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SQUARE-MEAN PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF CLASS r IN THE α-NORM UNDER THE LIGHT OF MEASURE THEORY

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Abstract. The main objective of this work is to study the existence and uniqueness of the square-mean (μ, ν)-pseudo almost automorphic solution of class r in the α-norm for a stochastic partial functional differential equation. For this purpose, we use the Banach contraction principle and the techniques of fractional powers of an operator to obtain the required results. An illustrative example is provided.

Keywords: (μ, ν)-pseudo almost automorphic functions, ergodicity, measure theory, partial functional differential equations, stochastic evolution equations, stochastic processes.

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1 Introduction

In this paper, we study the existence and uniqueness of square-mean (μ, ν)-pseudo almost automorphic solutions in the α-norm for the following stochastic differential equation

$$\mathrm{d}x(t) = [-Ax(t) + L(x_t) + f(t)]\mathrm{d}t + g(t)\mathrm{d}W(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

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where $-A\colon D(A) \subset H \to H$ is the infinitesimal generator of a compact analytic semi-group $(T(t))_{t\geq 0}$ on $L^2(P,H)$. Let $H_\alpha = (D(A^\alpha), \|.\|_\alpha)$, and let $\mathcal{C}_\alpha = C([-r,0], H_\alpha)$, with $0 < \alpha < 1$, denote the space of continuous functions from $[-r,0]$ to H_α, where A^α is the fractional α-power of A. (The operator A^α and the space H_α will be described later.) We define the norm on $\mathcal{C}_\alpha$ by

$$\|\varphi\|_{\mathcal{C}_\alpha} = \|A^\alpha \varphi\|_{C([-r,0],L^2(P,H))}.$$

For every $t \geq 0$ and $x \in \mathcal{C}_\alpha = C([-r,0], H_\alpha)$, where $r > 0$, the history function x_t is defined by $x_t(\theta) = x(t+\theta)$ for $-r \leq \theta \leq 0$. Here, L is a bounded linear operator from $\mathcal{C}_\alpha$ into $L^2(P,H)$, and $f\colon \mathbb{R} \to L^2(P,H)$ and $g\colon \mathbb{R} \to L^2(P,H)$ are two stochastic processes. Finally, $W(t)$ is a two-sided standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) | u, v \leq t\}$. Here, we assume that $(H, \|.\|)$ is a real separable Hilbert space and $L^2(P,H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 \mathrm{d}P < +\infty.$$

The concept of almost automorphy is a generalization of the classical periodicity. This notion was introduced in the literature by Bochner [8, 9]. For a comprehensive treatment of almost automorphic functions, we refer the reader to the books by G. M. N'Guérékata [23, 24] (see also [3]).

The notion of square-mean almost automorphic stochastic processes was introduced by Fu and Liu; for more details see [11] and the references therein. In addition, these authors defined the space of pseudo-almost automorphic stochastic processes to study the existence, uniqueness, and stability of solutions in the square-mean sense.

The aim of this work is to extend the results obtained in [17, 21, 29], where the authors studied equation (1.1) in the deterministic setting. It is well-known that stochastic modelling plays a crucial role in many fields, including physics, engineering, economics, and the social sciences. Accordingly, stochastic differential systems have attracted considerable research attention in recent years, with particular interest in their quantitative and qualitative properties, such as the existence, uniqueness and stability of their solutions. For further details, the reader is referred to [14, 19, 18] and the references therein. In this context, recent contributions have focused on square-mean pseudo almost automorphic solutions for abstract differential equations similar to equation (1.1); see, for instance, [4, 5, 15, 14, 22] and the references therein.

In [13], M. A. Diop *et al.* studied the existence, uniqueness and stability of the square-mean μ-pseudo almost periodic and automorphic solutions of a stochastic evolution equation. In [20], the authors used the Banach contraction principle and the techniques of fractional powers of operators to study the existence and uniqueness of square-mean almost automorphic mild solutions for a class of stochastic differential equations in a real separable Hilbert space. More recently, in [30], I. Zabsonre and M. Kiema studied the existence and uniqueness of the square-mean (μ,ν)-pseudo almost automorphic solutions of infinite class for a stochastic evolution equation. However, to the best of the authors' knowledge, the existence of square-mean (μ,ν)-pseudo almost automorphic solutions of class r in the α-norm for equation (1.1) has not yet been addressed in the literature, which constitutes the main motivation of this paper.

This paper is organized as follows. In Section 2, we recall some preliminary results on analytic semi-groups and fractional powers associated to their generators. In Section 3, we present the spectral decomposition of the phase space and the variation of constants formula. In Section 4, we study square-mean (μ,ν)-ergodic processes of class r. In Section 5, we study square-mean

(μ,ν)-pseudo almost automorphic processes. In Section 6, we discuss the existence and uniqueness of square-mean (μ,ν)-pseudo almost automorphic solutions of class r. The final section is dedicated to an application.

2 Analytic semi-group

Let $(L^2(P,H),\|.\|)$ be a Banach space, let α be a constant such that $0<\alpha<1$ and let $-A$ be the infinitesimal generator of a bounded analytic semi-group of linear operators $(T(t))_{t\geq 0}$ on $L^2(P,H)$. We assume without loss of generality that $0\in\rho(A)$. Note that if the assumption $0\in\rho(A)$ is not satisfied, instead of the operator A one can consider the operator $(A-\sigma I)$ with σ large enough such that $0\in\rho(A-\sigma I)$. This allows us to define the fractional power A^α as a closed linear invertible operator with domain $D(A^\alpha)$ dense in $L^2(P,H)$. The closedness of A^α implies that $D(A^\alpha)$ is a Banach space when endowed with the graph norm of A^α, that is, $|x|=\|x\|+\|A^\alpha x\|$. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $\|x\|_\alpha=\|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $\|.\|_\alpha$ is a Banach space, which we denote by $L^2(P,H_\alpha)$. For $0<\beta\leq\alpha<1$ the imbedding $L^2(P,H_\alpha)\hookrightarrow L^2(P,H_\beta)$ is compact if the resolvent operator of A is compact. Also, the following properties are well-known.

Proposition 2.1 ([26]) *Let $0<\alpha<1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semi-group $(T(t))_{t\geq 0}$ on the Banach space $L^2(P,H)$ such that $0\in\rho(A)$. Then, we have*

(i) $T(t)\colon H\to D(A^\alpha)$ *for every* $t>0$,

(ii) $T(t)A^\alpha x=A^\alpha T(t)x$ *for every* $x\in D(A^\alpha)$ *and* $t\geq 0$,

(iii) *for every* $t>0$ *the operator* $A^\alpha T(t)$ *is bounded on* H *and there exist* $M_\alpha>0$ *and* $\omega>0$ *such that* $\|A^\alpha T(t)\|\leq M_\alpha e^{-\omega t}t^{-\alpha}$ *for* $t>0$,

(iv) *if* $0<\alpha\leq\beta<1$, *then* $D(A^\beta)\hookrightarrow D(A^\alpha)$,

(v) *there exists* $N_\alpha>0$ *such that* $\|(T(t)-I)A^{-\alpha}\|\leq N_\alpha t^\alpha$ *for* $t>0$.

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}t^{\alpha-1}T(t)\mathrm{d}t,$$

where the integral converges in the uniform operator topology for every $\alpha>0$. Consequently, if $T(t)$ is compact for each $t>0$, then $A^{-\alpha}$ is compact.

3 Spectral decomposition

To equation (1.1) we associate the following initial value problem

$$\begin{cases} \mathrm{d}u_t=[Au_t+Lu_t+f(t)]\mathrm{d}t+g(t)\mathrm{d}W(t) \text{ for } t\geq 0,\\ u_0=\varphi\in\mathcal{C}_\alpha, \end{cases} \tag{3.1}$$

where $f, g\colon \mathbb{R}^+ \to L^2(P,H)$ are two continuous stochastic processes.

For each $t \geq 0$ we define the linear operator $\mathcal{U}(t)$ on $\mathcal{C}_\alpha$ by $\mathcal{U}(t) = v_t(.,\varphi)$, where $v(.,\varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}v(t) = -Av(t) + L(v_t) \text{ for } t \geq 0, \\ v_0 = \varphi \in \mathcal{C}_\alpha. \end{cases}$$

Proposition 3.1 ([1]) *Let $\mathcal{A}_\mathcal{U}$ be defined on $\mathcal{C}_\alpha$ by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) = \Big\{\varphi \in \mathcal{C}_\alpha \,:\, \varphi' \in \mathcal{C}_\alpha, \varphi(0) \in D(A), \varphi(0)' \in \overline{D(A)} \text{ and } \varphi(0)' = -A\varphi(0) + L(\varphi)\Big\}, \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \in D(\mathcal{A}_\mathcal{U}). \end{cases}$$

Then, $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of the semi-group $(\mathcal{U}(t))_{t\geq 0}$ on $\mathcal{C}_\alpha$.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in L^2(P,H)\},$$

where the function $X_0 c$ is defined by

$$(X_0c)(\theta) = \begin{cases} 0, & \text{if } \theta \in [-r,0), \\ c, & \text{if } \theta = 0. \end{cases}$$

Consider the extension $\widetilde{\mathcal{A}_\mathcal{U}}$ of $\mathcal{A}_\mathcal{U}$ defined on $\mathcal{C}_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \Big\{\varphi \in C^1([-r,0], L^2(P,H_\alpha)) : \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)}\Big\}, \\ \widetilde{\mathcal{A}_\mathcal{U}}\varphi = X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

We make the following assumption:

(H$_0$) The operator $-A$ is the infinitesimal generator of an analytic semi-group $(T(t))_{t\geq 0}$ on the Banach space $L^2(P,H)$ and satisfies $0 \in \rho(A)$.

Lemma 3.2 ([2]) *Assume that* (**H**$_0$) *holds. Then, $\widetilde{\mathcal{A}_\mathcal{U}}$ satisfies the Hile–Yosida condition on $\mathcal{C}_\alpha \oplus \langle X_0 \rangle$, that is, there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}_\mathcal{U}})$ and*

$$\|(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-n}\|_\alpha \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Definition 3.3 *We say a semi-group $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if $\sigma(\mathcal{A}_\mathcal{U}) \cap i\mathbb{R} = \emptyset$.*

For the sequel, we make the following assumption:

(H$_1$) $(T(t))_{t\geq 0}$ is compact on $\overline{D(A)}$ for $t > 0$.

The following result on the spectral decomposition of the phase space $\mathcal{C}_\alpha$ is obtained.

Proposition 3.4 *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. If the semi-group* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic, then the space* $\mathcal{C}_\alpha$ *can be decomposed as a direct sum* $\mathcal{C}_\alpha = S \oplus U$ *of two* $\mathcal{U}(t)$ *invariant closed subspaces* S *and* U *such that the restriction of* $(\mathcal{U}(t))_{t\geq 0}$ *to* U *is a group and there exist positive constants* $\overline{M}$ *and* ω *such that*

$$\|\mathcal{U}(t)\varphi\|_\alpha \leq \overline{M}e^{-\omega t}\|\varphi\|_\alpha \ \text{ for } t \geq 0 \text{ and } \varphi \in S,$$

$$\|\mathcal{U}(t)\varphi\|_\alpha \leq \overline{M}e^{\omega t}\|\varphi\|_\alpha \ \text{ for } t \leq 0 \text{ and } \varphi \in U.$$

The subspaces S *and* U *are called the stable and unstable space, respectively. Moreover, by* Π^s *and* Π^u *we denote the projection operators on* S *and* U*, respectively.*

4 Square-mean (μ, ν)-ergodic of class r

In the following, $\mathcal{N}$ denotes the Lebesgue σ-field of $\mathbb{R}$, while $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$. Note that $L^2(P,H)$ is a Hilbert space endowed with the following norm

$$\|x\|_{L^2} = \left(\int_\Omega \|x\|^2 \mathrm{d}P\right)^{\frac{1}{2}}.$$

Definition 4.1 ([13]) *Let* $x\colon \mathbb{R} \to L^2(P,H)$ *be a stochastic process.*

(i) *x said to be stochastically bounded in square-mean sense, if there exists* $M > 0$ *such that*

$$\mathbb{E}\|x(t)\|^2 \leq M \text{ for all } t \in \mathbb{R}.$$

(ii) *x said to be stochastically continuous in square-mean sense if*

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|^2 \leq M \text{ for all } \ t, s \in \mathbb{R}.$$

We denote by $SBC(\mathbb{R}, L^2(P,H))$ *the space of all the stochastically bounded continuous processes.*

Remark 4.2 ([13]) $(SBC(\mathbb{R}, L^2(P,H)), \|.\|_\infty)$ *is a Banach space, where*

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} (\mathbb{E}(\|x(t)\|^2))^{\frac{1}{2}}.$$

Definition 4.3 *Let* $\mu, \nu \in \mathcal{M}$*. A stochastic process* f *is said to be* α-(μ,ν)*-ergodic in square-mean sense, if* $f \in SBC(\mathbb{R}, L^2(P, H_\alpha))$ *and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|_\alpha^2 \mathrm{d}\mu(t) = 0.$$

We denote by $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu)$ *the space of all such process.*

Definition 4.4 *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be α-(μ,ν)-ergodic of class r in square-mean sense, if $f \in SBC(\mathbb{R}, L^2(P, H_\alpha))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(t)\|_\alpha^2 \mathrm{d}\mu(t) = 0.$$

We denote by $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ the space of all such process.

For $\mu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote by μ_a the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$\mu_a(A) = \mu(\{a + b : b \in A\}) \text{ for } A \in \mathcal{N}. \tag{4.1}$$

In what follows, we will need the following assumptions on $\mu, \nu \in \mathcal{M}$.

($\mathbf{H}_2$) Let μ,$\nu \in \mathcal{M}$ be such that

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \delta < +\infty.$$

($\mathbf{H}_3$) For all $a, b, c \in \mathbb{R}$ such that $0 \le a < b < c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \ge \delta_0 \ \Rightarrow\ \mu(a+\delta, b+\delta) \ge \alpha_0 \mu(\delta, c+\delta).$$

($\mathbf{H}_4$) For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval I such that

$$\mu\big(\{a+\tau : a \in A\}\big) \le \beta\mu(A) \text{ for } A \in \mathcal{N} \text{ satisfying } A \cap I = \emptyset.$$

Proposition 4.5 *Assume that ($\mathbf{H}_2$) holds. Then, the space $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.*

Proof. We can see that $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ is a vector subspace of $SBC(\mathbb{R}; L^2(P, H_\alpha))$. To complete the proof it is enough to prove that $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ is closed in $SBC(\mathbb{R}, L^2(P, H_\alpha))$. Let $(f_n)_n$ be a sequence in $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ such that $\lim_{n\to+\infty} f_n = f$ uniformly in $\mathbb{R}$. From $\nu(\mathbb{R}) = +\infty$, it follows that $\nu([-\tau,\tau]) > 0$ for τ sufficiently large. Let $\|f\|_{\infty,\alpha}^2 = \sup_{t\in\mathbb{R}} \mathbb{E}\|f(t)\|_\alpha^2$, and let $n_0 \in \mathbb{N}$ be such that for all $n \ge n_0$ we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\quad \le \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f_n(\theta) - f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\qquad + \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f_n(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\quad \le \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta\in\mathbb{R}} \mathbb{E}\|f_n(\theta) - f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t)
\end{aligned}$$

$$+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)$$

$$\leq \|f_n-f\|_{\infty,\alpha}^2\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t).$$

This implies that

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\leq\delta\varepsilon \text{ for any } \varepsilon>0.$$

The proof is complete. □

The following theorem is a characterization of square-mean α-(μ,ν)-ergodic processes.

Theorem 4.6 *Suppose that* ($\mathbf{H}_2$) *holds. Let* $\mu,\nu\in\mathcal{M}$ *and let* I *be a bounded* (*possibly empty*) *interval. Assume that* $f\in SBC(\mathbb{R},L^2(P,H_\alpha))$. *The following conditions are equivalent*:

(i) $f\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$,

(ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)=0,$

(iii) *for any* $\varepsilon>0$ *we have*

$$\lim_{\tau\to+\infty}\frac{\mu\big(\{t\in[-\tau,\tau]\setminus I:\mathbb{E}\|f(\theta)\|_\alpha^2>\varepsilon\}\big)}{\nu([-\tau,\tau]\setminus I)}=0.$$

Proof. The proof follows the same arguments as in the proof of [12, Theorem 2.22].

First, we will show the equivalence (i) ⇔ (ii). Let $A=\mu(I)$ and

$$B=\int_I\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t).$$

Since the interval I is bounded and the process f is stochastically bounded and continuous, both A and B are finite. For $\tau>0$ such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$ we have

$$\begin{aligned}&\frac{1}{\nu([-\tau,\tau])\setminus I}\int_{[-\tau,\tau]\setminus I}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta\|_\alpha^2\right)\mathrm{d}\mu(t)\\&=\frac{1}{\nu[-\tau,\tau]-A}\left[\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)-B\right]\\&=\frac{\nu([-\tau,\tau])}{\nu[-\tau,\tau]-A}\left[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)-\frac{B}{\nu([-\tau,\tau])}\right].\end{aligned}$$

From above equalities and the fact that $\nu(\mathbb{R}) = +\infty$, we deduce (ii) is equivalent to

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) = 0,$$

which is precisely condition (i).

Now, we will prove the implication (iii) $\Rightarrow$ (ii). Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon = \left\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon \right\}$$

and

$$B_\tau^\varepsilon = \left\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \le \varepsilon \right\}.$$

Assume that (iii) holds. Then,

$$\lim_{\tau\to+\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]) \setminus I)} = 0. \tag{4.2}$$

From the equality

$$\begin{aligned} &\int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\ &\qquad = \int_{A_\tau^\varepsilon} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\|^p \right) \mathrm{d}\mu(t) + \int_{B_\tau^\varepsilon} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t), \end{aligned}$$

we deduce that for τ sufficient large we have

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\ &\qquad \le \|f\|_{\infty,\alpha}^2 \times \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)} + \varepsilon \frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)}. \end{aligned}$$

Since $\mu(\mathbb{R}) = \nu(\mathbb{R}) = +\infty$, and by ($\mathbf{H}_2$), it follows that for all $\varepsilon > 0$ we have

$$\frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \le \delta\varepsilon.$$

Consequently, (ii) holds.

Finally, we will show the implication (ii) $\Rightarrow$ (iii). From (ii) we have

$$\int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \ge \int_{A_\tau^\varepsilon} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t).$$

Hence,

$$\frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \ge \varepsilon \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)}$$

and

$$\frac{1}{\varepsilon\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \left(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \ge \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)}.$$

Since this holds for sufficiently large τ, we obtain (4.2), that is, (iii). $\square$

Definition 4.7 *Let $f \in SBC(\mathbb{R}, L^2(P, H_\alpha))$ and $\tau \in \mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t) = f(t+\tau)$ for $t \in \mathbb{R}$. A subset $\mathscr{F}$ of $SBC(\mathbb{R}, L^2(P, H_\alpha))$ is said to be translation invariant if for all $f \in \mathscr{F}$, we have $f_\tau \in \mathscr{F}$ for all $\tau \in \mathbb{R}$.*

Definition 4.8 *Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoted by $\mu_1 \sim \mu_2$, if there exist constants $\alpha, \beta > 0$ and a bounded (possibly empty) interval I such that $\alpha\mu_1(A) \le \mu_2(A) \le \beta\mu_1(A)$ for every $A \in \mathcal{N}$ satisfying $A \cap I = \emptyset$.*

Remark 4.9 *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 4.10 *Let $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_1, \nu_1, r) = \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded (possibly empty) interval I such that $\alpha_1\mu_1(A) \le \mu_2(A) \le \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \le \nu_2(A) \le \beta_2\nu_1(A)$ for each $A \in \mathcal{N}$ satisfying $A \cap I = \emptyset$. Clearly, we can then write

$$\frac{1}{\beta_2\nu_1(A)} \le \frac{1}{\nu_2(A)} \le \frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field, for sufficiently large τ we obtain

$$\begin{aligned} &\frac{\alpha_1\mu_1\big(\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\beta_2\mu_2([-\tau,\tau] \setminus I)} \\ &\qquad \le \frac{\mu_2\big(\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\nu_2([-\tau,\tau] \setminus I)} \\ &\qquad \le \frac{\beta_1\mu_1\big(\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 > \varepsilon\}\big)}{\alpha_2\nu([-\tau,\tau] \setminus I)}. \end{aligned}$$

By Theorem 4.6, we deduce that $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_1, \nu_1, r) = \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_2, \nu_2, r)$. □

For $\mu, \nu \in \mathcal{M}$ we set $cl(\mu, \nu) = \big\{\overline{\omega}_1, \overline{\omega}_2 \in \mathcal{M} : \mu_1 \sim \overline{\omega}_1, \nu_1 \sim \overline{\omega}_2\big\}$.

Lemma 4.11 ([7]) *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_4)$. Then, the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 4.12 ([7]) *Condition $(\mathbf{H}_3)$ implies that for all $\sigma > 0$ we have*

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma, \tau+\sigma])}{\mu([-\tau,\tau])} < +\infty.$$

Theorem 4.13 *Assume that $(\mathbf{H}_4)$ holds. Then, $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ is invariant under translations.*

Proof. The proof is inspired by the proof of [6, Theorem 3.5]. Let $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$, there exists $a_0 > 0$ such that $\nu([-\tau - |a|, \tau + |a|]) > 0$ for $|a| > a_0$. Set

$$M_a(\tau) = \frac{1}{\nu_a([\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu_a(t) \text{ for all } \tau > 0 \text{ and } a \in \mathbb{R},$$

where ν_a is the positive measure define by equation (4.1). By Lemma 4.11, it follows that ν and ν_a are equivalent, as are μ and μ_a. Hence, by Theorem 4.10 we have $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_a, \nu_a, r) = \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. Therefore, $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_a, \nu_a, r)$, that is, $\lim_{t\to+\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.

For $A \in \mathcal{N}$ we denote by χ_A the characteristic function of A. By the definition of μ_a, we obtain

$$\int_{[-\tau,\tau]} \chi_A(t)\mathrm{d}\mu_a(t) = \int_{[-\tau,\tau]} \chi_A(t)\mathrm{d}\mu_a(t+a) = \int_{[-\tau+a,\tau+a]} \chi_A(t)\mathrm{d}\mu_a(t).$$

Since $t \mapsto \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2$ is the pointwise limit of an increasing sequence of function (see [27, Theorem 1.17, p. 15]), we deduce that

$$\int_{[-\tau,\tau]} \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \mathrm{d}\mu_a(t) = \int_{[-\tau+a,\tau+a]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^2 \mathrm{d}\mu(t).$$

Let $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. Then, we have $|a| + a = 2a^+$, $|a| - a = 2a^-$ and $[-\tau + a - |a|, \tau + a + |a|] = [-\tau - 2a^-, \tau + 2a^+]$. Therefore, we obtain

$$M_a(\tau + |a|) = \frac{1}{\nu([-\tau - 2a^-, \tau + 2a^+])} \int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 \mathrm{d}\mu(t). \tag{4.3}$$

From (4.3) and the following inequality

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 \mathrm{d}\mu(t) \\ &\qquad \le \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 \mathrm{d}\mu(t), \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 \mathrm{d}\mu(t) \\ &\qquad \le \frac{\nu([-\tau - 2a^-, \tau + 2a^+])}{\nu([-\tau,\tau])} \times M_a(\tau + |a|). \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|_\alpha^2 \mathrm{d}\mu(t) \\ &\qquad \le \frac{\nu([-\tau - 2|a|, \tau + 2|a|])}{\nu([-\tau,\tau])} \times M_a(\tau + |a|). \end{aligned} \tag{4.4}$$

From equations (4.3)–(4.4) and Lemma 4.12 we deduce that

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|_\alpha^2\mathrm{d}\mu(t)=0,$$

which is equivalent to

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta-a)\|_\alpha^2\mathrm{d}\mu(t)=0.$$

Hence, $f_a\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. We have proved that if $f\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$, then $f_{-a}\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$ for all $a\in\mathbb{R}$, that is, $\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$ is invariant under translations. □

Proposition 4.14 *The space $SPAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ is invariant under translations, that is, $f_a\in PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ for all $a\in\mathbb{R}$ and $f\in SPAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$.*

5 Square-mean (μ,ν)-pseudo almost automorphic process

This section is devoted to defining square-mean α-(μ,ν)-pseudo almost automorphic processes and studying their properties.

Definition 5.1 ([10]) *A continuous stochastic process $f\colon\mathbb{R}\to L^2(P,H)$ is said to be almost automorphic process in the square-mean sense, if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$ there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g\colon\mathbb{R}\to L^2(P,H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n)-g(t)\|^2=0\ \textit{for each}\ t\in\mathbb{R}$$

and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n)-f(t)\|^2=0\ \textit{for each}\ t\in\mathbb{R}.$$

We denote by $SAA(\mathbb{R},L^2(P,H))$ the space of all such stochastic processes.

Lemma 5.2 ([10]) *The space $SAA(\mathbb{R},L^2(P,H))$ of square-mean almost automorphic stochastic processes equipped with the norm $\|.\|_\infty$ is a Banach space.*

Definition 5.3 ([10]) *A bounded continuous stochastic process $f\colon\mathbb{R}\to L^2(P,H)$ is said to be compact almost automorphic process in the square-mean sense, if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$ there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g\colon\mathbb{R}\to L^2(P,H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n)-g(t)\|^2=0\ \textit{for all}\ \ t\in\mathbb{R}$$

and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n)-f(t)\|^2=0\ \textit{for all}\ \ t\in\mathbb{R},$$

uniformly on compact subsets of $\mathbb{R}$. We denote by $SAA_c(\mathbb{R},L^2(P,H))$ the space of all such stochastic processes.

Lemma 5.4 ([10]) *The space $SAA_c(\mathbb{R}, L^2(P,H))$ equipped with the norm $\|.\|_\infty$ is a Banach space.*

Definition 5.5 ([10]) *A function $f\colon \mathbb{R}\times L^2(P,H) \to L^2(P,H)$, $(t,x)\mapsto f(t,x)$, which is jointly continuous, is said to be almost automorphic in the square-mean sense in $t\in\mathbb{R}$ for each $x\in L^2(P,H)$, if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$ there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g\colon \mathbb{R}\times L^2(P,H)\to L^2(P,H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n,x)-g(t,x)\|^2 = 0 \text{ for all } t\in\mathbb{R}$$

and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n,x)-f(t,x)\|^2 = 0 \text{ for all } t\in\mathbb{R}.$$

We denote by $SAA(\mathbb{R}\times L^2(P,H), L^2(P,H))$ the space of all such stochastic processes.

Definition 5.6 ([10]) *A bounded function $f\colon \mathbb{R}\times L^2(P,H)\to L^2(P,H)$, $(t,x)\mapsto f(t,x)$, which is jointly continuous, is said to be compact almost automorphic process in the square-mean sense, if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$ there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g\colon \mathbb{R}\times L^2(P,H)\to L^2(P,H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n,x)-g(t,x)\|^2 = 0 \text{ for all } t\in\mathbb{R}$$

and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n,x)-f(t,x)\|^2 = 0 \text{ for all } t\in\mathbb{R},$$

uniformly on compact subsets of $\mathbb{R}$. We denote by $AA_c(\mathbb{R}\times L^2(P,H), L^2(P,H))$ the space of all such stochastic processes.

Definition 5.7 *Let $\mu,\nu\in\mathcal{M}$. A continuous stochastic process $f\colon\mathbb{R}\to L^2(P,H)$ is said to be α-(μ,ν)-square-mean pseudo almost automorphic process, if it can be decomposed as follows $f=g+\phi$, where $g\in SAA(\mathbb{R}, L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu)$. We denote by $SPAA(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu)$ the space of all such stochastic processes.*

Definition 5.8 *Let $\mu,\nu\in\mathcal{M}$. A continuous stochastic process $f\colon\mathbb{R}\to L^2(P,H)$ is said to be compact α-(μ,ν)-square-mean pseudo almost automorphic process, if it can be decomposed as follows $f=g+\phi$, where $g\in SAA_c(\mathbb{R}, L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu)$. We denote by $SPAA_c(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu)$ the space of all such stochastic processes.*

Definition 5.9 *Let $\mu,\nu\in\mathcal{M}$. A bounded continuous stochastic process $f\colon\mathbb{R}\to L^2(P,H_\alpha)$ is said to be α-(μ,ν)-square-mean pseudo almost automorphic process of class r (respectively, compact α-(μ,ν)-square-mean pseudo almost automorphic process of class r), if it can decomposed as follows $f=g+\phi$, where $g\in SAA(\mathbb{R}, L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$ (respectively, $g\in SAA_c(\mathbb{R}, L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$). We denote by $SPAA(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$ (respectively, $SPAA_c(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$) the space of all such stochastic processes.*

Theorem 5.10 *Let $\mu,\nu\in\mathcal{M}$ and let $f\in SPAA(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$ be such that $f=g+\phi$, where $g\in SAA(\mathbb{R}, L^2(P,H_\alpha))$ and $\phi\in\mathscr{E}(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$. If $SPAA(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$ is invariant under translations, then $\overline{\{f(t) : t\in\mathbb{R}\}}\supset\{g(t) : t\in\mathbb{R}\}$.*

The proof of Theorem 5.10 is similar to the proof of [6, Theorem 4.1].

Theorem 5.11 *Let $\mu, \nu \in \mathcal{M}$ and assume that* $(\mathbf{H}_2)$ *holds. The space $SPAA(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.*

The proof of Theorem 5.11 is similar to the proof of [6, Theorem 4.9].

Next, we study the composition of square-mean α-(μ, ν)-pseudo almost automorphic processes.

Definition 5.12 *Let $\mu, \nu \in \mathcal{M}$. A function $f\colon \mathbb{R} \times L^2(P, H_\alpha) \to L^2(P, H_\alpha)$, $(t, x) \mapsto f(t, x)$, which is jointly continuous, is said to be square-mean α-(μ, ν)-pseudo almost automorphic of class r in $t \in \mathbb{R}$ for any $x \in L^2(P, H_\alpha)$, if it can decomposed as follows $f = g + \phi$, where $g \in SAA(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H_\alpha))$ and $\phi \in \mathscr{E}(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H_\alpha), \mu, \nu, r)$. We denote by $SPAA(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H_\alpha), \mu, \nu, r)$ the set of all such stochastically continuous processes.*

Lemma 5.13 *Assume that* $(\mathbf{H}_2)$ *holds and let $f \in SBC(\mathbb{R}, L^2(\Omega, H))$. Then, $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])} = 0,$$

where

$$M_{\tau,\varepsilon} = \left\{ t \in [-\tau, \tau] : \sup_{\theta \in [-r, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \geq \varepsilon \right\}.$$

Proof. Suppose that $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. Then, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [-r, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f)} \left(\sup_{\theta \in [-r, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\quad + \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau] \setminus \mathcal{M}_{\tau,\varepsilon}(f)} \left(\sup_{\theta \in [-r, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\geq \frac{1}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f)} \left(\sup_{\theta \in [-r, t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\geq \frac{\varepsilon}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f)} \mathrm{d}\mu(t) \\
&\geq \frac{\varepsilon \mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])}.
\end{aligned}$$

Consequently,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])} = 0.$$

Now, suppose that $f \in SBC(\mathbb{R}, L^2(P, H_\alpha))$ is such that for any $\varepsilon > 0$ we have

$$\lim_{\tau \to +\infty} \frac{\mu(\mathcal{M}_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])} = 0.$$

We assume that $\mathbb{E}\|f(t)\|_\alpha^2 \leq N$ for all $t \in \mathbb{R}$. Using (**H**$_2$), we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f)} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\quad + \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus \mathcal{M}_{\tau,\varepsilon}(f)} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\leq \frac{N}{\nu([-\tau,\tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f)} \mathrm{d}\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus \mathcal{M}_{\tau,\varepsilon}(f)} \mathrm{d}\mu(t) \\
&\leq \frac{N\mu(\mathcal{M}_{\tau,\varepsilon})}{\nu([-\tau,\tau])} + \frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.
\end{aligned}$$

This implies that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|f(\theta)\|_\alpha^2 \right) \mathrm{d}\mu(t) \leq \delta\varepsilon \;\text{ for any } \varepsilon > 0.$$

Therefore, $f \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. ☐

Theorem 5.14 ([18]) *Let $f\colon \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$, be almost automorphic in square-mean sense in $t \in \mathbb{R}$ for each $x \in L^2(P,H)$, and assume that f satisfies the Lipschitz condition in the following sense*:

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 \leq L\|x-y\|^2$$

for all $x, y \in L^2(P,H)$ and for each $t \in \mathbb{R}$, where L is independent of t. Then, for any square-mean almost automorphic process $x\colon \mathbb{R} \to L^2(P,H)$ the stochastic process $F\colon \mathbb{R} \to L^2(P,H)$ given by $F(t) = f(t, x(t))$ is square-mean almost automorphic.

Theorem 5.15 *Let $\mu, \nu \in \mathcal{M}$. Also, let $\phi = \phi_1 + \phi_2 \in PAA(\mathbb{R} \times L^2(P,H_\alpha), L^2(P,H_\alpha), \mu, \nu, r)$ with $\phi_1 \in SAA(\mathbb{R} \times L^2(P,H_\alpha), L^2(P,H_\alpha))$ and $\phi_2 \in \mathscr{E}(\mathbb{R} \times L^2(P,H), L^2(P,H_\alpha), \mu, \nu, r)$. Finally, let $h \in PAA(\mathbb{R}, L^2(P,H_\alpha), \mu, \nu, r)$. Assume that*:

(i) *$\phi_1(t,x)$ is uniformly continuous on any bounded subset uniformly for $t \in \mathbb{R}$,*

(ii) *there exists a non negative function $L_\phi \in L^p(\mathbb{R})$ $(1 \leq p \leq +\infty)$ such that*

$$\mathbb{E}\|\phi(t,x) - \phi(t,y)\|_\alpha^2 \leq L_\phi(t)\mathbb{E}\|x-y\|_\alpha^2 \tag{5.1}$$

for all $t \in \mathbb{R}$ and for all $x, y \in L^2(P,H_\alpha)$.

If

$$\beta = \lim_{\tau \longrightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [-r,t]} L_\phi(\theta) \right) \mathrm{d}\mu(t) < +\infty, \tag{5.2}$$

then the function $t \mapsto \phi(t, h(t))$ *belongs to* $SPAA(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H), \mu, \nu, r)$.

Proof. Suppose that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$, where $\phi_1 \in SAA(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H_\alpha))$, $\phi_2 \in \mathscr{E}(\mathbb{R} \times L^2(P, H_\alpha), L^2(P, H_\alpha), \mu, \nu, r)$, $h_1 \in SAA(\mathbb{R}, L^2(P, H_\alpha))$ and $h_2 \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + [\phi(t, h(t)) - \phi(t, h_1(t))] + \phi_2(t, h_1(t)).$$

From [14], it follows that $\phi_1(., h(.)) \in SAA(\mathbb{R}, L^p(\Omega, H))$. To complete the proof it remains to show that both $\phi(., h(.)) - \phi(., h_1(.))$ and $\phi_2(., h(.))$ belong to $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. Clearly, $t \mapsto \phi(t, h(t)) - \phi(t, h_1(t))$ is bounded and continuous. Assume that $\mathbb{E}\|\phi(t, h(t)) - \phi(t, h_1(t))\|_\alpha^2 \leq N$ for all $t \in \mathbb{R}$. Since $h(t), h_1(t)$ are bounded, we can choose a bounded subset $B \subset \mathbb{R}$ such that $h(\mathbb{R})$, $h_1(\mathbb{R}) \subset B$. Under assumption (ii), for a given $\varepsilon > 0$ if $\mathbb{E}\|x - y\|_\alpha^2 \leq \varepsilon$, then

$$\mathbb{E}\|\phi(t, x) - \phi(t, y)\|_\alpha^2 \leq \varepsilon L_\phi(t) \text{ for all } t \in \mathbb{R}.$$

Since $\eta \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$, Lemma 5.13 yields that

$$\lim_{\tau \longrightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \mu(M_{\tau,\varepsilon}(\eta)) = 0.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau, \tau])} \int_{M_{\tau,\varepsilon}(\eta)} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\quad + \frac{1}{\nu([-\tau, \tau])} \int_{M_{\tau,\varepsilon}(\eta) \setminus [-\tau,\tau]} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) \\
&\leq \frac{N}{\nu([-\tau, \tau])} \int_{M_{\tau,\varepsilon}(\eta)} \mathrm{d}\mu(t) + \frac{\varepsilon}{\nu([-\tau, \tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(\eta)} \left(\sup_{\theta \in [-r,t]} |L_\phi(\theta)| \right) \mathrm{d}\mu(t) \\
&\leq \frac{N}{\nu([-\tau, \tau])} \int_{M_{\tau,\varepsilon}(\eta)} \mathrm{d}\mu(t) + \frac{\varepsilon}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in [-r,t]} |L_\phi(\theta)| \right) \mathrm{d}\mu(t) \\
&\leq \frac{N\mu(M_{\tau,\varepsilon}(\eta))}{\nu([-\tau, \tau])} + \frac{\varepsilon}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in [-r,t]} |L_\phi(\theta)| \right) \mathrm{d}\mu(t).
\end{aligned}$$

We deduce that for any $\varepsilon > 0$ we have

$$\lim_{\tau \longrightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) \leq \varepsilon\beta.$$

This shows that $t \mapsto \phi(t, h(t)) - \phi(t, h_1(t))$ is (μ, ν)-ergodic of class r.

Now, to complete the proof we need to show that $t \mapsto \phi_2(t, h(t))$ is (μ, ν)-ergodic of class r. Since ϕ_2 is uniformly continuous on the compact set $\wedge = \overline{\{h(t) : t \in \mathbb{R}\}}$ with respect to the second variable x, we deduce that for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in \mathbb{R}$ and $\xi_1, \xi_2 \in \wedge$ one has $\mathbb{E}\|\phi_2(t, \xi_1(t)) - \phi_2(\xi, \xi_2(t))\|_\alpha^2 \leq \varepsilon$, provided that $\mathbb{E}\|\xi_1 - \xi_2\|^2 \leq \delta$. Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset \wedge$ such that

$$\wedge \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i, \delta).$$

And then

$$\mathbb{E}\|\phi_2(t, h_1(t))\|_\alpha^2 \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \mathbb{E}\|\phi_2(t, z_i)\|_\alpha^2.$$

Since

$$\lim_{\tau \longrightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi_2(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) = 0$$

for every $i \in \{1, \ldots, n(\varepsilon)\}$, for any $\varepsilon > 0$ we get

$$\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi_2(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) \leq \varepsilon.$$

This implies that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [-r,t]} \mathbb{E}\|\phi_2(\theta, h_1(\theta))\|_\alpha^2 \right) \mathrm{d}\mu(t) = 0.$$

Consequently, $t \mapsto \phi_2(t, h_1(t))$ is (μ, ν)-ergodic of class r. □

6 Square-mean pseudo almost automorphic solution of class r

Lemma 6.1 (Ito's Isometry, see [25]) *Let $W : [0, T] \times \Omega \to \mathbb{R}$ denote the canonical real-valued Wiener process defined up to time $T > 0$ and let $X : [0, T] \times \Omega \to \mathbb{R}$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then,*

$$\mathbb{E}\left[\left(\int_0^T X_t \mathrm{d}W_t \right)^2 \right] = \mathbb{E}\left[\int_0^T X_t^2 \mathrm{d}t \right],$$

where $\mathbb{E}$ denotes expectation with respect to the classical Wiener measure.

We make the following assumption:

($\mathbf{H}_5$) g is a stochastically bounded process.

Theorem 6.2 *Assume that* ($\mathbf{H}_0$), ($\mathbf{H}_1$) *and* ($\mathbf{H}_5$) *hold and that the semi-group $(U(t))_{t \geq 0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then there exists a unique bounded solution u of equation* (1.1) *on $\mathbb{R}$ given by*

$$u_t = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s(\widetilde{B}_\lambda X_0 f(s)) \mathrm{d}s$$

$$+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s$$
$$+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)$$
$$+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s),$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_{\mathcal{U}})^{-1}$ *for* $\lambda > \widetilde{\omega}$, *and* Π^s, Π^u *are projections of* $\mathcal{C}_\alpha$ *onto the stable and unstable subspaces, respectively.*

Proof. Let

$$u_t = v(t) + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)$$
$$+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s),$$

where

$$v(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s.$$

Let us first prove that u_t exists. The existence of v was proved in [1]. Now, we show that the limit

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)$$

exists. Using Ito's isometry for the stochastic integral for $t \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\right\|_\alpha^2 \\
&\le \mathbb{E}\left(\int_{-\infty}^{t}\overline{M}^2\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\|(\widetilde{B}_\lambda X_0 g(s))\|^2\mathrm{d}s\right) \\
&\le \overline{M}^2\mathbb{E}\left(\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\|(\widetilde{B}_\lambda X_0 g(s))\|^2\mathrm{d}s\right) \\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\left(\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2\mathrm{d}s\right) \\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{\infty}\mathbb{E}\left(\int_{t-n}^{t-n+1}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2\mathrm{d}s\right) \\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\Bigg[\mathbb{E}\left(\int_{t-1}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2\mathrm{d}s\right)
\end{aligned}$$

$$+\sum_{n=2}^{\infty}\mathbb{E}\left(\int_{t-n}^{t-n+1}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}\|g(s)\|^2\mathrm{d}s\right)\Bigg].$$

Then, using Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\right\|_\alpha^2\\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\Bigg[\left(\int_{t-1}^{t}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}\mathrm{d}s\right)^{\frac{1}{2}}\times\mathbb{E}\left(\int_{t-1}^{t}\|g(s)\|^4\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\sum_{n=2}^{\infty}\left(\int_{t-n}^{t-n+1}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}\mathrm{d}s\right)^{\frac{1}{2}}\times\mathbb{E}\left(\int_{t-n}^{t-n+1}\|g(s)\|^4\mathrm{d}s\right)^{\frac{1}{2}}\Bigg]\\
&\le \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\Bigg[\left(\int_0^{4w}e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\sum_{n=2}^{\infty}\left(\int_{4w(n-1)}^{4wn}e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\Bigg]\\
&\le \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\Bigg[\left(\int_0^{4w}e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\sum_{n=2}^{\infty}\left(\int_{4w(n-1)}^{4wn}e^{-s}(n-1)^{-4\alpha}(4w)^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\Bigg]\\
&\le \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\Bigg[\left(\int_0^{4w}e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\sum_{n=2}^{\infty}\left((n-1)^{-4\alpha}\int_{4w(n-1)}^{4wn}e^{-s}(4w)^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\Bigg]\\
&\le \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\left(\int_0^{4w}e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}\sum_{n=2}^{\infty}\left(\int_{4w(n-1)}^{4wn}e^{-s}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\le \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\left(\int_0^{4w}e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(e^{4w}-1)^{\frac{1}{2}}\sum_{n=2}^{\infty}e^{-2wn}
\end{aligned}$$

$$\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\left(\int_0^{4w} e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}} + \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(e^{4w}+1)^{\frac{1}{2}}\sum_{n=2}^{\infty}e^{-2wn}.$$

Since the series

$$\sum_{n=2}^{\infty}e^{-2wn}$$

is convergent, it follows that

$$\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\right\|_\alpha^2 \leq K, \tag{6.1}$$

where

$$K = \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\left(\int_0^{4w} e^{-s}s^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}} + \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(e^{4w}+1)^{\frac{1}{2}}\sum_{n=2}^{\infty}e^{-2wn}.$$

Using Ito's isometry for the stochastic integral, for sufficiently large n and $\sigma \leq t$ we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{\sigma}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\right\|_\alpha^2 \\
&\leq \overline{M}^2\widetilde{M}^2|\Pi^s|^2\left[\left(\int_{\sigma-1}^{\sigma}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}\mathrm{d}s\right)^{\frac{1}{2}} \times \mathbb{E}\left(\int_{t-1}^{t}\|g(s)\|^4\mathrm{d}s\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \sum_{n=2}^{\infty}\left(\int_{t-n}^{t-n+1}\frac{e^{-4w(t-s)}}{(t-s)^{4\alpha}}\mathrm{d}s\right)^{\frac{1}{2}} \times \mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}\|g(s)\|^4\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1-4\alpha}{2}}}\left[\left(\int_{4w(t-\sigma)}^{4w(t-\sigma+1)} e^{-s}(t-\sigma)^{-4\alpha}(4w)^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \sum_{n=2}^{\infty}\left(\int_{4w(t-\sigma+n-1)}^{4w(t-\sigma+n)} e^{-s}(t-\sigma+n-1)^{-4\alpha}(4w)^{-4\alpha}\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(t-\sigma)^{-2\alpha}\left(\int_{4w(t-\sigma)}^{4w(t-\sigma+1)} e^{-s}\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad + \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(t-\sigma)^{-2\alpha}\left(\int_{4w(t-\sigma+n-1)}^{4w(t-\sigma+n)} e^{-s}\mathrm{d}s\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(t-\sigma)^{-2\alpha}(1-e^{-4w})^{\frac{1}{2}}e^{-2w(t-\sigma)}\\
&\quad+\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(e^{4w}+1)^{\frac{1}{2}}e^{-2w(t-\sigma)}\sum_{n=2}^{\infty}e^{-2wn}\\
&\leq K_1(t-\sigma)^{-2\alpha}e^{-2w(t-\sigma)}+K_2e^{-2w(t-\sigma)},
\end{aligned}$$

where

$$K_1=\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(1-e^{-4w})^{\frac{1}{2}}$$

and

$$K_2=\frac{\overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\|g(s)\|^2}{(4w)^{\frac{1}{2}}}(e^{4w}+1)^{\frac{1}{2}}.$$

Set $F(n,s,t)=\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))$ for $n\in\mathbb{N}$ and $s\leq t$. It follows that for sufficiently large n and m and $\sigma\leq t$ we have

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t}F(n,s,t)\mathrm{d}W(s)-\int_{-\infty}^{t}F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2\\
&\leq \mathbb{E}\left\|\int_{-\infty}^{\sigma}F(n,s,t)\mathrm{d}W(s)+\int_{\sigma}^{t}F(n,s,t)\mathrm{d}W(s)\right.\\
&\qquad\left.-\int_{-\infty}^{\sigma}F(m,s,t)\mathrm{d}W(s)-\int_{\sigma}^{t}F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2\\
&\leq 3\mathbb{E}\left\|\int_{-\infty}^{\sigma}F(n,s,t)\mathrm{d}W(s)\right\|_\alpha^2+3\mathbb{E}\left\|\int_{-\infty}^{\sigma}F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2\\
&\qquad+3\mathbb{E}\left\|\int_{\sigma}^{t}F(n,s,t)\mathrm{d}W(s)-\int_{\sigma}^{t}F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2\\
&\leq 6K_1(t-\sigma)^{-2\alpha}e^{-\omega 2(t-\sigma)}+K_2e^{-2w(t-\sigma)}\\
&\qquad+3\mathbb{E}\left\|\int_{\sigma}^{t}F(n,s,t)\mathrm{d}W(s)-\int_{\sigma}^{t}F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2.
\end{aligned}$$

Since

$$\lim_{n\to+\infty}\mathbb{E}\left\|\int_{\sigma}^{t}F(n,s,t)\mathrm{d}W(s)\right\|_\alpha^2$$

exists, we get

$$\limsup_{n,m\to+\infty}\mathbb{E}\left\|\int_{-\infty}^{t}F(n,s,t)\mathrm{d}W(s)-\int_{-\infty}^{t}F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2$$

$$\leq 6\big(K_1(t-\sigma)^{-2\alpha}e^{-2\omega(t-\sigma)} + K_2e^{-2w(t-\sigma)}\big).$$

If $\sigma \to -\infty$, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\left\|\int_{-\infty}^{t} F(n,s,t)\mathrm{d}W(s) - \int_{-\infty}^{t} F(m,s,t)\mathrm{d}W(s)\right\|_\alpha^2 = 0.$$

We deduce that

$$\lim_{n\to+\infty} \mathbb{E}\left\|\int_{-\infty}^{t} F(n,s,t)\mathrm{d}W(s)\right\|_\alpha^2 = \lim_{n\to+\infty} \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s))\mathrm{d}W(s)\right\|_\alpha^2$$

exists. Therefore, the limit

$$\lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s))\mathrm{d}W(s)$$

exists. Moreover, from equation (6.1) we can see that the function

$$\eta_1 \colon t \mapsto \lim_{n\to+\infty} \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s))\mathrm{d}W(s)\right\|_\alpha^2$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$\eta_2 \colon t \mapsto \lim_{n\to+\infty} \mathbb{E}\left\|\int_{t}^{+\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_nX_0g(s))\mathrm{d}W(s)\right\|_\alpha^2$$

is well-defined and bounded on $\mathbb{R}$. □

Theorem 6.3 *Assume that* ($\mathbf{H}_3$) *holds. Let* $\mu,\nu \in \mathcal{M}$ *and* $\phi \in SPAA_c(\mathbb{R}, L^2(P,H_\alpha),\mu,\nu,r)$. *Then, the function* $t \mapsto \phi_t$ *belongs to* $SPAA_c(C([-r,0],L^2(P,H_\alpha)),\mu,\nu,r)$.

Proof. Assume that $\phi = g+h$, where $g \in SAA_c(\mathbb{R},L^2(P,H_\alpha))$ and $h \in \mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. Then, we can see that $\phi_t = g_t + h_t$ and that g_t is a square-mean almost automorphic process. Firstly, we show that $g_t \in SAA_c(\mathbb{R},L^2(P,H_\alpha))$. Let $(s_m)_{m\in\mathbb{N}}$ be a sequence of real numbers. Fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v \in SBC(\mathbb{R},L^2(P,H_\alpha))$ such that $g(s+s_n) \longrightarrow v(s)$ uniformly on compact subsets of $\mathbb{R}$. Let $K \subset [-L,L]$. For $\varepsilon > 0$ fix $N_{\varepsilon,L} \in \mathbb{N}$ such that $\mathbb{E}\|g(s+s_n)-v(s)\|_\alpha^2 \leq \varepsilon$ for $s \in [-L,L]$ whenever $n \geq N_{\varepsilon,L}$. For $t \in K$ and $n \geq N_{\varepsilon,L}$ we have

$$\mathbb{E}\|g_{t+s_n} - v(s)\|_\alpha^2 \leq \sup_{\theta\in[-L,L]} \mathbb{E}\|g(\theta+s_n) - v(s)\|_\alpha^2 \leq \varepsilon.$$

Hence, g_{t+s_n} converges to v_t uniformly in K. Similarly, we can show that v_{t+s_n} converges to v_t uniformly in K. Thus, the function $s \mapsto g_s$ belongs $SAA_c(\mathbb{R},L^2(P,H_\alpha))$.

Finally, we show that $h_t \in \mathscr{E}(\mathbb{R},L^p(P,H_\alpha),\mu,\nu,r)$. Let

$$M_a = \frac{1}{\nu_a([-\tau,\tau])}\int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|_\alpha^2 \mathrm{d}\mu_a(t),$$

where μ_a and ν_a are the positive measures defined by equation (4.1). By Lemma 4.11, it follows that μ and μ_a are equivalent, and so are ν and ν_a. By Theorem 4.10, we get $\mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r) = \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_a, \nu_a, r)$. Therefore, $h \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu_a, \nu_a, r)$, that is, $\lim_{\tau\to\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.

On the other, hand for $\tau > 0$ we have

$$
\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\left(\sup_{\theta\in[-r,0]}\mathbb{E}\|h(\theta+\xi)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-2r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-2r,t-r]}\mathbb{E}\|h(\theta)\|_\alpha^2+\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-2r,t-r]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\qquad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{\tau-r}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\qquad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{\tau+}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t+r)\\
&\qquad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\left(\frac{1}{\nu([-\tau-r,\tau+r])}\int_{-\tau-r}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t+r)\right)\\
&\qquad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\left(\sup_{\theta\in[-r,0]}\mathbb{E}\|h(\theta+\xi)\|_\alpha^2\right)\mathrm{d}\mu(t)\\
&\quad\leq \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\times M_r(\tau+r)+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|h(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t).
\end{aligned}
$$

By Lemmas 4.11 and 4.12 this shows that ϕ_t belongs to $SPAA_c(C[-r,0], L^2(P, H_\alpha)), \mu, \nu, r)$. The proof is complete. $\square$

Theorem 6.4 *Assume that* ($\mathbf{H}_5$) *holds. Let $f, g \in SAA_c(\mathbb{R}, L^2(P, H_\alpha))$, and let Ψ be the mapping defined for $t \in \mathbb{R}$ by*

$$\begin{aligned}\Psi(f,g)(t) = \Bigg[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\Bigg](0).\end{aligned}$$

Then, $\Psi(f,g) \in SAA_c(\mathbb{R}, L^2(P, H_\alpha))$.

Proof. We can see that $\Psi(f,g) \in SBC(\mathbb{R}, L^2(P, H_\alpha))$. In fact,

$$\begin{aligned}&\mathbb{E}\|\Psi(f,g)(t)\|_\alpha^2\\ &= \mathbb{E}\Bigg\|\Bigg[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\ &\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\ &\quad+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\\ &\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\Bigg](0)\Bigg\|_\alpha^2\\ &\le 4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\Bigg\|_\alpha^2\\ &\quad+4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\Bigg\|_\alpha^2\\ &\quad+4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\Bigg\|_\alpha^2\\ &\quad+4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\Bigg\|_\alpha^2.\end{aligned}$$

Using Ito's isometry for the stochastic integral, we obtain

$$
\begin{aligned}
&E\|\Psi(f,g)(t)\|_\alpha^2\\
&\leq 4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\|f(s)\|^2\mathrm{d}s\right)\\
&\quad+4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{t}^{+\infty}\frac{e^{2w(t-s)}}{(s-t)^{2\alpha}}|\Pi^u|^2\|f(s)\|^2\mathrm{d}s\right)\\
&\quad+4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\|g(s)\|^2\mathrm{d}s\right)\\
&\quad+4\mathbb{E}\left(\overline{M}^2\widetilde{M}^2\int_{t}^{+\infty}\frac{e^{2w(t-s)}}{(s-t)^{2\alpha}}|\Pi^u|^2\|g(s)\|^2\mathrm{d}s\right)\\
&\leq\frac{8\Delta(\|f\|_\infty^2+\|g\|_\infty^2)}{(2w)^{1-2\alpha}}\left(\int_0^{+\infty}e^{-s}s^{-2\alpha}\mathrm{d}s\right)\\
&=\frac{8\Delta(\|f\|_\infty^2+\|g\|_\infty^2)}{(2w)^{1-2\alpha}}\Gamma(1-2\alpha)<+\infty,
\end{aligned}
\tag{6.2}
$$

where $\Delta=\max(\overline{M}^2\widetilde{M}^2|\Pi^s|^2,\overline{M}^2\widetilde{M}^2|\Pi^u|^2)$.

For a given sequence $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v,h\in SBC(\mathbb{R},L^2(P,H))$ such that $f(t+s_n)$ converges to $v(t)$ and $g(t+s_n)$ converges to $h(t)$ uniformly on compact subsets of $\mathbb{R}$. Let

$$
\begin{aligned}
w(t+s_n)=\Bigg[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s+s_n))\mathrm{d}s\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s+s_n))\mathrm{d}s\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0g(s+s_n))\mathrm{d}W(s)\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0g(s+s_n))\mathrm{d}W(s)\Bigg].
\end{aligned}
$$

By equation (6.2) and the Lebesgue dominated convergence theorem, it follows that $w(t+s_n)$ converges to

$$
\begin{aligned}
z(t)=\Bigg[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0v(s))\mathrm{d}s\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0v(s))\mathrm{d}s\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0h(s))\mathrm{d}W(s)
\end{aligned}
$$

$$+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))\mathrm{d}W(s)\Bigg].$$

Now, it remains to show that this convergence is uniform on all compact subset of $\mathbb{R}$. Let $K\subset\mathbb{R}$ be an arbitrary compact set and let $\varepsilon>0$. Fix $L>0$ and $N_\varepsilon\in\mathbb{N}$ such that $K\subset[-\frac{L}{2},\frac{L}{2}]$ with

$$\int_{\frac{L}{2}}^{+\infty}e^{-s}s^{-2\alpha}\mathrm{d}s<\varepsilon,$$

$$\mathbb{E}\|f(s+s_n)-v(s)\|^2\le\varepsilon \text{ for } n\ge N_\varepsilon \text{ and } s\in[-L,L] \tag{6.3}$$

and

$$\mathbb{E}\|g(s+s_n)-h(s)\|^2\le\varepsilon \text{ for } n\ge N_\varepsilon \text{ and } s\in[-L,L]. \tag{6.4}$$

Then, for each $t\in K$, we get

$$\begin{aligned}
&\mathbb{E}\|w(t+s_n)-z(t)\|_\alpha^2\\
&=\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))\mathrm{d}s\\
&\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s+s_n))\mathrm{d}s\\
&\quad+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))\mathrm{d}W(s)\\
&\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s+s_n))\mathrm{d}W(s)\\
&\quad-\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))\mathrm{d}s\\
&\quad-\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 v(s))\mathrm{d}s\\
&\quad-\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))\mathrm{d}W(s)\\
&\quad-\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))\mathrm{d}W(s)\Bigg\|_\alpha^2\\
&\le 4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s))\mathrm{d}s\Bigg\|_\alpha^2\\
&\quad+4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s))\mathrm{d}s\Bigg\|_\alpha^2\\
&\quad+4\mathbb{E}\Bigg\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(g(s+s_n)-h(s))\mathrm{d}W(s)\Bigg\|_\alpha^2
\end{aligned}$$

$$
\begin{aligned}
&+ 4\mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0(g(s+s_n)-h(s))\mathrm{d}W(s)\right\|_\alpha^2 \\
&= 4(\gamma_1+\gamma_2+\gamma_3+\gamma_4).
\end{aligned}
$$

Firstly, we evaluate γ_1. We have

$$
\begin{aligned}
\gamma_1 &= \mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s))\mathrm{d}s\right\|_\alpha^2 \\
&\le \overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s \\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s \\
&\quad + \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\gamma_2 &= \mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s))\mathrm{d}s\right\|_\alpha^2 \\
&\le \overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{2\omega(t-s)}}{(s-t)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s \\
&\le \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{-\infty}^{-L}\frac{e^{2\omega(s-t)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s \\
&\quad + \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{-L}^{t}\frac{e^{2\omega(s-t)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s.
\end{aligned}
$$

Secondly, by Ito's isometry for the stochastic integral, we obtain

$$
\begin{aligned}
\gamma_3 &= \mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(g(s+s_n)-h(s))\mathrm{d}W(s)\right\|_\alpha^2 \\
&\le \overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s \\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s \\
&\quad + \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s.
\end{aligned}
$$

Finally, applying Ito's isometry for stochastic integrals once again, we get

$$
\gamma_4 = \mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^s(t-s)\Pi^u(\widetilde{B}_\lambda X_0(g(s+s_n)-h(s))\mathrm{d}W(s)\right\|^2
$$

$$
\begin{aligned}
&\leq \overline{M}^2\widetilde{M}^2\int_{-\infty}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}|\Pi^u|^2\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s\\
&\leq \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{-\infty}^{-L}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s\\
&\qquad+\overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{-L}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\mathbb{E}\|w(t+s_n)-z(t)\|_\alpha^2\\
&\leq 4\Bigg[\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}\frac{e^{2\omega(s-t)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s\Bigg]\\
&\leq 4\Bigg(\frac{2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2}{(2w)^{1-2\alpha}}\int_{t+L}^{+\infty}e^{-s}s^{-2\alpha}\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2\big(|\Pi^s|^2+|\Pi^u|^2\big)\int_{-L}^{+\infty}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(f(s+s_n)-v(s))\|^2\mathrm{d}s\\
&\quad+\overline{M}^2\widetilde{M}^2\big(|\Pi^s|^2+|\Pi^u|^2\big)\int_{-L}^{+\infty}\frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}\mathbb{E}\|(g(s+s_n)-h(s))\|^2\mathrm{d}s\Bigg)\\
&\leq 4\Bigg(\frac{2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2}{(2w)^{1-2\alpha}}\int_{\frac{L}{2}}^{+\infty}e^{-s}s^{-2\alpha}\mathrm{d}s\\
&\quad+\frac{2\varepsilon\overline{M}^2\widetilde{M}^2\big(|\Pi^s|^2+|\Pi^u|^2\big)}{(2w)^{1-2\alpha}}\int_{0}^{+\infty}e^{-s}s^{-2\alpha}\mathrm{d}s\Bigg)\\
&\leq \frac{8\overline{M}^2\widetilde{M}^2}{(2w)^{1-2\alpha}}\left(\varepsilon|\Pi^s|^2+\frac{\big(|\Pi^s|^2+|\Pi^u|^2\big)}{(2w)^{1-2\alpha}}\Gamma(1-2\alpha)\right)\varepsilon.
\end{aligned}
$$

Since the last estimate is independent of $t\in K$, this proves that the convergence is uniform on K.

Proceeding as before, one can similarly prove that $z(t-s_n)$ converges to w uniformly on compact subsets of $\mathbb{R}$. The proof is complete. $\square$

Theorem 6.5 *Assume that* ($\mathbf{H}_2$) *and* ($\mathbf{H}_5$) *hold. Let* $f, g \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$. *Then,* $\Psi(f,g) \in \mathscr{E}(\mathbb{R}, L^2(P, H_\alpha), \mu, \nu, r)$.

Proof. Let

$$\begin{aligned}\Psi(f,g)(t) = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\\&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s).\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}\|\Psi(f,g)(t)\|_\alpha^2 = \mathbb{E}\Bigg\|&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\mathrm{d}s\\&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\\&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\mathrm{d}W(s)\Bigg\|_\alpha^2.\end{aligned}$$

Consequently, by Ito's isometry for stochastic integrals, we have

$$\begin{aligned}&\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|\Psi(f,g)(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)\\&\leq\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\left[4\overline{M}^2\widetilde{M}^2\mathbb{E}\left(\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\|f(s)\|^2\mathrm{d}s\right.\right.\right.\\&\quad+\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\|f(s)\|^2\mathrm{d}s+\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\|g(s)\|^2\mathrm{d}s\\&\quad\left.\left.\left.+\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\|g(s)\|^2\mathrm{d}s\right)\right]\right)\mathrm{d}\mu(t)\\&\leq 4\overline{M}^2\widetilde{M}^2\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|f(s)\|^2\mathrm{d}s\right)\mathrm{d}\mu(t)\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\mathbb{E}\|f(s)\|^2\mathrm{d}s\right)\mathrm{d}\mu(t)\\
&+\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}|\Pi^s|^2\mathbb{E}\|g(s)\|^2\mathrm{d}s\right)\mathrm{d}\mu(t)\\
&+\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}|\Pi^u|^2\mathbb{E}\|g(s)\|^2\mathrm{d}s\right)\mathrm{d}\mu(t)\\
\leq{}&\Delta\left[\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}\big(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2\big)\mathrm{d}s\right)\mathrm{d}\mu(t)\right.\\
&\left.+\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{2w(\theta-s)}}{(s-\theta)^{2\alpha}}\big(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2\big)\mathrm{d}s\right)\mathrm{d}\mu(t)\right],
\end{aligned}
$$

where $\Delta=\max(4\overline{M}^2\widetilde{M}^2|\Pi^s|^2,4\overline{M}^2\widetilde{M}^2|\Pi^u|^2)$.

By Fubini's theorem, we have

$$
\begin{aligned}
&\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}\frac{e^{-2w(\theta-s)}}{(\theta-s)^{2\alpha}}\big(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2\big)\mathrm{d}s\right)\mathrm{d}\mu(t)\\
&\quad\leq\int_{-\tau}^{\tau}\left(\int_{0}^{+\infty}\sup_{\theta\in[t-r,t]}\frac{e^{-2ws}}{s^{2\alpha}}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)\mathrm{d}s\right)\mathrm{d}\mu(t)\\
&\quad\leq\int_{0}^{+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\left(\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)\mathrm{d}\mu(t)\right)\mathrm{d}s.
\end{aligned}
$$

Moreover, by the Lebesgue dominated convergence theorem and Theorem 4.13, we deduce that

$$
\lim_{\tau\to+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)\mathrm{d}\mu(t)=0
$$

for all $s\in\mathbb{R}^+$, and

$$
\begin{aligned}
&\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)\mathrm{d}\mu(t)\\
&\qquad\leq\frac{e^{-2ws}}{s^{2\alpha}}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\big(\|f\|_\infty^2+\|g\|_\infty^2\big).
\end{aligned}
$$

Since the functions f and g are bounded, we infer that

$$
s\mapsto\frac{e^{-2ws}}{s^{2\alpha}}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\big(\|f\|_\infty^2+\|g\|_\infty^2\big)
$$

belongs to $L^1([0,+\infty])$. In view of the Lebesgue dominated convergence theorem, it follows that

$$
\lim_{\tau\to+\infty}\int_{0}^{+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\left(\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}\big(\mathbb{E}\|f(\theta-s)\|^2+\mathbb{E}\|g(\theta-s)\|^2\big)\mathrm{d}\mu(t)\right)\mathrm{d}s=0.
$$

Similarly, we obtain

$$\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2ws}}{s^{2\alpha}}\frac{1}{\nu([-\tau,\tau])}\left(\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}\left(\mathbb{E}\|f(\theta+s)\|^2+\mathbb{E}\|g(\theta+s)\|^2\right)\mathrm{d}\mu(t)\right)\mathrm{d}s=0.$$

Consequently,

$$\lim_{\tau\to+\infty}\frac{1}{\nu([(-\tau,\tau)])}\int_{-\tau}^{\tau}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|\Psi(f,g)(\theta)\|_\alpha^2\right)\mathrm{d}\mu(t)=0.$$

Thus, we obtain the desired result. □

To prove the existence of a square-mean compact pseudo almost automorphic solution of class r, we need the following condition:

(H$_6$) $f,g\colon\mathbb{R}\to L^2(P,H)$ are compact α-$cl(\mu,\nu)$-pseudo almost automorphic of class r.

Theorem 6.6 *Assume that* (**H**$_0$)*,* (**H**$_1$)*,* (**H**$_3$) *and* (**H**$_6$) *hold. Then, equation* (1.1) *has a unique compact α-cl(μ,ν)-pseudo almost automorphic solution of class r.*

Proof. Since f and g are (μ,ν)-pseudo almost automorphic functions, we can write $f=f_1+f_2$ and $g=g_1+g_2$, where $f_1,g_1\in SAA_c(\mathbb{R},L^2(P,H_\alpha))$ and $f_2,g_2\in\mathscr{E}(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. Using Theorem 6.2, Theorem 6.4 and Theorem 6.5, we get the desired result. □

Our next objective is to show the existence of square-mean α-(μ,ν)-pseudo almost automorphic solutions of class r for the following problem

$$\mathrm{d}x(t)=[-Ax(t)+L(x_t)+f(t,u_t)]\mathrm{d}t+g(t,u_t)\mathrm{d}W(t)\text{ for }t\in\mathbb{R},\tag{6.5}$$

where $f,g\colon\mathbb{R}\times\mathcal{C}_\alpha\to L^2(P,H)$ are two continuous stochastic processes. To do this we will need the following assumptions.

(H$_7$) Let $\mu,\nu\in\mathcal{M}$, and let $f\colon\mathbb{R}\times C([-r,0],L^2(P,H_\alpha))\to L^2(P,H)$ be a square-mean $cl(\mu,\nu)$-pseudo almost automorphic of class r such that there exists a positive constant L_f such that
$$\mathbb{E}\|f(t,\phi_1)-f(t,\phi_2)\|^2\le L_f\mathbb{E}\|\phi_1-\phi_2\|_\alpha^2$$
for all $t\in\mathbb{R}$ and $\phi_1,\phi_2\in C([-r,0],L^2(P,H_\alpha))$.

(H$_8$) Let $\mu,\nu\in\mathcal{M}$, and let $g\colon\mathbb{R}\times C([-r,0],L^2(P,H_\alpha))\to L^2(P,H_\alpha)$ be a square-mean $cl(\mu,\nu)$-pseudo almost automorphic of class r such that there exists a positive constant L_g such that
$$\mathbb{E}\|g(t,\phi_1)-g(t,\phi_2)\|^2\le L_g\mathbb{E}\|\phi_1-\phi_2\|_\alpha^2$$
for all $t\in\mathbb{R}$ and $\phi_1,\phi_2\in C([-r,0],L^2(P,H_\alpha))$.

(H$_9$) The unstable space is trivial, that is, $U\equiv\{0\}$.

Theorem 6.7 *Assume that* $(\mathbf{H}_0)$–$(\mathbf{H}_4)$ *and* $(\mathbf{H}_7)$–$(\mathbf{H}_9)$ *hold. If*

$$\frac{2\overline{M}^2\widetilde{M}^2|\Pi^s|^2(L_f+L_g)}{(2w)^{1-2\alpha}}\,\Gamma(1-2\alpha)<1,$$

then equation (6.5) *has a unique compact* α-$cl(\mu,\nu)$*-square-mean pseudo almost automorphic solution of class* r.

Proof. Let x be a function in $SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. From Theorem 6.3 the function $t\to x_t$ belongs to $SPAA_c(C([-r,0],L^2(P,H_\alpha)),\mu,\nu,r)$. Hence, Theorem 5.15 implies that the function $g(.)=f(.,x)$ is in $SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. Since by $(\mathbf{H}_9)$ the unstable space is trivial, that is, $U\equiv\{0\}$, if follows that $|\Pi^u|=0$. Consider the mapping

$$\mathcal{H}\colon SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)\to SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$$

defined for $t\in\mathbb{R}$ by

$$(\mathcal{H}x)(t)=\Bigg[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s,x_s))\mathrm{d}s + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0g(s,x_s))\mathrm{d}W(s)\Bigg](0).$$

From Theorem 6.2, Theorem 6.4 and Theorem 6.5, it suffices to show now that the operator $\mathcal{H}$ has a unique fixed point in $SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. Let $x_1,x_2\in SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. Then, we have

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{H}x_1)-(\mathcal{H}x_2)\|_\alpha^2\\
&\le 2\mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0(f(s,x_{1s})-f(s,x_{2s}))\big)\mathrm{d}s\right\|_\alpha^2\\
&\quad+2\mathbb{E}\left\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}_\lambda X_0(g(s,x_{1s})-g(s,x_{2s}))\big)\mathrm{d}W(s)\right\|_\alpha^2.
\end{aligned}$$

By Ito's isometry, it follows that

$$\begin{aligned}
\mathbb{E}\|(\mathcal{H}x_1)-(\mathcal{H}x_2)\|_\alpha^2 &\le 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}L_f\mathbb{E}\|x_{1s}-x_{2s}\|_\alpha^2\mathrm{d}s\\
&\quad+2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}\frac{e^{-2w(t-s)}}{(t-s)^{2\alpha}}L_g\mathbb{E}\|x_{1s}-x_{2s}\|_\alpha^2\mathrm{d}s\\
&\le\frac{2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\big(L_f+L_g\big)}{(2w)^{1-2\alpha}}\left(\int_0^{+\infty}e^{-s}s^{-2\alpha}\mathrm{d}s\right)\|x_1-x_2\|_{\infty,\alpha}^2\\
&\le\frac{2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\big(L_f+L_g\big)}{(2w)^{1-2\alpha}}\Gamma(1-2\alpha)\|x_1-x_2\|_{\infty,\alpha}.
\end{aligned}$$

This means that $\mathcal{H}$ is a strict contraction. Thus, by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $SPAA_c(\mathbb{R},L^2(P,H_\alpha),\mu,\nu,r)$. We conclude that equation (6.5) has one and only one compact α-$cl(\mu,\nu)$-square-mean pseudo almost automorphic solution of class r. □

7 Application

As an illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \mathrm{d}z(t,x) \\ \quad = -\dfrac{\partial^2}{\partial x^2}z(t,x)\mathrm{d}t + \left[\displaystyle\int_{-r}^0 G(\theta)z(t+\theta,x)\mathrm{d}\theta + \sin\left(\dfrac{1}{2+\cos t+\cos\sqrt{2}t}\right)\right. \\ \quad\quad \left. + \arctan(t) + \displaystyle\int_{-r}^0 h\left(\theta, \dfrac{\partial}{\partial x}z(t+\theta,x)\right)\mathrm{d}\theta\right]\mathrm{d}t + \left[x\sin\left(\dfrac{1}{2+\cos t+\cos\sqrt{3}t}\right)\right. \\ \quad\quad \left. + \cos(t) + \displaystyle\int_{-r}^0 h\left(\theta, \dfrac{\partial}{\partial x}z(t+\theta,x)\right)\mathrm{d}\theta\right]\mathrm{d}W(t) \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi], \\ z(t,0)=z(t,\pi)=0 \quad \text{for } t\in\mathbb{R} \text{ and } x\in[0,\pi], \end{cases} \tag{7.1}$$

where $G\colon [-r,0]\to\mathbb{R}$ is a continuous function and $h\colon [-r,0]\times\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous with the respect to the second argument. For example, we can take

$$G(\theta) = \frac{\theta^2-1}{(\theta^2+1)^2} \text{ for } \theta\in[-r,0]$$

and

$$h(\theta,x) = \theta^2 + \sin\left(\frac{x}{4}\right) \text{ for } (\theta,x)\in[-r,0]\times\mathbb{R}.$$

We can see that G is continuous, and $|h(\theta,x_1)-h(\theta,x_2)| \leq \frac{1}{4}|x_1-x_2|$, which implies that h is Lipschitz continuous with the respect to its second argument. $W(t)$ is a two-sided standard Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\big\{W(u)-W(v) \mid u,v\leq t\big\}$.

To rewrite equation (7.1) in abstract form, we introduce the space $H = L^2((0,\pi))$. Let $A\colon D(A)\to L^2((0,\pi))$ be defined by

$$\begin{cases} D(A) = H^2(0,\pi)\cap H^1(0,\pi), \\ Ay(t) = y''(t) \text{ for } t\in(0,\pi) \text{ and } y\in D(A). \end{cases}$$

Then, the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 :\ n\geq 1\},$$

and the associated eigenfunctions $(\zeta_n)_{n\geq 1}$ are given by

$$\zeta_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns),\ s\in[0,\pi].$$

Consequently, the operator A takes the form

$$Ay = \sum_{n=1}^{\infty} n^2(y,\zeta_n)\zeta_n,\ y\in D(A).$$

For each

$$y \in D(A^{\frac{1}{2}}) = \left\{ y \in H : \sum_{n=1}^{\infty} n(y, \zeta_n)\zeta_n \in H \right\},$$

we define the fractional power

$$A^{\frac{1}{2}} \colon D(A^{\frac{1}{2}}) \subset H \to H$$

by

$$A^{\frac{1}{2}} y = \sum_{n=1}^{\infty} n(y, \zeta_n)\zeta_n, \ y \in D(A^{\frac{1}{2}}).$$

It is well-known that $-A$ is the generator of a compact analytic semi-group $(T(t))_{t\geq 0}$ on H which is given by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u, \zeta_n)\zeta_n, \ u \in H.$$

This means that ($\mathbf{H}_0$) and ($\mathbf{H}_2$) are satisfied.

Here, we choose $\alpha = \frac{1}{2}$. Moreover, we define $f \colon \mathbb{R} \times \mathcal{C}_{\frac{1}{2}} \to L^2((0,\pi))$ and $L \colon \mathcal{C}_{\frac{1}{2}} \to L^2(P, H)$ as follows

$$\begin{aligned} f(t,\phi)(x) &= x \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t} \right) + \arctan(t) \\ &\quad + \int_{-r}^{0} h\left(\theta, \frac{\partial}{\partial x}\phi(\theta)(x) \right) \mathrm{d}\theta \text{ for } x \in (0,\pi) \text{ and } t \in \mathbb{R} \\ g(t,\phi)(x) &= x \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{3}t} \right) + \cos(t) \\ &\quad + \int_{-r}^{0} h\left(\theta, \frac{\partial}{\partial x}\phi(\theta)(x) \right) \mathrm{d}\theta \text{ for } x \in (0,\pi) \text{ and } t \in \mathbb{R} \end{aligned}$$

and

$$L(\phi)(x) = \int_{-r}^{0} G(\theta)\phi(\theta)(x)\mathrm{d}\theta \text{ for } -r \leq \theta \text{ and } x \in (0,\pi).$$

Lemma 7.1 ([28]) *If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in L^2(P, H)$ and $|y'| = |A^{\frac{1}{2}} y|$.*

Let us set $v(t) = z(t,x)$. Then, equation (7.1) takes the following abstract form

$$\mathrm{d}v(t) = [-Av(t) + L(v_t) + f(t, v_t)]\mathrm{d}t + g(t, v_t)\mathrm{d}W(t) \text{ for } t \in \mathbb{R}. \tag{7.2}$$

Consider the measures μ and ν whose Randon–Nikodym derivatives ρ_1 and ρ_2 are given by

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0, \\ e^t & \text{for } t \leq 0 \end{cases}$$

and $\rho_2(t) = |t|$ for $t \in \mathbb{R}$, respectively. Then, $\mathrm{d}\mu(t) = \rho_1(t)\mathrm{d}t$ and $\mathrm{d}\mu(t) = \rho_2(t)\mathrm{d}t$, where $\mathrm{d}t$ denotes the Lebesgue measure on $\mathbb{R}$. In other words,

$$\mu(A) = \int_A \rho_1(t)\mathrm{d}t \text{ and } \nu(A) = \int_A \rho_2(t)\mathrm{d}t \text{ for } A \in \mathcal{N}.$$

From [7], it follows that $\mu, \nu \in \mathcal{M}$ satisfy condition ($\mathbf{H}_4$). Moreover,

$$A^{\frac{1}{2}}\left(x \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\right) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)$$

and

$$A^{\frac{1}{2}}\left(x \sin\left(\frac{1}{2+\cos t+\cos\sqrt{3}t}\right)\right) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{3}t}\right).$$

Then,

$$t \mapsto \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) \quad \text{and} \quad t \mapsto \sin\left(\frac{1}{2+\cos t+\cos\sqrt{3}t}\right)$$

are α-almost automorphic. We have also

$$\lim_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-r}^{0} e^t \mathrm{d}t + \int_0^\tau \mathrm{d}t}{\displaystyle 2\int_0^\tau t\mathrm{d}t} = \limsup_{\tau\to+\infty} \frac{1+e^{-\tau}+\tau}{\tau^2} = 0 < +\infty,$$

which implies that ($\mathbf{H}_2$) is satisfied.

Obviously,$-1 \le \sin\theta \le 1$ for $\theta \in \mathbb{R}$. Hence, by Lemma 7.1, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}|\cos(\theta)|^2_{\frac{1}{2}} \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}|A^{\frac{1}{2}}\cos(\theta)|^2 \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}|\sin(\theta)|^2 \mathrm{d}\mu(t) \\
&\le \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \mathrm{d}\mu(t) \\
&\le \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau \to +\infty.
\end{aligned}$$

This means that $t \mapsto \cos(t)$ is α-(μ,ν)-ergodic of class r. Similarly, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}|\arctan(\theta)|^2_{\frac{1}{2}} \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}|A^{\frac{1}{2}}\arctan(\theta)|^2 \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\left|\frac{1}{1+\theta^2}\right|^2 \mathrm{d}\mu(t) \\
&\le \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \mathrm{d}\mu(t) \\
&\le \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau \to +\infty.
\end{aligned}$$

Thus, $t \mapsto \arctan(t)$ is α-(μ,ν)-ergodic of class r. Consequently, f and g are uniformly compact (μ,ν)-pseudo almost automorphic of class r. Moreover, L is a bounded linear operator from $\mathcal{C}_{\frac{1}{2}}$ to $L^2(P, L^2(0,\pi))$.

Let L_h be the Lipschitz constant of h. Then, for every $\phi_1, \phi_2 \in \mathcal{C}_{\frac{1}{2}}$ we have

$$\begin{aligned}
&\mathbb{E}\|f(t,\phi_1)(x) - f(t,\phi_2)(x)\|^2 \\
&= \mathbb{E}\left\|\int_{-r}^{0} h\Big(\theta, \frac{\partial}{\partial x}\phi_1(\theta)(x)\Big) - h\Big(\theta, \frac{\partial}{\partial x}\phi_2(\theta)(x)\Big)\mathrm{d}\theta\right\|^2 \\
&\le L_h^2 \int_{-r}^{0} \mathbb{E}\left\|\frac{\partial}{\partial x}\phi_1(\theta)(x) - \frac{\partial}{\partial x}\phi_2(\theta)(x)\right\|^2 \mathrm{d}\theta \\
&\le L_h^2 \int_{-r}^{0} \sup_{-r\le\theta\le 0} \mathbb{E}\left\|\frac{\partial}{\partial x}\phi_1(\theta)(x) - \frac{\partial}{\partial x}\phi_2(\theta)(x)\right\|^2 \mathrm{d}x \\
&\le rL_h^2 \sup_{-r\le\theta\le 0} \mathbb{E}\|\phi_1(\theta)(x) - \phi_2(\theta)(x)\|_\alpha^2 \\
&\le rL_h^2\lambda\mathbb{E}\|\phi_1(x) - \phi_2(x)\|_\alpha^2 \text{ for certain } \lambda \in \mathbb{R}_+.
\end{aligned}$$

Similarly,

$$\mathbb{E}\|g(t,\phi_1)(x) - g(t,\phi_2)(x)\|^2 \le rL_h^2\lambda\mathbb{E}\|\phi_1(x) - \phi_2(x)\|_\alpha^2 \text{ for certain } \lambda \in \mathbb{R}_+.$$

Consequently, we conclude that f and g are Lipschitz continuous and α-$cl(\mu,\nu)$-pseudo almost automorphic of class r.

Moreover, since h is Lipschitz continuous and therefore bounded, there exists a positive real number M_1 such that $\|h(\theta,x)\| \le M_1$. Consequently, for $t \in \mathbb{R}^+$ and $x \in [0,\pi]$, we have

$$\begin{aligned}
\mathbb{E}\|g(t,\phi)(x)\|^2 &\le 3\pi^2 + 3 + 3\int_{-r}^{0} \mathbb{E}\left\|h\Big(\theta, \frac{\partial}{\partial x}\phi(\theta)(x)\Big)\right\|^2 \mathrm{d}\theta \\
&\le 3(\pi^2 + 1 + rM_1^2) = M_2.
\end{aligned}$$

This implies that g satisfies ($\mathbf{H}_5$).

For hyperbolycity, we assume that:

($\mathbf{H}_{10}$) $\displaystyle\int_{-r}^{0} |G(\theta)|\mathrm{d}\theta < 1.$

Lemma 7.2 ([16]) *Assume that* ($\mathbf{H}_{10}$) *holds. Then, the semi-group* $(\mathcal{U}(t))_{t\ge 0}$ *is hyperbolic and the unstable space is trivial, that is,* $U \equiv \{0\}$.

We can see that

$$\int_{-r}^{0} |G(\theta)|\mathrm{d}\theta = \int_{-r}^{0}\left|\frac{\theta^2 - 1}{(\theta^2+1)^2}\right|\mathrm{d}\theta = \left[\frac{\theta}{\theta^2+1}\right]_{-r}^{0} = \frac{r}{r^2+1} < 1$$

and

$$\int_{-r}^{0} |G(\theta)|\mathrm{d}\theta$$
$$= \int_{-r}^{0} \left| \frac{\theta^2 - 1}{(\theta^2+1)^2} \right| \mathrm{d}\theta = \int_{-r}^{-1} \frac{\theta^2 - 1}{(\theta^2+1)^2} \mathrm{d}\theta + \int_{-1}^{0} \frac{-\theta^2 + 1}{(\theta^2+1)^2} \mathrm{d}\theta = 1 - \frac{r}{r^2+1} < 1,$$

if $r \geq 1$.

Theorem 7.3 *Assume that* $(\mathbf{H}_7)$–$(\mathbf{H}_{10})$ *hold. If* $Lip(h)$ *is small enough, then equation* (7.2) *has a unique compact* α-$cl(\mu, \nu)$*-square-mean pseudo almost automorphic solution* v *of class* r.

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