbb{R})$*,* $f(t,x(t)) \in C(J,\mathbb{R})$*. Since*

$$\frac{|1-x_1x_2|}{(1+x_1^2)(1+x_2^2)} \le \frac{\sqrt{(1+x_1^2)(1+x_2^2)}}{(1+x_1^2)(1+x_2^2)} = \frac{1}{\sqrt{(1+x_1^2)(1+x_2^2)}} \le 1\,,$$

we have

$$|f(t,x_1)-f(t,x_2)| \le t^2 \cdot \frac{|x_1-x_2|\,|1-x_1x_2|}{(1+x_1^2)(1+x_2^2)} < t^2\,|x_1-x_2|\,.$$

Now we have satisfied all the assumptions of Theorem 3.2, and it follows that IVP (6.1)–(6.2) *has a unique solution* $x(t) \in C^1([0,1],\mathbb{R})$.

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