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A NOTE ON ALMOST PERIODIC FUNCTIONS WITH VALUES IN LOCALLY CONVEX SPACES

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Abstract. It is well known that the almost periodic functions $F : \mathbb{R}^n \to Y$ form a vector space with the usual operations if Y is a Fréchet space. We extend this result to the almost periodic functions with values in an arbitrary locally convex space Y.

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1 Introduction and preliminaries

The almost periodic functions $F : \mathbb{R}^n \to Y$, where Y is a Fréchet space, form a vector space with the usual operations (see $[1, 7]$). In this short note, we will extend this result to the almost periodic functions with values in an arbitrary locally convex space Y, in a more general context, and present an illustrative application to the integral equations.

Let us recall, if Y is a locally convex space and $\mathcal{W}$ is a base of balanced neighbourhoods of zero in Y, then a continuous function $F : \mathbb{R}^n \to Y$ is said to be almost periodic if, for every $W \in \mathcal{W}$, there exists a number $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists a point $\tau \in B(\mathbf{t}_0, l)$ such that $F(\mathbf{t}+\tau)-F(\mathbf{t}) \in W$ for all $\mathbf{t} \in \mathbb{R}^n$. If this is the case, the range of $F(\cdot)$ is totally bounded in Y and $F(\cdot)$ is uniformly continuous; the space of all almost periodic functions, denoted by $AP(\mathbb{R}^n : Y)$, is translation invariant, closed under uniform convergence and closed under reflexions at zero (cf. [10] and [11] for some pioneering results established by G. M. N'Guérékata in this direction). It is also worth noting that, in the research articles [5] and [8], the almost periodicity in some non locally convex spaces, namely the so-called p-Fréchet spaces, where $0 < p < 1$, has been studied.

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In our recent research article [4], we have considered several new classes of (metrically) ρ-almost periodic type functions $F : I \times X \to Y$, where $\emptyset \neq I \subseteq \mathbb{R}^n$, X is an arbitrary non-empty set and Y is a locally convex space. In that paper, we have specifically introduced the following notion:

Definition 1.1 *Suppose that (C1) holds, where:*

(C1) $\emptyset \neq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq \mathbb{R}^n$, $I' + I \subseteq I$, Y is a locally convex space over the field of complex numbers, the topology on Y is induced by the fundamental system $\circledast_Y$ of seminorms, ρ is a binary relation on Y, X is an arbitrary non-empty set and $\mathcal{B}$ is a collection of certain non-empty subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

Let $F : I \times X \to Y$. Then we say that $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that

$$\kappa\big(F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x}\big) \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B.$$

If X is a topological space, then we say that $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I', \rho)$-almost periodic if $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic and continuous; furthermore, if $\rho = c\mathrm{I}$, where $c \in \mathbb{C} \setminus \{0\}$, then we say that $F(\cdot;\cdot)$ is (pre-)Bohr $(\mathcal{B}, I', c)$-almost periodic.

We omit the term "$\mathcal{B}$" from the notation if $X = \{0\}$, the term "ρ" from the notation if $\rho = \mathrm{I}$ and the term "I'" from the notation if $I' = I$.

We will use the following result (cf. [4, Theorem 2.4(ii)]):

Theorem 1.2 *Suppose that $k \in \mathbb{N}$, X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y_i is a Fréchet space for $1 \leq i \leq k$ and the function $F_i : \mathbb{R}^n \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic for $1 \leq i \leq k$. Then the function $(F_1, ..., F_k)(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic.*

2 The main results

If Y is a Fréchet space, then the Bochner criterion provides a necessary and sufficient condition for a bounded continuous function $F : \mathbb{R}^n \to Y$ to be almost periodic. This result is essentially important in the proof of the fact that the almost periodic functions $F : \mathbb{R}^n \to Y$, where Y is a Fréchet space, form a vector space with the usual operations (see [7, Theorem 9, Theorem 10] for more details in this direction; let us also note that G. M. N'Guérékata has considered the almost periodic functions with values in Fréchet spaces in his monograph [12]). In this paper, we extend the above-mentioned result to the almost periodic functions with values in an arbitrary locally convex space Y following a completely different approach with the use of local Banach spaces.

Unless stated otherwise, we will always assume henceforth that (C1) holds. If $\kappa \in \circledast_Y$, then we define $N_\kappa := \{y \in Y : \kappa(y) = 0\}$ and $\rho_\kappa := \{(x + N_\kappa, y + N_\kappa) : (x, y) \in \rho\}$. Let us consider now the vector space $Y_\kappa := \{y + N_\kappa : y \in Y\}$, equipped with the norm $\|y + N_\kappa\|_\kappa := \kappa(y)$ for all $y \in Y$, and the corresponding canonical mapping $q_\kappa : Y \to Y_\kappa$ defined by $q_\kappa(y) := y + N_\kappa$ for all $y \in Y$. It is clear that ρ_κ is a binary relation on Y_κ. Further on, the local Banach space for the seminorm κ is the completion $(\hat{Y_\kappa}, \|\cdot\hat{\|}_\kappa)$ of the normed space $(Y_\kappa, \|\cdot\|_\kappa)$, see [9, pp. 289–290] for more details on the subject.

We will use the following interesting construction of the completion $(\hat{Y_\kappa}, \|\cdot\hat{\|}_\kappa)$ given in the research report [13] by F. J. Sayas:

Let c_κ be the vector space of all Cauchy sequences $(x_m)_{m\in\mathbb{N}}$ in Y_κ and let $c_{0,\kappa}$ be the vector space of all sequences $(x_m)_{m\in\mathbb{N}}$ in Y_κ that converge to zero. Then the vector space $\hat{Y_\kappa} := \{(x_m)_{m\in\mathbb{N}} + c_{0,\kappa} : (x_m)_{m\in\mathbb{N}} \in c_\kappa\}$, equipped with the norm $\|(x_m)_{m\in\hat{\mathbb{N}}} + c_{0,\kappa}\|_\kappa := \lim_{m\to+\infty} \|x_m\|_\kappa$, $(x_m)_{m\in\mathbb{N}} \in c_\kappa$ is a Banach space, and this space is exactly the completion of $(Y_\kappa, \|\cdot\|_\kappa)$ since the mapping $K_\kappa : Y_\kappa \to \hat{Y_\kappa}$, given by $K_\kappa(x) := (x, x, ..., x, ...) + c_{0,\kappa}$ for all $x \in Y_\kappa$, is an isometry and $K_\kappa(Y_\kappa)$ is dense in $\hat{Y_\kappa}$. If $T \in L(Y)$ and for each $\kappa \in \circledast_Y$ there exists $c > 0$ such that $\kappa(Ty) \leq c\kappa(y)$ for all $y \in Y$, then we can simply prove that $T(N_\kappa) \subseteq N_\kappa$ and, in this case, we can define the bounded linear operator $T_\kappa \in L(Y_\kappa)$ by $T_\kappa(y + N_\kappa) := Ty + N_\kappa$, $y \in Y$. If we define $\hat{T_\kappa}((x_m)_{m\in\mathbb{N}} + c_{0,\kappa}) := (T_\kappa x_m)_{m\in\mathbb{N}} + c_{0,\kappa}$, $(x_m)_{m\in\mathbb{N}} \in c_\kappa$, then we have

$$\hat{T_\kappa} \in L(\hat{Y_\kappa}) \quad \text{and} \quad \hat{T_\kappa}\big(K_\kappa(x)\big) = K_\kappa\big(T_\kappa x\big), \quad x \in Y_\kappa. \tag{2.1}$$

Further on, we define $\hat{\rho_\kappa} := \{(K_\kappa(x+N_\kappa), K_\kappa(y+N_\kappa)) : (x, y) \in \rho\}$, then $\hat{\rho_\kappa}$ is a binary relation on $\hat{Y_\kappa}$ and, if $\rho = T \in L(Y)$ satisfies the above properties, then $\hat{\rho_\kappa}$ is equal to the part of the operator $\hat{T_\kappa}$ in the dense subspace $K_\kappa(Y_\kappa)$ of $\hat{Y_\kappa}$.

Now we are able to formulate the main result of this paper:

Theorem 2.1 *Suppose that* (C1) *holds and $F : I \times X \to Y$. Then the following hold:*

(i) If the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic, then for each $\kappa \in \circledast_Y$ we have that the function $F_\kappa : I \times X \to \hat{Y_\kappa}$, given by $F_\kappa(\mathbf{t}; x) := (K_\kappa \circ q_\kappa)(F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ is pre-$(\mathcal{B}, I', \hat{\rho_\kappa})$-almost periodic. Moreover, if X is a topological space and $F(\cdot;\cdot)$ is continuous, then $F_\kappa(\cdot;\cdot)$ is continuous as well.

(ii) Suppose that for each $\kappa \in \circledast_Y$ the function $F_\kappa : I \times X \to \hat{Y_\kappa}$, given by $F_\kappa(\mathbf{t}; x) := (K_\kappa \circ q_\kappa)(F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ is pre-$(\mathcal{B}, I', \hat{\rho_\kappa})$-almost periodic. Then the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic provided that (ii.1) *or* (ii.2) *holds, where:*

(ii.1) $\rho = T \in L(Y)$ and for each $\kappa \in \circledast_Y$ there exists $c > 0$ such that $\kappa(Ty) \leq c\kappa(y)$ for all $y \in Y$;

(ii.2) $\kappa(\cdot)$ is a norm on Y for every $\kappa \in \circledast_Y$.

Proof. Suppose that the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic and $\kappa \in \circledast_Y$ is fixed. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. Then we know that there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that $\kappa(F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x}) \leq \epsilon$, $\mathbf{t} \in I$, $x \in B$. This implies $K_\kappa(y_{\mathbf{t};x} + N_\kappa) \in \hat{\rho_\kappa}(F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in B$. Therefore, it suffices to show that

$$\Big\| K_\kappa\big(F(\mathbf{t} + \tau; x) + N_\kappa\big) - K_\kappa\big(y_{\mathbf{t};x} + N_\kappa\big) \Big\|_{\hat{Y_\kappa}} \leq \epsilon, \quad \mathbf{t} \in I, \ x \in B.$$

Since $K_\kappa(\cdot)$ is an isometry, this is equivalent to saying that $\|F(\mathbf{t} + \tau; x) - (y_{\mathbf{t};x} + N_\kappa\|_\kappa \leq \epsilon$ for all $\mathbf{t} \in I$ and $x \in B$, *i.e.*, $\kappa(F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x}) \leq \epsilon$ for all $\mathbf{t} \in I$ and $x \in B$, which is true.

Clearly, if X is a topological space and $F(\cdot;\cdot)$ is continuous, then the continuity of $F_\kappa(\cdot;\cdot)$ follows immediately from the fact that $q_\kappa(\cdot)$ is continuous and $K_\kappa(\cdot)$ is an isometry.

Let us assume now that the requirements in part (ii) hold. Then, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $G_{\kappa,x}(\mathbf{t}) \in \hat{\rho_\kappa}(K_\kappa(F(\mathbf{t};x) + N_\kappa))$ such that:

$$\big\|K_\kappa\big(F(\mathbf{t}+\tau;x) + N_\kappa\big) - G_{\kappa,x}(\mathbf{t})\big\|_{\hat{Y_\kappa}} \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B. \tag{2.2}$$

Let $\mathbf{t} \in I$ and $x \in B$ be fixed. Then there exists $(Y_1(\mathbf{t};x), Y_2(\mathbf{t};x)) \in \rho$ such that $K_\kappa(F(\mathbf{t};x) + N_\kappa) = K_\kappa(Y_1(\mathbf{t};x) + N_\kappa)$ and $G_{\kappa,x}(\mathbf{t}) = K_\kappa(Y_2(\mathbf{t};x) + N_\kappa)$. Since $K_\kappa(\cdot)$ is an injective isometry, the above implies $F(\mathbf{t};x) + N_\kappa = Y_1(\mathbf{t};x) + N_\kappa$; keeping in mind (2.2) and these equalities, we simply get

$$\big\|F(\mathbf{t}+\tau;x) - Y_2(\mathbf{t};x) + N_\kappa\big\|_\kappa \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B, \tag{2.3}$$

i.e.,

$$\kappa\big(F(\mathbf{t}+\tau;x) - Y_2(\mathbf{t};x)\big) \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B.$$

Let (ii.1) hold. Then $T(N_\kappa) \subseteq N_\kappa$, $TY_1(\mathbf{t};x) = Y_2(\mathbf{t};x)$ and therefore

$$\kappa\big(F(\mathbf{t}+\tau;x) - Y_2(\mathbf{t};x)\big) = \kappa\big(F(\mathbf{t}+\tau;x) - TF(\mathbf{t};x)\big),\ \mathbf{t} \in I,\ x \in B.$$

This simply implies that $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic. On the other hand, if (ii.2) holds, $\mathbf{t} \in I$ and $x \in B$, then we have $F(\mathbf{t};x) = Y_1(\mathbf{t};x)$, where the pair $(Y_1(\mathbf{t};x), Y_2(\mathbf{t};x))$ satisfies the above requirements. This yields $Y_2(\mathbf{t};x) \in \rho(F(\mathbf{t};x))$ and completes the proof in a routine manner. $\square$

We will also clarify the following result:

Theorem 2.2 *Suppose that* (C1) *holds,* $F : I \times X \to Y$*,* $\rho = T \in L(Y)$ *and for each* $\kappa \in \circledast_Y$ *there exists* $c > 0$ *such that* $\kappa(Ty) \leq c\kappa(y)$ *for all* $y \in Y$*. Then* $F(\cdot;\cdot)$ *is pre-*$(\mathcal{B}, I', T)$*-almost periodic if and only if for each* $\kappa \in \circledast_Y$ *the function* $F_\kappa : I \times X \to \hat{Y_\kappa}$*, given by* $F_\kappa(\mathbf{t};x) := (K_\kappa \circ q_\kappa)(F(\mathbf{t};x))$*,* $\mathbf{t} \in I$*,* $x \in X$*, is pre-*$(\mathcal{B}, I', \hat{T_\kappa})$*-almost periodic.*

Proof. If $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', T)$-almost periodic, then Theorem 2.1(i) immediately yields that for each $\kappa \in \circledast_Y$ the function $F_\kappa(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \hat{T_\kappa})$-almost periodic since $\hat{T_\kappa}$ is an extension of $\hat{\rho_\kappa}$. To prove the converse statement, it suffices to repeat verbatim the corresponding part of the proof of Theorem 2.1(ii); let us only emphasize that we can use the fact that $K_\kappa(\cdot)$ is an isometry and the equality stated in the second part of (2.1). $\square$

As an immediate corollary of Theorem 2.2, we have the following result:

Corollary 2.3 *Suppose that* (C1) *holds,* X *is a topological space,* $F : I \times X \to Y$ *is continuous,* $\rho = T \in L(Y)$ *and for each* $\kappa \in \circledast_Y$ *there exists* $c > 0$ *such that* $\kappa(Ty) \leq c\kappa(y)$ *for all* $y \in Y$*. Then* $F(\cdot;\cdot)$ *is Bohr* $(\mathcal{B}, I', T)$*-almost periodic if and only if for each* $\kappa \in \circledast_Y$ *the function* $F_\kappa : I \times X \to \hat{Y_\kappa}$*, given by* $F_\kappa(\mathbf{t};x) := (K_\kappa \circ q_\kappa)(F(\mathbf{t};x))$*,* $\mathbf{t} \in I$*,* $x \in X$*, is Bohr* $(\mathcal{B}, I', \hat{T_\kappa})$*-almost periodic.*

Keeping in mind Theorem 1.2, we can state the following important corollary:

Theorem 2.4 *Suppose that X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y is a locally convex space, $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. If the functions $F : \mathbb{R}^n \times X \to Y$ and $G : \mathbb{R}^n \times X \to Y$ are Bohr $\mathcal{B}$-almost periodic, then the function $(\alpha F + \beta G)(\cdot;\cdot)$ is likewise Bohr $\mathcal{B}$-almost periodic.*

Proof. The proof immediately follows from Corollary 2.3 with $T = \mathrm{I}$ and $I = I' = \mathbb{R}^n$, by observing that for each $\kappa \in \circledast_Y$ we have $(K_\kappa \circ q_\kappa)((\alpha F + \beta G)(\mathbf{t}; x)) = \alpha(K_\kappa \circ q_\kappa)(F(\mathbf{t}; x)) + \beta(K_\kappa \circ q_\kappa)(G(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$. $\square$

If Y_1, ..., Y_k are locally convex spaces, then $Y_1 \times ... \times Y_k$ is a locally convex space and the fundamental system of seminorms which defines the topology on $Y_1 \times ... \times Y_k$ is given by $\kappa(y_1, ..., y_k) := \kappa_1(y_1) + ... + \kappa_k(y_k)$, $y_1 \in Y_1, ..., y_k \in Y_k$, where $\kappa_j(\cdot)$ runs through the all seminorms in $\circledast_j$ for $1 \leq j \leq k$. Now we are able to formulate the following result:

Theorem 2.5 *Suppose that $k \in \mathbb{N}$, X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y_i is a locally convex space for $1 \leq i \leq k$ and the function $F_i : \mathbb{R}^n \times X \to Y_i$ is Bohr $\mathcal{B}$-almost periodic for $1 \leq i \leq k$. Then the function $(F_1, ..., F_k) : \mathbb{R}^n \times X \to Y_1 \times ... \times Y_k$ is Bohr $\mathcal{B}$-almost periodic.*

Proof. If the function $F_i : \mathbb{R}^n \times X \to Y_i$ is Bohr $\mathcal{B}$-almost periodic for $1 \leq i \leq k$, then the function $F_i' : \mathbb{R}^n \times X \to Y_1 \times ... \times Y_i \times ... \times Y_k$ given by $F_i'(\mathbf{t}; x) := (0, 0, ..., F_i(\mathbf{t}; x), ..., 0)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ is Bohr $\mathcal{B}$-almost periodic for $1 \leq i \leq k$. Applying Theorem 2.4 and the mathematical induction, it readily follows that the function $(F_1, ..., F_k) : \mathbb{R}^n \times X \to Y_1 \times ... \times Y_k$ is Bohr $\mathcal{B}$-almost periodic since $(F_1, ..., F_k)(\mathbf{t}; x) = F_1'(\mathbf{t}; x) + ... + F_k'(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. $\square$

It is clear that Theorem 2.2 and Corollary 2.3 enable one to reformulate many structural results already known for the Bohr $(\mathcal{B}, I', c)$-almost periodic functions with values in complex Banach spaces to the Bohr $(\mathcal{B}, I', c)$-almost periodic functions with values in locally convex spaces ($c \in \mathbb{C} \setminus \{0\}$). We can formulate the Maak criterion for almost periodic functions with values in locally convex spaces (see, *e.g.*, [2]) and we can consider the approximation of almost periodic functions with values in locally convex spaces by the trigonometric polynomials in the local Banach spaces (see, *e.g.*, [6] and references quoted therein); for example, we have the following result:

Theorem 2.6 *Suppose that $F : \mathbb{R}^n \to Y$ is a bounded, continuous function. Then $F(\cdot)$ is Bohr almost periodic if and only if for each $\epsilon > 0$ and $\kappa \in \circledast$ there exists a trigonometric polynomial $P : \mathbb{R}^n \to Y$ such that*

$$\sup_{\mathbf{t} \in \mathbb{R}^n} \kappa\big(F(\mathbf{t}) - P(\mathbf{t})\big) \leq \epsilon.$$

More details will appear somewhere else.

3 An application

In this section, we will provide an illustrative application of our results to the following integral equation:

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s})u(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{3.1}$$

where $X = Y$ is a sequentially complete locally convex space, $f : \mathbb{R}^n \to Y$ is almost periodic, $a \in L^1((0,\infty)^n)$ and $u : \mathbb{R}^n \to Y$ is an unknown function; we can also consider the corresponding semilinear problem but we will skip all related details for the sake of brevity.

It can be easily shown that $AP(\mathbb{R}^n : Y)$ is a sequentially complete locally convex space when equipped with the family of seminorms $(\|\cdot\|_{\kappa,\infty})_{\kappa\in\circledast_Y}$, where $\|f\|_{\kappa,\infty} := \sup_{\mathbf{t}\in\mathbb{R}^n} \kappa(f(\mathbf{t}))$ for all $\kappa \in \circledast_Y$. Further on, the mapping $\Pi : AP(\mathbb{R}^n : Y) \to AP(\mathbb{R}^n : Y)$, defined by

$$\big[\Pi u\big](t) := f(\mathbf{t}) + \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s})u(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,\ u \in AP\big(\mathbb{R}^n : Y\big),$$

is well-defined due to Theorem 2.4 and a simple argumentation concerning the invariance of almost periodicity under the action of the infinite convolution product considered above (see, *e.g.*, [4] and [6]). Suppose now that $\int_{(0,+\infty)^n} |a(\mathbf{s})|\, d\mathbf{s} < 1$, then the mapping $\Pi(\cdot)$ has a unique fixed point due to the well-known fixed point theorem of A. Deleanu and G. Marinescu [3, Theorem 1, p. 92]. This implies the required conclusion.

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