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EXISTENCE RESULTS FOR RENORMALIZED SOLUTIONS TO NON-COERCIVE NONLINEAR ELLIPTIC EQUATIONS INVOLVING A HARDY POTENTIAL AND WITH L^1-DATA

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Abstract. Here, we prove the existence of a renormalized solution for a class of nonlinear elliptic Dirichlet problems which contain a nonlinear convection term that satisfies an optimal growth condition, as well as a zero-order perturbation term known as the Hardy potential. Additionally, it is important to note that the right-hand side is assumed to be an L^1-function.

Keywords: Nonlinear elliptic equation, Convection term, Hardy potential, Renormalized solution, L^1-data.

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1 Introduction

For any $N > 2$, $1 < p < \infty$, and an arbitrary bounded open subset Ω of $\mathbb{R}^N$ containing the origin 0, we study in this paper the following nonlinear and noncoercive elliptic Dirichlet problem

$$\begin{cases} -\operatorname{div}(b(x,v,\nabla v)+\mathcal{B}(x,v)) \;=\; \lambda\frac{|v|^{s-1}v}{|x|^p}+f & \text{in } \Omega, \\ v \;=\; 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $b : \Omega\times\mathbb{R}\times\mathbb{R}^N \longmapsto \mathbb{R}^N$ is a Carathéodory function which satisfies the classical Leray–Lions assumptions:

$$b(x,\eta,\xi)\cdot\xi \;\geq\; \alpha|\xi|^p, \tag{1.2}$$

$$|b(x,\eta,\xi)| \;\leq\; C(\mathcal{L}(x)+|\eta|^{p-1}+|\xi|^{p-1}), \tag{1.3}$$

$$(b(x,\eta,\xi)-b(x,\eta,\xi'))\cdot(\xi-\xi') \;\geq\; 0, \tag{1.4}$$

for almost every $x \in \Omega$, for every $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^N$, α and C are positive real numbers, and $\mathcal{L} \in L^{p'}(\Omega)$, with $p' = \dfrac{p}{p-1}$.

$\mathcal{B} : \Omega\times\mathbb{R} \longmapsto \mathbb{R}^N$ is a Carathéodory function such that

$$|\mathcal{B}(x,\eta)| \leq c_0(x)|\eta|^\gamma \text{ with } c_0(x) \in L^{\frac{N}{p-1}}(\Omega) \text{ and } 0 \leq \gamma \leq p-1. \tag{1.5}$$

$$f \in L^1(\Omega),\ 0 \leq s < (p-1)\left(1-\frac{p}{N}\right) \text{ and } \lambda \geq 0. \tag{1.6}$$

Numerous researchers, as highlighted in papers like [1,2,14,15], have delved into comprehending the impact of the Hardy potential on the existence and nonexistence of solutions.

Notably, in [8], authors studied both the existence and the summability of solutions for the given problem

$$\begin{cases} -\operatorname{div}(M(x)\,\nabla v) \;=\; \lambda\,\dfrac{v}{|x|^2}+f & \text{in } \Omega, \\ v \;=\; 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Here $M : \Omega \to \mathbb{R}^{N^2}$ is a bounded and measurable matrix, *i.e.*, there exist $\alpha,\beta > 0$ such that

$$\alpha\,|\xi|^2 \leq M(x)\xi\cdot\xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega,\ \forall\xi \in \mathbb{R}^N,$$

$\lambda \geq 0$, and the data f belongs to $L^m(\Omega)$ with $m > 1$. More in detail the authors, in the paper we quoted before, proved that the problem (1.7) has no weak solution when the data f belongs to $L^1(\Omega)$. Moreover, the weak solution of the problem (1.7) is unbounded even if $m > \frac{N}{2}$. Related results, for problems involving the Hardy potential, can be seen in the book [15] and [14], in a more general framework. Furthermore, for problems involving a lower order term with natural growth are covered extensively in the literature (see, for instance, [7,16] and the references therein).

M. F. Betta, O. Guibé, and A. Mercaldo studied the following nonlinear elliptic Neumann problem [5]:

$$\begin{cases} -\operatorname{div}(b(x,v,\nabla v)+\mathcal{B}(x,v)) = f & \text{in } \Omega, \\ (b(x,v,\nabla v)+\mathcal{B}(x,v))\cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Their research has compellingly demonstrated the existence of renormalized solutions to the above problem. Furthermore, their approach provides a means to prove the existence of solutions for an operator containing a zero-order term and to deduce the stability of renormalized solutions. Interested readers are encouraged to explore the details in the cited references for a more comprehensive understanding.

Moving on, let's address the Dirichlet problems brought up by the nonlinear convection term. When f is a Radon measure with bounded variation defined on Ω, T. Del Vecchio and M.R. Posteraro demonstrated, using the symmetrization method, the existence of weak solutions for a class of nonlinear and noncoercive problem involving a lower order term, whose prototype is

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v + c_0(x)|v|^{\gamma}) + d(x)|\nabla v|^{\mu} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

In this context, the functions $d(x)$ and $c_0(x)$ belong to $L^N(\Omega)$ and $L^{\frac{N}{p-1}}(\Omega)$ respectively. Moreover, in the case when $\lambda = \mu = p-1$ they supposed that $\|d\|_{L^N(\Omega)}$ or $\|c_0\|_{L^{\frac{N}{p-1}}(\Omega)}$ is small enough.

In another notable study [12], O. Guibé and A. Mercaldo studied the problem (1.9) in the general framework of Lorentz spaces. The authors successfully demonstrated the existence of renormalized solutions under the conditions $0 \leq \gamma \leq p-1$ and $0 \leq \mu \leq p-1$.

In the present paper, our main contribution is the incorporation of both the nonlinear convection and the Hardy potential terms in the same equation and to give an existence result of renormalized solutions.

Proving the existence of renormalized solutions for the nonlinear noncoercive elliptic problem (1.1) with data in $L^1(\Omega)$ presents several obstacles.

Firstly, since no assumed smallness hypothesis on $\|c_0\|_{L^{\frac{N}{p-1}}(\Omega)}$, the operator

$$v \longmapsto -\operatorname{div}\left(b(x,v,\nabla v) + \mathcal{B}(x,v)\right)$$

is not, in general, coercive. Secondly, the lack of compactness, mainly attributed to the impact of the singular term (the Hardy potential), generally obstructs the existence of a solution. This means that we will be dealing with all the difficulties previously described, at the same time. To our knowledge, the analysis of the combined effect of the convection term satisfies only a growth property and the Hardy potential has not been previously explored.

The paper is organized as follows. In Section 2, we provide some preliminaries on the definitions of the Lorentz-Marcinkiewicz space and the notion of a renormalized solution. Section 3 will be devoted to give the proof of the existence result.

2 Definitions of renormalized solutions

Before giving the definition of such a notion of solution, we need a few notations and definitions. For $k > 0$, denote by $\mathcal{T}_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k, that is

$$\mathcal{T}_k(l) = \min\{k, \max\{-k, l\}\}.$$

We recall the basic tools on the Lorentz-Marcinkiewicz space that we need in our study. For $1 < q < \infty$ and $1 < s < \infty$ the Lorentz space $L^{q,s}(\Omega)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,s}(\Omega)} = \left(\int_0^{\mathrm{meas}(\Omega)} \left[f^*(l)l^{\frac{1}{q}}\right]^s \frac{\mathrm{d}l}{l}\right)^{1/s} < +\infty.$$

Here f^* denotes the decreasing rearrangement of f, *i.e.*, the decreasing function defined by

$$f^*(l) = \inf\{s \geq 0 : \mathrm{meas}\{x \in \Omega : |f(x)| > s\} < l\} \quad t \in [0, \mathrm{meas}(\Omega)].$$

We recall also that, for any $1 \leq r < +\infty$ the Lorentz-Marcinkiewicz space $L^{r,\infty}(\Omega)$ is the set of measurable functions f on Ω such that

$$\|f\|_{L^{r,\infty}(\Omega)} = \sup_l l(\mathrm{meas}\{x \in \Omega : |f| > l\})^{\frac{1}{r}} < +\infty, \tag{2.1}$$

endowed with the norm defined by (2.1). Moreover for any p and q such that
$1 \leq q < r < p \leq +\infty$, the following chain of continuous inclusions in Lebesgue spaces holds true

$$L^p(\Omega) \subset L^{r,\infty}(\Omega) \subset L^q(\Omega) \subset L^1(\Omega). \tag{2.2}$$

For references about rearrangements see, for example, [9]. The following technical lemma will be useful in the sequel.

Lemma 2.1 *Assume that Ω is an open subset of $\mathbb{R}^N$ with finite measure and that $1 < p < N$. Let u be a measurable function satisfying $\mathcal{T}_k(v) \in W_0^{1,p}(\Omega)$, for every positive k, and such that*

$$\int_\Omega |\nabla \mathcal{T}_k(v)|^p \, \mathrm{d}x \leq Mk + L, \quad \forall k > 0,$$

where L and M are given constants. Then, there exist a constant C which depending only on N and p such that

$$\||v|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} \leq C(N,p)\left[M + (\mathrm{meas}(\Omega))^{\frac{1}{p}} L^{\frac{1}{p'}}\right], \tag{2.3}$$

and

$$\||\nabla v|^{p-1}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \leq C(N,p)\left[M + (\mathrm{meas}(\Omega))^{\frac{1}{p^*}} L^{\frac{1}{p'}}\right], \tag{2.4}$$

where $p^* = \dfrac{Np}{N-p}$.

Proof. See [12]. $\square$

Consider a measurable function $v : \Omega \to \mathbb{R}$ that is finite almost everywhere and satisfies $\mathcal{T}_k(v) \in W_0^{1,p}(\Omega)$ for all $k > 0$. According to the uniqueness result found in [4], Lemma 2.1, there exists a unique measurable function $G : \Omega \to \mathbb{R}^N$ such that:

$$\nabla \mathcal{T}_k(v) = G\chi_{\{|v| \leq k\}} \quad \text{a.e. in } \Omega, \text{ for all } k > 0. \tag{2.5}$$

We define the gradient ∇v of function v to be this function G and represent it as $\nabla v = G$. It is important to note that this definition differs from the definition of the distributional gradient. However, if $G \in \left(L^1_{\mathrm{loc}}(\Omega)\right)^N$, then $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$, and in this case, G corresponds to the distributional gradient of v.

Conversely, there are examples of functions $G \notin L^1_{\text{loc}}(\Omega)$ (resulting in the distributional gradient of G being undefined) for which the gradient ∇v is defined in the previous sense.

We are now in a position to introduce the notion of renormalized solutions.

Definition 2.1 *We say that a function $v : \Omega \to \mathbb{R}$ is a renormalized solution of Problem* (1.1) *if it satisfies the following conditions:*

$$v \text{ is measurable and finite almost everywhere in } \Omega, \tag{2.6}$$

$$\mathcal{T}_k(v) \in W_0^{1,p}(\Omega) \quad \forall k > 0, \tag{2.7}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_\Omega |\nabla \mathcal{T}_n(v)|^p \,\mathrm{d}x = 0, \tag{2.8}$$

and if, for any $h \in W^{1,\infty}(\mathbb{R})$ such that $supp(h)$ is compact we have

$$\begin{aligned} &\int_\Omega b(x,v,\nabla v)\cdot\nabla v h'(v)\varphi \,\mathrm{d}x + \int_\Omega b(x,v,\nabla v)\cdot\nabla\varphi h(v)\,\mathrm{d}x \\ &\quad + \int_\Omega \mathcal{B}(x,v)\cdot\nabla v h'(v)\varphi\,\mathrm{d}x + \int_\Omega \mathcal{B}(x,v)\cdot\nabla\varphi h(v)\,\mathrm{d}x \\ &\quad = \lambda\int_\Omega \frac{|v|^{s-1}v}{|x|^p} h(v)\varphi\,\mathrm{d}x + \int_\Omega f h(v)\varphi\,\mathrm{d}x \qquad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \end{aligned} \tag{2.9}$$

In the following we will indicate by $\mathcal{C}_i$ any positive constant which depends only on data and whose values may change from line to line.

Remark 2.1 *In* (2.9)*, every term is well-defined. It is worth noting that growth assumption* (1.5) *on the function $\mathcal{B}$, along with the conditions* (2.6)–(2.8)*, allows us to establish that any renormalized solution v satisfies*

$$\lim_{n\to+\infty} \frac{1}{n} \int_\Omega |\mathcal{B}(x,v)||\nabla \mathcal{T}_n(v)| \,\mathrm{d}x = 0. \tag{2.10}$$

To prove this, we first observe that the growth assumption (1.5) *implies that*

$$\int_\Omega |\mathcal{B}(x,v)||\nabla \mathcal{T}_n(v)|\,\mathrm{d}x \le \int_\Omega c_0(x)|\mathcal{T}_n(v))|^\gamma|\,\nabla \mathcal{T}_n(v)|\,\mathrm{d}x.$$

By employing Hölder's and Sobolev's inequalities, we can deduce that

$$\begin{aligned} \int_\Omega c_0(x)|\mathcal{T}_n(v)|^\gamma|\nabla\mathcal{T}_n(v)|\,\mathrm{d}x &\le (\mathrm{meas}(\Omega))^{\frac{p-1-\gamma}{p^*}} \|c_0\|_{L^{\frac{N}{p-1}}(\Omega)} \|\mathcal{T}_n(v)\|^\gamma_{L^{p^*}(\Omega)} \|\nabla\mathcal{T}_n(v)\|_{L^p(\Omega)} \\ &\le \mathcal{S}^\gamma (\mathrm{meas}(\Omega))^{\frac{p-1-\gamma}{p^*}} \|c_0\|_{L^{\frac{N}{p-1}}(\Omega)} \|\nabla\mathcal{T}_n(v)\|^{\gamma+1}_{L^p(\Omega)}. \end{aligned}$$

Therefore Young inequality leads to

$$\frac{1}{n}\int_\Omega c_0(x)|\mathcal{T}_n(v)|^\gamma|\nabla\mathcal{T}_n(v)|\,\mathrm{d}x \le \frac{\mathcal{C}_1}{n}\|c_0\|_{L^{\frac{N}{p-1}}(\Omega)} + \frac{\mathcal{C}_2}{n}\|\nabla\mathcal{T}_n(v)\|^p_{L^p(\Omega)}. \tag{2.11}$$

By combining (2.11) *and* (2.8)*, we can derive* (2.10)*.*

Remark 2.2 *We point out that the term*

$$\lambda \int_\Omega \frac{|v|^{s-1}v}{|x|^p} h(v)\varphi \, \mathrm{d}x$$

is well-defined. Indeed, let $n > 0$ such that $supp(h) \subset [-2n, 2n]$. We have

$$\lambda \int_\Omega \frac{|v|^{s-1}v}{|x|^p} h(v)\varphi \, \mathrm{d}x = \lambda \int_\Omega \frac{|\mathcal{T}_{2n}(v)|^{s-1}\mathcal{T}_{2n}(v)}{|x|^p} h(v)\varphi \, \mathrm{d}x.$$

By applying Hölder's and Hardy's inequalities, we can conclude that:

$$\begin{aligned}\lambda \int_\Omega \frac{|\mathcal{T}_{2n}(v)|^s}{|x|^p} |h(v)\varphi| \, \mathrm{d}x &\leq 2\lambda n \|\varphi\|_{L^\infty(\Omega)} \left(\int_\Omega \frac{|\mathcal{T}_{2n}(v)|^p}{|x|^p} \, \mathrm{d}x \right)^{\frac{s}{p}} \left(\int_\Omega \frac{\mathrm{d}x}{|x|^p} \right)^{\frac{p-s}{p}} \\ &\leq 2\lambda n \|\varphi\|_{L^\infty(\Omega)} \mathcal{H}^{\frac{s}{p}} \left(\int_\Omega |\nabla \mathcal{T}_{2n}(v)|^p \, \mathrm{d}x \right)^{\frac{s}{p}} \left(\int_\Omega \frac{\mathrm{d}x}{|x|^p} \right)^{\frac{p-s}{p}}.\end{aligned}$$

Ultimately, by (2.7) *and since $\dfrac{1}{|x|^p} \in L^1(\Omega)$, it results*

$$\frac{|v|^{s-1}v}{|x|^p} h(v)\varphi \in L^1(\Omega). \tag{2.12}$$

3 Existence

This section is devoted to establish the following existence theorem.

Theorem 3.1 *Under assumptions* (1.2) $-$ (1.6) *there exists a renormalized solution of equation* (1.1).

Proof. We split the proof into four steps.

First Step: A priori estimates

For $\varepsilon > 0$, let us define

$$\begin{aligned}\mathcal{B}_\varepsilon(x, l) &= \mathcal{B}(x, \mathcal{T}_{\frac{1}{\varepsilon}}(l)) \quad \forall l \in \mathbb{R}. \\ f^\varepsilon &= \mathcal{T}_{\frac{1}{\varepsilon}}(f).\end{aligned} \tag{3.1}$$

Let $v_\varepsilon \in W_0^{1,p}(\Omega)$ be a weak solution to the following approximate problem

$$\begin{cases} -\operatorname{div}(b(x, v_\varepsilon, \nabla v_\varepsilon) + \mathcal{B}_\varepsilon(x, v_\varepsilon)) = \lambda \dfrac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon} + f^\varepsilon & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The existence of a solution v_ε of (3.2) is a well-known result (see, e.g., [13]), namely for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ it satisfies

$$\int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon) \cdot \nabla\varphi \, \mathrm{d}x + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon) \cdot \nabla\varphi \, \mathrm{d}x = \lambda \int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon} \varphi \, \mathrm{d}x + \int_\Omega f^\varepsilon \varphi \, \mathrm{d}x. \tag{3.3}$$

Now we give the following lemma which gives some *a priori* estimates results on v_ε and ∇v_ε.

Lemma 3.1 *Assume that the assumptions* $(1.2)-(1.6)$ *hold. Then, there exist two positive constants* c_1 *and* c_2 *depend only on* $N, p, \alpha, \operatorname{meas}(\Omega), \|c_0\|_{L^{\frac{N}{p-1}}(\Omega)}$ *such that every weak solution of the problem* (3.3) *satisfies*

$$\||v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} \leq c_1, \tag{3.4}$$

$$\||\nabla v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \leq c_2. \tag{3.5}$$

Proof. We first establish an estimate of the level sets of the functions $|v_\varepsilon|$. In the second step we prove that $\mathcal{T}_k(v_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$.

Step 1: v_ε *is finite a.e. in* Ω.

Let us consider the real valued function $\psi_p : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_p(l) = \int_0^l \frac{\mathrm{d}r}{\left(\beta_\varepsilon^{\frac{1}{p-1}} + |r|\right)^p},$$

where $\beta_\varepsilon > 1$ will be suitable chosen. We use $\varphi = \psi_p(v_\varepsilon) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ in (3.3), we get

$$\begin{aligned}
&\int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla v_\varepsilon \psi_p'(v_\varepsilon)\,\mathrm{d}x + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon)\cdot\nabla v_\varepsilon\psi_p'(v_\varepsilon)\,\mathrm{d}x \\
&= \lambda \int_\Omega \frac{|T_\varepsilon(v_\varepsilon)|^{s-1}T_\varepsilon(v_\varepsilon)}{|x|^p+\varepsilon}\psi_p(v_\varepsilon)\,\mathrm{d}x + \int_\Omega f^\varepsilon\psi_p(v_\varepsilon)\,\mathrm{d}x.
\end{aligned} \tag{3.6}$$

Using (1.2), (1.3), (2.6) and Young's inequality, we get that

$$\begin{aligned}
\alpha \int_\Omega \frac{|\nabla v_\varepsilon|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |v_\varepsilon|)^p}\,\mathrm{d}x &\leq \int_\Omega c_0(x)|v_\varepsilon|^\gamma|\,\nabla v_\varepsilon||\psi'(v_\varepsilon)|\,\mathrm{d}x \\
&\quad + \frac{\lambda}{\beta_\varepsilon(p-1)}\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}\,\mathrm{d}x + \frac{\|f\|_{L^1(\Omega)}}{\beta_\varepsilon(p-1)} \\
&\leq \frac{1}{p'\alpha^{\frac{1}{p-1}}}\|c_0\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha}{p}\int_\Omega \frac{|\nabla v_\varepsilon|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |v_\varepsilon|)^p}\,\mathrm{d}x + \frac{1}{\beta_\varepsilon(p-1)}M_\varepsilon,
\end{aligned}$$

where

$$M_\varepsilon = \lambda\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s\,\mathrm{d}x}{|x|^p+\varepsilon} + \|f^\varepsilon\|_{L^1(\Omega)},$$

i.e.,

$$\int_\Omega \frac{|\nabla v_\varepsilon|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |v_\varepsilon|)^p}\,\mathrm{d}x \leq \frac{1}{\alpha^{p'}}\|c_0\|^{p'}_{L^{p'}(\Omega)} + \frac{p'}{\alpha\beta_\varepsilon(p-1)}M_\varepsilon.$$

Let us define $\beta_\varepsilon = 1 + \dfrac{p'}{\alpha(p-1)}M_\varepsilon$. As $\beta_\varepsilon > 1$, we have that

$$\begin{aligned}
\int_\Omega \frac{|\nabla v_\varepsilon|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |v_\varepsilon|)^p}\,\mathrm{d}x &\leq \frac{1}{\alpha^{p'}}\|c_0\|^{p'}_{L^{p'}(\Omega)} + \frac{\beta_\varepsilon - 1}{\beta_\varepsilon} \\
&\leq \frac{1}{\alpha^{p'}}\|c_0\|^{p'}_{L^{p'}(\Omega)} + 1.
\end{aligned}$$

Applying the Poincaré inequality, we get that

$$\int_\Omega \frac{|\nabla v_\varepsilon|^p}{(\beta_\varepsilon^{\frac{1}{p-1}} + |v_\varepsilon|)^p}\,\mathrm{d}x \geq \Lambda \int_\Omega \left[\ln\left(1 + \frac{|v_\varepsilon|}{\beta_\varepsilon^{\frac{1}{p-1}}}\right)\right]^p \mathrm{d}x.$$

Note that, for any $h > 0$, we have

$$\begin{aligned}
\operatorname{meas}\left\{|v_\varepsilon| \geq h\beta_\varepsilon^{\frac{1}{p-1}}\right\} &= \frac{1}{[\ln(1+h)]^p}\int_{\{|v_\varepsilon|\geq h\beta_\varepsilon^{\frac{1}{p-1}}\}} [\ln(1+h)]^p\;\mathrm{d}x \\
&\leq \frac{1}{[\ln(1+h)]^p}\int_{\left\{|v_\varepsilon|\geq h\beta_\varepsilon^{\frac{1}{p-1}}\right\}} \left[\ln\left(1 + \frac{|v_\varepsilon|}{\beta_\varepsilon^{\frac{1}{p-1}}}\right)\right]^p \mathrm{d}x \\
&\leq \frac{1}{[\ln(1+h)]^p}\int_\Omega \left[\ln\left(1 + \frac{|v_\varepsilon|}{\beta_\varepsilon^{\frac{1}{p-1}}}\right)\right]^p \mathrm{d}x \\
&\leq \frac{\mathcal{C}_3}{[\ln(1+h)]^p}.
\end{aligned}$$

Then, for any $\eta > 0$, we deduce that

$$\operatorname{meas}\{|v_\varepsilon| \geq \sigma\} \leq \frac{1}{\eta^p},$$

where

$$\sigma = \left(\exp(\eta\mathcal{C}_3^{1/p}) - 1\right)\beta_\varepsilon^{\frac{1}{p-1}}. \tag{3.7}$$

Step 2: $\mathcal{T}_k(v_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$.

We take $\mathcal{T}_k(v_\varepsilon)$ as test function in (3.3) we obtain

$$\begin{aligned}
&\int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla\mathcal{T}_k(v_\varepsilon)\,\mathrm{d}x + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon)\cdot\nabla\mathcal{T}_k(v_\varepsilon)\,\mathrm{d}x \\
&= \lambda\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon}\mathcal{T}_k(v_\varepsilon)\,\mathrm{d}x + \int_\Omega f^\varepsilon\mathcal{T}_k(v_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Let us notice that we require to distinguish two cases.

If $\gamma < p-1$, by (1.2), (1.5), Hölder inequality and Young inequality we get

$$\begin{aligned}
\alpha\int_\Omega |\nabla\mathcal{T}_k(v_\varepsilon)|^p\;\mathrm{d}x &\leq \int_\Omega c_0(x)|v_\varepsilon|^\gamma\,|\nabla\mathcal{T}_k(v_\varepsilon)|\,\mathrm{d}x, + k\,\lambda\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}\,\mathrm{d}x + k\|f^\varepsilon\|_{L^1(\Omega)} \\
&\leq \int_\Omega c_0(x)|v_\varepsilon|^\gamma\,|\nabla\mathcal{T}_k(v_\varepsilon)|\,\mathrm{d}x + k\,M_\varepsilon \\
&\leq \mathcal{C}_4\,\|\nabla\mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^{\gamma p'} + \frac{\alpha}{2p}\,\|\nabla\mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p + k\,M_\varepsilon \\
&\leq \frac{\mathcal{C}_4^{r'}}{r'(\frac{\alpha r}{2p})^{r'/r}} + \frac{\alpha}{p}\,\|\nabla\mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p + k\,M_\varepsilon,
\end{aligned}$$

where

$$r = \frac{p-1}{\gamma} \quad \text{and} \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Then, we obtain that

$$\|\nabla \mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p \le M_1 + M'_\varepsilon\, k \qquad \forall k > 0,$$

with

$$M_1 = \frac{p'\,\mathcal{C}_4^{r'}}{\alpha\, r'(\frac{\alpha\, r}{2p})^{r'/r}} \quad \text{and} \quad M'_\varepsilon = \frac{p'}{\alpha} M_\varepsilon.$$

From Lemma 2.1, it follows that

$$\||v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} \le C(N,p)\left[M'_\varepsilon + (\text{meas }(\Omega))^{\frac{1}{p}} M_1^{\frac{1}{p'}}\right],$$

$$\||\nabla v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \le C(N,p)\left[M'_\varepsilon + (\text{meas }(\Omega))^{\frac{1}{p^*}} M_1^{\frac{1}{p'}}\right].$$

If $\gamma = p-1$, by (1.2), (1.5), Hölder and Young inequalities, we get

$$\begin{aligned}
\alpha \int_\Omega \left|\nabla \mathcal{T}_k(v_\varepsilon)\right|^p \mathrm{d}x &\le \int_\Omega c_0(x)\,|v_\varepsilon|^{p-1}\,|\nabla \mathcal{T}_k(v_\varepsilon)|\,\mathrm{d}x + k\,M_\varepsilon \\
&\le \int_{\{|v_\varepsilon|>\sigma\}} c_0(x)\,|v_\varepsilon|^{p-1}\,|\nabla \mathcal{T}_k(v_\varepsilon)|\,\mathrm{d}x \\
&\quad + \int_{\{|v_\varepsilon|\le\sigma\}} c_0(x)\,|v_\varepsilon|^{p-1}\,|\mathcal{T}_k(v_\varepsilon)|\,\mathrm{d}x + k\,M_\varepsilon \\
&\le \mathcal{S}^{p-1}\|c_0\|_{L^{\frac{N}{p-1}}(\{|v_\varepsilon|\ge\sigma\})}\,\|\nabla \mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p + \mathcal{C}_6 \\
&\quad + \frac{\alpha}{p}\,\|\nabla \mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p + k\,M_\varepsilon\;,
\end{aligned}$$

where σ is the number defined by (3.7).

Now, we choose $\eta = \bar{\eta}$ such that

$$\text{meas}\left\{|v_\varepsilon| > (\exp(\bar{\eta}\,\mathcal{C}_3^{1/p}) - 1)\,\beta_\varepsilon^{\frac{1}{p-1}}\right\} \le \frac{1}{\bar{\eta}^p} < \tau,$$

for some $\tau > 0$, implies

$$\frac{p'}{\alpha}\mathcal{S}^{p-1}\|c_0\|_{L^{\frac{N}{p-1}}(\{|v_\varepsilon| > \exp(\bar{\eta}\,\mathcal{C}_3^{1/p}) - 1)\,\beta_\varepsilon^{\frac{1}{p-1}}\})} \le \frac{1}{2}.$$

Therefore, we get

$$\|\nabla \mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p \le M_2 + M'_\varepsilon k, \qquad \forall k > 0,$$

where

$$M_2 = \frac{p'\,\mathcal{C}_6}{\alpha}.$$

Again, thanks to Lemma 2.1, we derive the following estimates

$$\||v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} \le C(N,p)\left[M'_\varepsilon + (\text{meas }(\Omega))^{\frac{1}{p}} M_2^{\frac{1}{p'}}\right],$$

$$\||\nabla v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \leq C(N,p)\left[M'_\varepsilon + (\text{meas }(\Omega))^{\frac{1}{p^*}} M_2^{\frac{1}{p'}}\right].$$

On the other hand we have

$$M_\varepsilon \leq \lambda \int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p}\,\mathrm{d}x + \|f^\varepsilon\|_{L^1(\Omega)}. \tag{3.8}$$

By (2.2), Hölder inequality and since $s < (p-1)\left(1 - \frac{p}{N}\right)$, we get

$$\begin{aligned}
\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p}\,\mathrm{d}x &\leq \left(\int_\Omega |v_\varepsilon|^{p-1}\,\mathrm{d}x\right)^{\frac{s}{p-1}} \left(\int_\Omega \frac{\mathrm{d}x}{|x|^{\frac{p(p-1)}{p-s-1}}}\right)^{\frac{p-s-1}{p-1}} \\
&\leq \mathcal{C}_7 \||v_\varepsilon|^{p-1}\|_{L^1(\Omega)}^{\frac{s}{p-1}} \\
&\leq \mathcal{C}_8 \||v_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)}^{\frac{s}{p-1}} \\
&\leq \mathcal{C}_8\left[C(N,p)\left[M'_\varepsilon + (\text{meas }(\Omega))^{\frac{1}{p}} M_2^{\frac{1}{p'}}\right]\right]^{\frac{s}{p-1}} \\
&\leq \mathcal{C}_8 C(N,p)\left(\frac{p'}{\alpha}\right)^{\frac{s}{p-1}} M_\varepsilon^{\frac{s}{p-1}} + \mathcal{C}_8 C(N,p)(\text{meas }(\Omega))^{\frac{s}{p(p-1)}} M_2^{\frac{s}{p}} \\
&\leq \mathcal{C}_9 M_\varepsilon^{\frac{s}{p-1}} + \mathcal{C}_8 C(N,p)(\text{meas }(\Omega))^{\frac{s}{p(p-1)}} M_2^{\frac{s}{p}} \\
&\leq \mathcal{C}_{10} + \frac{1}{2\lambda} M_\varepsilon + \mathcal{C}_8 C(N,p)(\text{meas }(\Omega))^{\frac{s}{p(p-1)}} M_2^{\frac{s}{p}}.
\end{aligned}$$

Which, by (3.8), implies that

$$\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p}\,\mathrm{d}x \leq \mathcal{C}_{11}. \tag{3.9}$$

Thus, we deduce that

$$\|\nabla \mathcal{T}_k(v_\varepsilon)\|_{L^p(\Omega)}^p \leq Mk + L, \qquad \forall k > 0. \tag{3.10}$$

Finally, by (3.7) and Lemma 2.1, we get (3.4) and (3.5). ☐

Following the papers [4, 6], we deduce that there is a function $v_k \in W_0^{1,p}(\Omega)$ such that, up to a subsequence still denoted by v_ε one has

$$\mathcal{T}_k(v_\varepsilon) \rightharpoonup v_k \text{ weakly in } W_0^{1,p}(\Omega),$$

$$\mathcal{T}_k(v_\varepsilon) \to v_k \text{ strongly in } L^p(\Omega).$$

Let $\sigma > 0$ and $\eta > 0$ be fixed. For every $k > 0$, and every ε and δ we have

$$\{|v_\varepsilon - v_\delta| > \sigma\} \subseteq \{|v_\varepsilon| > k\} \cup \{|v_\delta| > k\} \cup \{|\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v_\delta)| > \sigma\}.$$

Let $\eta > 0$ be fixed, there exists $k > 0$ such that, for every ε and δ,

$$\text{meas}\left(\{|v_\varepsilon| > k\}\right) + \text{meas}\left(\{|v_\delta| > k\}\right) < \frac{\eta}{2}.$$

Once k is chosen, we deduce from (3.10) that $\mathcal{T}_k(v_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$, and so, up to a subsequence still denoted by $v_\varepsilon, \mathcal{T}_k(v_\varepsilon)$ is (strongly convergent in $L^p(\Omega)$ and hence) a Cauchy sequence in measure. Consequently, there exists ε_0 such that, for every ε and δ smaller than ε_0, we have

$$\operatorname{meas}\left(\{|\mathcal{T}_k\left(v_\varepsilon\right)-\mathcal{T}_k\left(v_\delta\right)|>\sigma\}\right)<\frac{\eta}{2}.$$

We have thus proved that up to a subsequence, v_ε is a Cauchy sequence in measure. Therefore, there exist a further subsequence, still denoted by v_ε, and a measurable function v, which is finite almost everywhere, such that

$$v_\varepsilon \to v \quad \text{a.e. in } \Omega. \tag{3.11}$$

Consequently, we have

$$\mathcal{T}_k(v_\varepsilon) \rightharpoonup \mathcal{T}_k(v) \ \text{ weakly in } W_0^{1,p}(\Omega), \tag{3.12}$$

$$\mathcal{T}_k(v_\varepsilon) \to \mathcal{T}_k(v) \ \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega, \tag{3.13}$$

$$\forall k>0, \qquad b(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v_\varepsilon)) \rightharpoonup \sigma_k \qquad \text{weakly in } (L^{p'}(\Omega))^N. \tag{3.14}$$

Second step: Energy formula

In this step, we aim to derive an energy estimate for the approximating solutions.

Using $\frac{1}{n}\mathcal{T}_n(v_\varepsilon)$ as test function in (3.3) leads to

$$\begin{aligned}&\frac{1}{n}\int_\Omega b(x,v_\varepsilon,\nabla v_\varepsilon)\cdot\nabla\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x+\frac{1}{n}\int_\Omega \mathcal{B}_\varepsilon(x,v_\varepsilon)\cdot\nabla\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x\\&=\frac{\lambda}{n}\int_\Omega\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon}\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x+\frac{1}{n}\int_\Omega f^\varepsilon\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Thanks to (1.2) and (1.5) we deduce that

$$\begin{aligned}\frac{\alpha}{n}\int_\Omega|\nabla\mathcal{T}_n(v_\varepsilon)|^p\,\mathrm{d}x&\le\frac{1}{n}\int_\Omega c_0(x)|\mathcal{T}_n(v_\varepsilon)|^\gamma|\nabla\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x\\&\quad+\frac{\lambda}{n}\int_\Omega\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x+\frac{1}{n}\int_\Omega f^\varepsilon\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

As before, we distinguish two cases.

If $\gamma<p-1$, we have, by Hölder, Sobolev and Young inequalities, that

$$\begin{aligned}&\frac{\alpha}{n}\int_\Omega|\nabla\mathcal{T}_n(v_\varepsilon)|^p\,\mathrm{d}x\\&\le\frac{1}{n}\|c_0\|_{L^{\frac{N}{p-1}}(\Omega)}\left(\int_\Omega|\mathcal{T}_n(v_\varepsilon)|^{\frac{\gamma}{p-1}p^*}\,\mathrm{d}x\right)^{\frac{p-1}{p^*}}\left(\int_\Omega|\nabla\mathcal{T}_n(v_\varepsilon)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\&+\frac{\lambda}{n}\int_\Omega\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x+\frac{1}{n}\int_\Omega f^\varepsilon\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x\\&\le\frac{\mathcal{C}_{14}}{n}+\frac{\alpha}{2n}\int_\Omega|\nabla\mathcal{T}_n(v_\varepsilon)|^p\,\mathrm{d}x+\frac{\lambda}{n}\int_\Omega\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x+\frac{1}{n}\int_\Omega f^\varepsilon\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Which implies that

$$\frac{\alpha}{2n}\int_\Omega|\nabla\mathcal{T}_n(v_\varepsilon)|^p\,\mathrm{d}x\le\frac{\mathcal{C}_{14}}{n}+\frac{\lambda}{n}\int_\Omega\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x+\frac{1}{n}\int_\Omega f^\varepsilon\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x.$$

If $\gamma = p-1$, we have, by Hölder and Sobolev inequalities, that

$$\begin{aligned}
\frac{\alpha}{n}\int_\Omega |\nabla \mathcal{T}_n(v_\varepsilon)|^p \,\mathrm{d}x &\leq \frac{1}{n}\int_{\Omega\cap\{|v_\varepsilon|\leq\sigma\}} c_0(x)|\mathcal{T}_n(v_\varepsilon)|^{p-1}|\nabla \mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x \\
&+ \frac{1}{n}\int_{\Omega\cap\{|v_\varepsilon|>\sigma\}} c_0(x)|\mathcal{T}_n(v_\varepsilon)|^{p-1}|\nabla \mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x \\
&+ \frac{\lambda}{n}\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x + \frac{1}{n}\int_\Omega f^\varepsilon \mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x \\
&\leq \frac{1}{n}\int_{\Omega\cap\{|v_\varepsilon|\leq\sigma\}} c_0(x)|\mathcal{T}_n(v_\varepsilon)|^{p-1}|\nabla \mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x \\
&+ \frac{\mathcal{S}^{p-1}}{n}\|c_0\|_{L^{\frac{N}{p-1}}(\{|v_\varepsilon|>\sigma\})}\int_\Omega |\nabla \mathcal{T}_n(v_\varepsilon)|^p\,\mathrm{d}x \\
&+ \frac{\lambda}{n}\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x + \frac{1}{n}\int_\Omega f^\varepsilon \mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x,
\end{aligned}$$

where σ is the number defined by (3.7).

Now, we choose σ such that

$$\mathcal{S}^{p-1}\|c_0\|_{L^{\frac{N}{p-1}}(\{|v_\varepsilon|>\sigma\})} < \frac{\alpha}{2}.$$

Thus, we get

$$\begin{aligned}
\frac{\alpha}{2n}\int_\Omega |\nabla \mathcal{T}_n(v_\varepsilon)|^p \,\mathrm{d}x &\leq \frac{\mathcal{C}_{15}}{n} + \frac{1}{n}\int_{\Omega\cap\{|v_\varepsilon|\leq\sigma\}} c_0(x)|\mathcal{T}_n(v_\varepsilon)|^{p-1}|\nabla \mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x \\
&+ \frac{\lambda}{n}\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x + \frac{1}{n}\int_\Omega f^\varepsilon \mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

For any $n > \sigma$, by applying Hölder and Sobolev inequalities, we derive

$$\frac{1}{n}\int_{\Omega\cap\{|v_\varepsilon|\leq\sigma\}} c_0(x)|\mathcal{T}_n(v_\varepsilon)|^{\gamma}|\nabla \mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x \leq \frac{\sigma^\gamma}{n}\|c_0\|_{L^{p'}(\Omega)}\|\nabla T_\sigma(v_\varepsilon)\|^p_{L^p(\Omega)},$$

which implies, thanks to (3.10), that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\Omega\cap\{|v_\varepsilon|\leq\sigma\}} c_0(x)|\mathcal{T}_\sigma(v_\varepsilon)|^{\gamma}|\nabla T_\sigma(v_\varepsilon)|\,\mathrm{d}x = 0. \tag{3.15}$$

By virtue of (3.11) we have that

$$\mathcal{T}_n(v_\varepsilon) \rightharpoonup \mathcal{T}_n(v), \text{ weak-* in } L^\infty(\Omega). \tag{3.16}$$

Additionally, since f^ε strongly converges to f in $L^1(\Omega)$, it follows that

$$\limsup_{\varepsilon\to 0}\frac{1}{n}\int_\Omega |f_\varepsilon|\,|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x = \frac{1}{n}\int_\Omega |f|\,|\mathcal{T}_n(v)|\,\mathrm{d}x.$$

Moreover, it is easy to prove, since v is finite a.e. in Ω, that

$$\frac{\mathcal{T}_n(v)}{n} \rightharpoonup 0 \text{ weak-* in } L^\infty(\Omega). \tag{3.17}$$

Therefore, we deduce that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_\Omega |f_\varepsilon|\,|\mathcal{T}_n(v_\varepsilon)|\,\mathrm{d}x=0. \tag{3.18}$$

Let E be a measurable subset of Ω such that

$$\forall\delta>0,\exists\eta(\delta)>0 \text{ such that meas }(E)\le\eta(\delta). \tag{3.19}$$

Observe that, by (3.4) and the Hölder inequality, and since $s<(p-1)\left(1-\frac{p}{N}\right)$, we have

$$\begin{aligned}
\int_E\frac{|v_\varepsilon|^s}{|x|^p}\,\mathrm{d}x &\le\left(\int_E|v_\varepsilon|^{p-1}\,\mathrm{d}x\right)^{\frac{s}{p-1}}\left(\int_E\frac{\mathrm{d}x}{|x|^{\frac{p(p-1)}{p-1-s}}}\right)^{\frac{p-1-s}{p-1}}\\
&\le\left(\int_\Omega|v_\varepsilon|^{p-1}\,\mathrm{d}x\right)^{\frac{s}{p-1}}\left(\int_E\frac{\mathrm{d}x}{|x|^{\frac{p(p-1)}{p-1-s}}}\right)^{\frac{p-1-s}{p-1}}\\
&\le\mathcal{C}_{16}\left(\int_E\frac{\mathrm{d}x}{|x|^{\frac{p(p-1)}{p-1-s}}}\right)^{\frac{p-1-s}{p-1}}.
\end{aligned}$$

By means of the absolute continuity of the Lebesgue integral, using (3.19), it follows

$$\left(\int_E\frac{\mathrm{d}x}{|x|^{\frac{p(p-1)}{p-1-s}}}\right)^{\frac{p-1-s}{p-1}}\le\delta.$$

Hence,

$$\int_E\frac{|v_\varepsilon|^s}{|x|^p}\,\mathrm{d}x\le\mathcal{C}_{16}\delta.$$

Consequently by Vitali's theorem, we deduce that

$$\frac{|v_\varepsilon|^s}{|x|^p}\to\frac{|v|^s}{|x|^p}\text{ strongly in }L^1(\Omega), \tag{3.20}$$

which, together with (3.16) and (3.17), yields that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{\lambda}{n}\int_\Omega\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^s}{|x|^p+\varepsilon}\mathcal{T}_n(v_\varepsilon)\,\mathrm{d}x=0,$$

and therefore, thanks to (3.15) and (3.18),

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{|v_\varepsilon|\le n\}}b(x,v_\varepsilon,\nabla v_\varepsilon)\cdot\nabla v_\varepsilon\,\mathrm{d}x=0. \tag{3.21}$$

Third step: The weak L^1-convergence of the truncated energy

In this step we prove that for any $k>0$,

$$\lim_{\varepsilon\to 0}\int_\Omega\left(b\left(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v_\varepsilon)\right)-b\left(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v)\right)\right)\cdot\nabla\left(\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v)\right)\,\mathrm{d}x=0. \tag{3.22}$$

The method which we use to prove (3.22) relies on similar techniques developed in [5]. For all $n \in \mathbb{N}$, we define the function $h_n : \mathbb{R} \longmapsto \mathbb{R}$ as:

$$h_n(l) = \begin{cases} 0, & |l| > 2n, \\ \frac{2n-|l|}{n}, & n < |l| \leq 2n, \\ 1, & |l| \leq n. \end{cases}$$

Let $k > 0$, if we take $\varphi = h_n(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))$ in (3.3) we get

$$\begin{aligned} &\int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla v_\varepsilon h_n'(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x + \int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))h_n(v_\varepsilon)\,\mathrm{d}x \\ &\quad + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon)\cdot\nabla v_\varepsilon h_n'(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon)\cdot\nabla(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))h_n(v_\varepsilon)\,\mathrm{d}x \\ &\quad = \lambda \int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon} h_n(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x + \int_\Omega f^\varepsilon h_n(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x. \end{aligned} \tag{3.23}$$

Now, we proceed by taking the limit in (3.23) first as $\varepsilon \to 0$, then as $n \to +\infty$.

Observe that, recalling the definition of the function h_n,

$$\left| \int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla v_\varepsilon h_n'(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x \right| \leq \frac{2k}{n} \int_{\Omega_{2n}} b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla v_\varepsilon\,\mathrm{d}x,$$

where

$$\Omega_{2n} = \{x \in \Omega \mid n \leq |v_\varepsilon(x)| \leq 2n\}.$$

Thus, by (3.21), we deduce that

$$\lim_{n\to+\infty} \limsup_{\varepsilon\to 0} \int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon)\cdot\nabla v_\varepsilon h_n'(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x = 0. \tag{3.24}$$

Thanks to (3.11) we obtain

$$\mathcal{B}_\varepsilon(x, v_\varepsilon)h_n(v_\varepsilon) \to \mathcal{B}(x, v)h_n(v) \text{ a.e. in } \Omega.$$

In addition, by (1.5), we have

$$|\mathcal{B}_\varepsilon(x, v_\varepsilon)h_n(v_\varepsilon)| \leq (2n)^\gamma c_0(x).$$

By Lebesgue's dominated convergence theorem, we conclude

$$\mathcal{B}_\varepsilon(x, v_\varepsilon)h_n(v_\varepsilon) \to \mathcal{B}(x, v)h_n(v) \text{ strongly in } L^{\frac{N}{p-1}}(\Omega).$$

So that, using (3.12), we deduce that

$$\lim_{n\to+\infty} \limsup_{\varepsilon\to 0} \int_\Omega \mathcal{B}_\varepsilon(x, \mathcal{T}_{2n}(v_\varepsilon))\cdot\nabla(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))h_n(v_\varepsilon)\,\mathrm{d}x = 0. \tag{3.25}$$

According to the definition of h_n, it follows that

$$\left| \int_\Omega \mathcal{B}_\varepsilon(x, \mathcal{T}_{2n}(v_\varepsilon))\cdot\nabla v_\varepsilon h_n'(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon) - \mathcal{T}_k(v))\,\mathrm{d}x \right| \leq \frac{2k}{n} \int_{\Omega_{2n}} |\mathcal{B}_\varepsilon(x, v_\varepsilon)||\nabla v_\varepsilon|\,\mathrm{d}x.$$

Thus, by (3.21) and Remark 2.1, we affirm that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_\Omega \mathcal{B}_\varepsilon(x,\mathcal{T}_{2n}(v_\varepsilon))\cdot\nabla v_\varepsilon h_n'(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v))\,\mathrm{d}x=0. \tag{3.26}$$

For $2n\le\frac{1}{\varepsilon}$ and for any $l\in\mathbb{R}$ we have

$$\frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(l)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(l)}{|x|^p+\varepsilon}h_n(l)=\frac{|\mathcal{T}_{2n}(l)|^{s-1}\mathcal{T}_{2n}(l)}{|x|^p+\varepsilon}h_n(l).$$

Moreover, due to (3.11), we have

$$\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v)\rightharpoonup 0 \text{ a.e. in } \Omega \text{ and weak-* in } L^\infty(\Omega).$$

By Vitali's theorem, since h_n is bounded by 1 and due to (3.20), we get that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\lambda\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon}h_n(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v))\,\mathrm{d}x=0. \tag{3.27}$$

Furthermore, as a consequence of (3.1), we deduce that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_\Omega f^\varepsilon h_n(v_\varepsilon)(\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v))\,\mathrm{d}x=0. \tag{3.28}$$

Then (3.24) $-$ (3.28) allow one to assure that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_\Omega b(x,v_\varepsilon,\nabla v_\varepsilon)\cdot\nabla(\mathcal{T}_k(v_\varepsilon)-\mathcal{T}_k(v))h_n(v_\varepsilon)\,\mathrm{d}x\le 0.$$

Note that, for any $n>k$, we have

$$h_n(v_\varepsilon)b\left(x,\mathcal{T}_{2n}\left(v_\varepsilon\right),\nabla\mathcal{T}_{2n}\left(v_\varepsilon\right)\right)\chi_{\{|v_\varepsilon|<k\}}=b\left(x,\mathcal{T}_k\left(v_\varepsilon\right),\nabla\mathcal{T}_k\left(v_\varepsilon\right)\right)\quad\text{a.e. in }\Omega.$$

From (3.11) and (3.14) it follows that for any $k<n$

$$\sigma_{2n}\cdot\nabla\mathcal{T}_k(v)=\sigma_k\cdot\nabla\mathcal{T}_k(v)\qquad\text{a.e. in }\Omega.$$

Therefore,

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\int_\Omega b(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v_\varepsilon))\cdot\nabla\mathcal{T}_k(v_\varepsilon)\,\mathrm{d}x\\ &\le\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_\Omega h_n b(x,\mathcal{T}_k(v_\varepsilon),\nabla\mathcal{T}_k(v_\varepsilon))\cdot\nabla\mathcal{T}_k(v_\varepsilon)\,\mathrm{d}x\\ &=\lim_{n\to+\infty}\int_\Omega h_n(v)\sigma_{2n}\cdot\nabla\mathcal{T}_k(v)\,\mathrm{d}x=\int_\Omega\sigma_k\cdot\nabla\mathcal{T}_k(v)\,\mathrm{d}x.\end{aligned} \tag{3.29}$$

Moreover, we have

$$\begin{aligned}&\int_\Omega\left(b\left(x,\mathcal{T}_k\left(v_\varepsilon\right),\nabla\mathcal{T}_k\left(v_\varepsilon\right)\right)-b\left(x,\mathcal{T}_k\left(v_\varepsilon\right),\nabla\mathcal{T}_k(v)\right)\right)\cdot\left(\nabla\mathcal{T}_k\left(v_\varepsilon\right)-\nabla\mathcal{T}_k(v)\right)\,\mathrm{d}x\\ &=\int_\Omega b\left(x,\mathcal{T}_k\left(v_\varepsilon\right),\nabla\mathcal{T}_k\left(v_\varepsilon\right)\right)\cdot\left(\nabla\mathcal{T}_k\left(v_\varepsilon\right)-\nabla\mathcal{T}_k(v)\right)\,\mathrm{d}x\\ &\quad-\int_\Omega b\left(x,\mathcal{T}_k\left(v_\varepsilon\right),\nabla\mathcal{T}_k(v)\right)\cdot\left(\nabla\mathcal{T}_k\left(v_\varepsilon\right)-\nabla\mathcal{T}_k(v)\right)\,\mathrm{d}x.\end{aligned}$$

From (1.4), it follows, for any $\varepsilon > 0$, that

$$0 \leq \int_\Omega \left(b\left(x, \mathcal{T}_k\left(v_\varepsilon\right), \nabla \mathcal{T}_k\left(v_\varepsilon\right)\right) - b\left(x, \mathcal{T}_k\left(v_\varepsilon\right), \nabla \mathcal{T}_k(v)\right)\right) \cdot \left(\nabla \mathcal{T}_k\left(v_\varepsilon\right) - \nabla \mathcal{T}_k(v)\right) \mathrm{d}x. \tag{3.30}$$

In addition, thanks to (3.11) and (1.3), we affirm that

$$b\left(x, \mathcal{T}_k\left(v_\varepsilon\right), \nabla \mathcal{T}_k(v)\right) \to b\left(x, \mathcal{T}_k(v), \nabla \mathcal{T}_k(v)\right) \text{ strongly in } (L^{p'}(\Omega))^N,$$

which implies, by (3.29) and (3.30), that the approximate solution v_ε verifies (3.22).

As a consequence of (3.22), we have

$$\lim_{\varepsilon \to 0} \int_\Omega b\left(x, \mathcal{T}_k\left(v_\varepsilon\right), \nabla \mathcal{T}_k\left(v_\varepsilon\right)\right) \cdot \nabla \mathcal{T}_k\left(v_\varepsilon\right) \mathrm{d}x = \int_\Omega \sigma_k \cdot \nabla \mathcal{T}_k(v) \, \mathrm{d}x.$$

By the monotone character of b and Minty's argument, we conclude that

$$\sigma_k = b(x, \mathcal{T}_k(v), \nabla \mathcal{T}_k(v)). \tag{3.31}$$

Using (3.12) and (3.14) leads to

$$b(x, v_\varepsilon, \nabla v_\varepsilon) \cdot \nabla \mathcal{T}_k(v_\varepsilon) \rightharpoonup b(x, v, \nabla v) \cdot \nabla \mathcal{T}_k(v) \text{ weakly in } L^1(\Omega). \tag{3.32}$$

Fourth Step: Passing to the limit

In this step we prove that u satisfies (2.6), (2.7), (2.8) and (2.9).

We first observe that, by the Fatou Lemma and (3.11), we have

$$\operatorname{meas}\left\{|v| \geq \sigma\right\} \leq \liminf_{\varepsilon \to 0} \operatorname{meas}\left\{|v_\varepsilon| \geq \sigma\right\} \leq \frac{1}{\eta^p},$$

thus, v is finite almost everywhere in Ω, that is, (2.6) holds. Moreover, thanks to the weak convergence of $\mathcal{T}_k(v_\varepsilon)$, we assert that (2.7) holds. In addition, the decay of the truncated energy (2.8) is a consequence of (3.21) and (3.32). Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let $h \in W^{1,\infty}(\mathbb{R})$ be a function with compact support contained in the interval $[-k, k]$, where $k > 0$. Taking $h(v_\varepsilon)\varphi$ as a test function in (3.3) we get

$$\begin{aligned} &\int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon) \cdot \nabla v_\varepsilon h'(v_\varepsilon) \varphi \, \mathrm{d}x + \int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon) \cdot \nabla \varphi h(v_\varepsilon) \, \mathrm{d}x \\ &\quad + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon) \cdot \nabla v_\varepsilon h'(v_\varepsilon) \varphi \, \mathrm{d}x + \int_\Omega \mathcal{B}_\varepsilon(x, v_\varepsilon) \cdot \nabla \varphi h(v_\varepsilon) \, \mathrm{d}x \\ &\quad = \lambda \int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1} \mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p + \varepsilon} h(v_\varepsilon) \varphi \, \mathrm{d}x + \int_\Omega f^\varepsilon h(v_\varepsilon) \varphi \, \mathrm{d}x. \end{aligned} \tag{3.33}$$

Thanks to (3.32) we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega b(x, v_\varepsilon, \nabla v_\varepsilon) \cdot \nabla v_\varepsilon h'(v_\varepsilon) \varphi \, \mathrm{d}x &= \lim_{\varepsilon \to 0} \int_\Omega b(x, \mathcal{T}_k(v_\varepsilon), \nabla \mathcal{T}_k(v_\varepsilon)) \cdot \nabla \mathcal{T}_k(v_\varepsilon) h'(v_\varepsilon) \varphi \, \mathrm{d}x \\ &= \int_\Omega b(x, v, \nabla v) \cdot \nabla v h'(v) \varphi \, \mathrm{d}x. \end{aligned}$$

Using (3.14), (3.31) and the fact that h is a compact support we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega b(x,v_\varepsilon,\nabla v_\varepsilon)\cdot\nabla\varphi h(v_\varepsilon)\,\mathrm{d}x=\int_\Omega b(x,v,\nabla v)\cdot\nabla\varphi h(v)\,\mathrm{d}x.$$

Due to (3.13) we have

$$\mathcal{B}_\varepsilon(x,v_\varepsilon)h(v_\varepsilon)\to\mathcal{B}(x,v)h(v)\qquad\text{a.e. in }\Omega,$$

and by the growth condition (1.5) we deduce that

$$|\mathcal{B}_\varepsilon(x,v_\varepsilon)h(v_\varepsilon)|\le k^\gamma c_0(x).$$

Therefore, by Lebesgue's convergence theorem, we conclude that

$$\mathcal{B}_\varepsilon(x,v_\varepsilon)h(v_\varepsilon)\to\mathcal{B}(x,v)h(v)\quad\text{strongly in }L^{\frac{N}{p-1}}(\Omega),$$

so that

$$\lim_{\varepsilon\to 0}\int_\Omega \mathcal{B}_\varepsilon(x,v_\varepsilon)\cdot\nabla\varphi h(v_\varepsilon)\,\mathrm{d}x=\int_\Omega \mathcal{B}(x,v)\cdot\nabla\varphi h(v)\,\mathrm{d}x.$$

Similarly, from (3.12) and the condition $p'<\frac{N}{p-1}$, we arrive at

$$\lim_{\varepsilon\to 0}\int_\Omega \mathcal{B}_\varepsilon(x,v_\varepsilon)\cdot\nabla v_\varepsilon h'(v_\varepsilon)\varphi\,\mathrm{d}x=\int_\Omega \mathcal{B}(x,v)\cdot\nabla\varphi h'(v)\varphi\,\mathrm{d}x.$$

Applying Vitali's theorem, thanks to (3.20), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{|\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)|^{s-1}\mathcal{T}_{\frac{1}{\varepsilon}}(v_\varepsilon)}{|x|^p+\varepsilon}h(v_\varepsilon)\varphi\,\mathrm{d}x=\lambda\int_\Omega\frac{|v|^{s-1}v}{|x|^p}h(v)\varphi\,\mathrm{d}x.$$

From (3.1) and (3.11) it follows that

$$\lim_{\varepsilon\to 0}\int_\Omega f^\varepsilon h(v_\varepsilon)\varphi\,\mathrm{d}x=\int_\Omega fh(v)\varphi\,\mathrm{d}x.$$

Therefore, by passing to the limit in (3.24), we ensure that the limit v satisfies (2.9) in the definition of renormalized solutions. Thus, we conclude that v is a renormalized solution to (1.1). □

Declarations

Conflict of interest: The authors declare no conflicts of interest.

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