Journal of Nonlinear Evolution Equations and Applications
Volume 2024, Number 5, pp. 67–80 (June 2024)

ISSN 2161-3680
http://www.jneea.com

(ω, c)–ASYMPTOTICALLY PERIODIC MILD SOLUTIONS TO SEMILINEAR TWO TERMS FRACTIONAL DIFFERENTIAL EQUATIONS

PIHIRE VINCENT OUENA *
Université Joseph KI-ZERBO, Département de Mathématiques, 03 BP 7021
Ouagadougou 03, Burkina Faso

MOUMINI KÉRÉ †
Ecole Normale Supérieure, Institut des Sciences et de Technologie, 01 BP 1757
Ouagadougou 01, Burkina Faso

Received April 27, 2024,

Accepted May 14, 2024

Communicated by Gaston M. N'Guérékata

Abstract. In this article, we first explore new properties of (ω, c)–asymptotically periodic functions. Then using the Banach fixed point principle and the Leray–Schauder alternative theorem, we prove the existence and uniqueness of (ω, c)–asymptotically periodic mild solutions to the abstract semilinear fractional differential equation of the form:

$$\begin{aligned} D_t^{\alpha} u(t) &= Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0, \\ u(0) &= u_0, \end{aligned}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}, u_0 \in \mathbb{X},\ f : \mathbb{R}_+ \times \mathbb{X} \to \mathbb{X}$ is (ω, c)–asymptotically periodic in $t \in \mathbb{R}_+$ and $D_t^{\alpha}(\cdot)(1 < \alpha < 2)$ is the Riemann-Liouville fractional derivative.

Keywords: (ω, c)–asymptotically periodic, mild solution, fractional differential equation, operator of sectorial type.

2010 Mathematics Subject Classification: 34A08, 47D60, 34C25.

---

*e-mail address: ouepiv@gmail.com
†e-mail address: moumik3000@gmail.com (corresponding author)

1 Introduction

The concept of (ω, c)–periodic functions was introduced by E. Alvaraz *et al.* [3], as a variant of Bloch periodicity [9], which contains among others the class of periodic and antiperiodic functions. This type of functions was motivated by the Mathieu's equation

$$y''(t) + [a - 2q\cos(2t)]y(t) = 0$$

arising in seasonally forced population dynamics modeling. In recent years, several authors have been interested in periodic functions and their applications. In 2019, J. R. Wang, L. Ren and Y. Zhou [17] studied (ω, c)–periodic solutions for time varying impulsive differential equations. In 2020 G. Mophou and G. M. N'Guérékata [15] studied an existence result of (ω, c)–periodic mild solutions to some fractional differential equations. In 2021 M. Kéré, G. M. N'Guérékata and E. R. Oueama [11] studied an existence result of (ω, c)–almost periodic mild solutions to some fractional differential equations. For more results of periodic functions, see [3, 10, 13, 1, 2, 14, 12, 6].

Recently in 2019, E. Alvarez, S. Castillo and M. Pinto [4] extended the concept to the one of (ω, c)–asymptotically periodic functions, see also [9]. A continuous function f is said to be (ω, c)–asymptotically periodic if it can be written as $f = g + h$ where g is a (ω, c)–periodic function and h is c–asymptotic. This new concept attracted authors like J. Larrouy and G. M. N'Guérékata [13]. They did an excellent work on (ω, c)–periodic and asymptotically (ω, c)–periodic mild solutions to fractional Cauchy problems. In their paper, they established some new properties of (ω, c)–periodic and asymptotically (ω, c)–periodic functions and studied the existence and uniqueness of mild solutions of these types to semilinear fractional differential equations.

In 2013, J. Q. Zhao, Y. K. Chang and G. M. N'Guérékata studied the asymptotic behavior of mild solutions to semilinear fractional differential equations (3.1)–(3.2) in Banach space [18].
In our paper, we consider the following equations:

$$\begin{aligned} D_t^{\alpha}u(t) &= Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0, \\ u(0) &= u_0, \end{aligned}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}, u_0 \in \mathbb{X}$, $f : \mathbb{R}_+ \times \mathbb{X}, \mathbb{X}$ is asymptotically (ω, c)–periodic in $t \in \mathbb{R}_+$ and $D_t^{\alpha}(\cdot)$ $(1 < \alpha < 2)$ stands for the Riemann–Liouville fractional derivative.

The main purpose is to study the existence and uniqueness of (ω, c)–asymptotically periodic mild solution. For this, we use two tools, namely the Banach fixed point principle and the well-known alternative theorem of Leray–Schauder. Theorems 3.2 and 3.3 are our main results.

2 Preliminaries

In what follows, we assume that $(\mathbb{X}, \|.\|)$ and $(\mathbb{Y}, \|.\|)$ are two complex Banach spaces, and we will denote by $C(\mathbb{R}, \mathbb{X})$ the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$, and $BC(\mathbb{R}, \mathbb{X})$ the collection of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$. The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by $\|f\|_{\infty} := \sup_{t\in\mathbb{R}} \|f(t)\|$ is a Banach space. The notation $B(\mathbb{X}, \mathbb{Y})$ stands for the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ endowed with the uniform operator topology and we abbreviate to $B(\mathbb{X})$ whenever $\mathbb{X} = \mathbb{Y}$.

Now we recall some definitions, properties about sectorial linear operators and their associated solution operators and some notions of (ω, c)–asymptotically periodic functions.

2.1 Sectorial linear operators and their associated solution operator

A closed and linear operator A is said to be sectorial if there exist $0 < \theta < \frac{\pi}{2}$, $M > 0$ and $\widetilde{\omega} \in \mathbb{R}$ such that its resolvent exists outside the sector $\widetilde{\omega} + S_\theta := \{\widetilde{\omega} + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \widetilde{\omega}\}$ and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \widetilde{\omega}|}, \quad \lambda \notin \widetilde{\omega} + S_\theta.$$

Definition 2.1 *([18]): Let $1 < \alpha < 2$ and A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$. The operator A is called a generator of a solution operator if there exist $\widetilde{\omega} \in \mathbb{R}$ and a strongly continuous function $E_\alpha : \mathbb{R}_+ \to B(\mathbb{X})$ such that $\{\lambda^\alpha : Re\lambda > \widetilde{\omega}\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} E_\alpha(t)x \, \mathrm{d}t,\ Re\lambda > \widetilde{\omega},\ x \in \mathbb{X}$. In this case, $E_\alpha(t)$ is called the solution operator generated by A.*
We note that, if A is sectorial of type $\widetilde{\omega} \in \mathbb{R}$ with $0 \leq \theta \leq \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$E_\alpha(t) := \frac{1}{2\pi i} \int_\phi e^{t\lambda} (\lambda^\alpha - A)^{-1} \lambda^{\alpha-1} \, \mathrm{d}\lambda$$

where ϕ is a suitable path lying outside the sector $\widetilde{\omega} + S_\theta$.

Lemma 2.2 *([5, 18]): Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator in a complex Banach space $\mathbb{X}$ satisfying $\widetilde{\omega} + S_\theta := \{\widetilde{\omega} + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ and*

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \widetilde{\omega}|}, \quad \lambda \notin \widetilde{\omega} + S_\theta$$

for some $M > 0$, $\widetilde{\omega} < 0$ and $0 < \theta < \pi(1 - \frac{\alpha}{2}) < \frac{\pi}{2}$.
Then, there exists $C > 0$ such that

$$\|E_\alpha(t)\|_{\mathbf{B}(X)} \leq \frac{CM}{1 + |\widetilde{\omega}| t^\alpha}, \quad t \geq 0\,.$$

Definition 2.3 *([16]): The derivative of order α of a function $u : \mathbb{R}_+ \to \mathbb{R}$ in the sense of Riemann–Liouville is defined as*

$$D_t^\alpha u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} u(s) \, \mathrm{d}s, \quad n - 1 < \alpha < n\,, n \in \mathbb{N}.$$

If $1 < n < 2$, then

$$D_t^\alpha u(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t \frac{(t-s)^{2-\alpha-1}}{\Gamma(2-\alpha)} u(s) \, \mathrm{d}s.$$

Definition 2.4 *([16]): The integral of order α of a function $u : \mathbb{R}_+ \to \mathbb{R}$ in the sense of Riemann-Liouville is defined as*

$$I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, \mathrm{d}s.$$

Remark 2.5 *([16]): If $\alpha \in \mathbb{R}_*^+$, then the integral of order α of u is also considered as the derivative of order $-\alpha$ in the sense of Riemann-Liouville :*

$$I_t^{\alpha} u(t) = D_t^{-\alpha} u(t).$$

2.2 (ω, c)-asymptotically periodic functions

Definition 2.6 *([3]): A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)–periodic if there exist $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$ such that*

$$f(t+\omega) = cf(t), \ \forall t \in \mathbb{R}.$$

ω is called the cperiod of f.
We denote by $P_{\omega,c}(\mathbb{X})$ the collection of all functions $f \in C(\mathbb{R}, \mathbb{X})$ which are (ω, c)–periodic.
When $c = 1$, we write $P_{\omega}(\mathbb{X})$ instead $P_{\omega,1}(\mathbb{X})$ and we say that f is ω–periodic.
We define $c^{t/\omega} := \exp\big((t/\omega)\log(c)\big)$ and we will use the notation $c^\wedge(t) := c^{t/\omega}$ and $|c|^\wedge(t) := |c|^{t/\omega}$.

The following proposition provides a characterization of (ω, c)–periodic functions.

Proposition 2.7 *([3]): Let $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)–periodic if and only if:*

$$f(t) = c^\wedge(t) u(t), \ u(t) \in P_{\omega}(\mathbb{X}).$$

Lemma 2.8 *([13]): Let $u \in P_{\omega,c}(\mathbb{R}, \mathbb{X})$. Then $u \in P_{-\omega,c^{-1}}(\mathbb{R}, \mathbb{X})$.*

We will prove the following

Theorem 2.9 *Let $u \in P_{\omega,c}(\mathbb{R}, \mathbb{X})$. Then $D_t^{\alpha} u(t) \in P_{\omega,c}(\mathbb{R}, \mathbb{X})$ if*

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \int_0^{\omega} (t+\omega-\tau)^{n-\alpha-1} u(\tau)\, \mathrm{d}\tau \in \mathbb{R}\,,$$

where $D_t^{\alpha}(\cdot)$ is the Riemann–Liouville fractional derivative.

Proof. We have

$$D_t^{\alpha} u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} u(s)\ \mathrm{d}s, \quad n-1 < \alpha < n\,,$$

therefore

$$D_t^{\alpha} u(t+\omega) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^{t+\omega} (t+\omega-s)^{n-\alpha-1} u(s)\ \mathrm{d}s.$$

Let us use $\xi = s - \omega$. With this notation,

$$\begin{aligned} D_t^\alpha u(t+\omega) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\omega}^{t} (t-\xi)^{n-\alpha-1} u(\xi+\omega)\ \mathrm{d}\xi \\ &= \frac{c}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\omega}^{t} (t-\xi)^{n-\alpha-1} u(\xi)\ \mathrm{d}\xi \\ &= \frac{c}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{0}^{t} (t-\xi)^{n-\alpha-1} u(\xi)\ \mathrm{d}\xi \\ &\quad + \frac{c}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\omega}^{0} (t-\xi)^{n-\alpha-1} u(\xi)\ \mathrm{d}\xi \\ &= c\, D_t^\alpha u(t) + \frac{c}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\omega}^{0} (t-\xi)^{n-\alpha-1} u(\xi)\ \mathrm{d}\xi. \end{aligned}$$

After a change of variable $\tau = \xi + \omega$, we obtain

$$D_t^\alpha u(t+\omega) = c\, D_t^\alpha u(t) + \frac{c}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{0}^{\omega} (t+\omega-\tau)^{n-\alpha-1} u(\tau-\omega)\ \mathrm{d}\tau.$$

Using Lemma 2.8, we have

$$D_t^\alpha u(t+\omega) = c\, D_t^\alpha u(t) + \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{0}^{\omega} (t+\omega-\tau)^{n-\alpha-1} u(\tau)\ \mathrm{d}\tau.$$

Since

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \int_{0}^{\omega} (t+\omega-\tau)^{n-\alpha-1} u(\tau)\, \mathrm{d}\tau \in \mathbb{R}\ ,$$

then

$$\frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{0}^{\omega} (t+\omega-\tau)^{n-\alpha-1} u(\tau)\ \mathrm{d}\tau = 0$$

therefore

$$D_t^\alpha u(t+\omega) = c\, D_t^\alpha u(t).$$

Finally, $D_t^\alpha u(t) \in P_{\omega,c}(\mathbb{R},\mathbb{X})$. □

Lemma 2.10 *Let* $u \in P_{\omega,c}(\mathbb{R},\mathbb{X})$. *Then* $I_t^\alpha u(t) \in P_{\omega,c}(\mathbb{R},\mathbb{X})$ *if*

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \int_{0}^{\omega} (t+\omega-\tau)^{n+\alpha-1} u(\tau)\, \mathrm{d}\tau \ \in\ \mathbb{R}$$

where $I_t^\alpha(\cdot)$ *is the Riemann–Liouville fractional integral.*

Proof. We know that $I_t^\alpha u(t) = D_t^{-\alpha} u(t)$, therefore

$$\begin{aligned} I_t^\alpha u(t+\omega) &= D_t^{-\alpha} u(t+\omega) \\ &= c\, D_t^{-\alpha} u(t) + \frac{1}{\Gamma(n+\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{0}^{\omega} (t+\omega-\tau)^{n+\alpha-1} u(\tau)\, \mathrm{d}\tau. \end{aligned}$$

Since

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \int_{0}^{\omega} (t+\omega-\tau)^{n+\alpha-1} u(\tau)\, \mathrm{d}\tau \in \mathbb{R},$$

we have
$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^\omega (t+\omega-\tau)^{n+\alpha-1}u(\tau)\,\mathrm{d}\tau = 0.$$

Therefore

$$\begin{aligned} I_t^\alpha u(t+\omega) &= c\,D_t^{-\alpha}u(t) \\ &= c\,I_t^\alpha u(t). \end{aligned}$$

Consequently, $I_t^\alpha u(t) \in P_{\omega,c}(\mathbb{R}, X)$. □

We define the following spaces of functions vanishing at infinity:

$$\begin{aligned} C_0(\mathbb{X}) &:= \left\{h \in C(\mathbb{R}_+, \mathbb{X}) : \lim_{t\to\infty} h(t) = 0\right\} \quad \text{and} \\ C_0(\Omega, \mathbb{X}) &:= \left\{h \in C(\mathbb{R}_+ \times \Omega, \mathbb{X}) : \lim_{t\to\infty} h(t,x) = 0 \text{ for all } x \text{ in any compact subset of } \Omega\right\}. \end{aligned}$$

Definition 2.11 *([4]): A function $h \in C(\mathbb{R}, \mathbb{X})$ is called c–asymptotic if $c^\wedge(-t)h(t) \in C_0(\mathbb{X})$,* i.e.*,*

$$\lim_{t\to\infty} c^\wedge(-t)h(t) = 0.$$

The collection of these functions will be denoted by $C_{0,c}(\mathbb{X})$.

Definition 2.12 *([4]): A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)–asymptotically periodic if $f = g + h$ where $g \in P_{\omega,c}(\mathbb{R}, \mathbb{X})$ and $h \in C_{0,c}(\mathbb{X})$. The collection of these functions (with the same c–period ω for the first component) will be denoted by $AP_{\omega,c}(\mathbb{X})$.*

The following proposition provides a characterization of asymptotically (ω, c)–periodic functions.

Proposition 2.13 *([4]): Let $f \in C(\mathbb{R}, \mathbb{X})$. Then f is (ω, c)–asymptotically periodic if and only if:*

$$f(t) := c^\wedge(t)u(t)$$

where $u \in AP_\omega(\mathbb{X})$.

Lemma 2.14 *([4]): Let $\alpha \in \mathbb{C}$. Then*

1. *$(f + g) \in AP_{\omega,c}(\mathbb{X})$ and $\alpha h \in AP_{\omega,c}(\mathbb{X})$ whenever $f, g, h \in AP_{\omega,c}(\mathbb{X})$*
2. *If $\tau \geq 0$ is constant, then $f_\tau(t) = f(t+\tau) \in AP_{\omega,c}(\mathbb{X})$ whenever $f \in AP_{\omega,c}(\mathbb{X})$*
3. *Let $g \in P_{\omega,c}(\mathbb{X})$ and $h \in C_{0,c}(\mathbb{X})$ such that $g, h \in C^1(\mathbb{R}, \mathbb{X})$. Then the derivative of $(f = g + h) \in AP_{\omega,c}(\mathbb{X})$ belongs to $AP_{\omega,c}(\mathbb{X})$.*

Theorem 2.15 *([4]): Let $f(t,x) := g(t,x) + h(t,x)$ where $g(t+\omega, cx) = cg(t,x)$ and $h \in C_{0,c}(\mathbb{X}, \mathbb{X})$. Let us assume the following conditions:*

(1) $h_t(z) = c^\wedge(-t)h(c^\wedge(t)z)$ *is uniformly continuous for* z *in any bounded subset of* $\mathbb{X}$ *uniformly for* $t \geq d$ *and* $h_t(z) \to 0$ *as* $t \to \infty$ *uniformly in* z.

(2) There exists $v \in BC(\mathbb{R}_+, \mathbb{R}_+)$ *such that*

$$\|f(t, u_1) - f(t, u_2)\| \leq v(t)\|u_1 - u_2\|, \ \forall\, u_1,\ u_2 \in \mathbb{X},\ t \in \mathbb{R}_+.$$

If $u \in AP_{\omega,c}(\mathbb{X})$, *then* $f(., u(.)) \in AP_{\omega,c}(\mathbb{X})$.

Theorem 2.16 *[4]):* $AP_{\omega,c}([d, \infty) \times \mathbb{X}, \mathbb{X})$ *is a Banach space with the norm*

$$\|f\|_{a\omega,c} := \sup_{t \geq d} \||c|^\wedge(-t)f(t)\|.$$

In the sequel, we will consider $t \in \mathbb{R}_+$. Hence we will use $\|f\|_{a\omega,c}$ as

$$\|f\|_{a\omega,c} := \sup_{t \geq 0} \||c|^\wedge(-t)f(t)\|.$$

Proposition 2.17 *([13]): Let* $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ *be a sectorial operator of type* $\widetilde{\omega} < 0$ *and* θ *in a complex Banach space* $\mathbb{X}$. *If* $|c| \geq 1$, *then there exists* $C > 0$ *depending solely on* α *and* θ *such that:*

$$\|E_\alpha(t)\|_{a\omega,c} \leq \frac{CM}{1 + |\widetilde{\omega}|t^\alpha},\ t\ \geq\ 0.$$

Theorem 2.18 *([13, 7]): Assume that* A *is sectorial of type* $\widetilde{\omega} < 0$. *If* $f : \mathbb{R}_+ \to \mathbb{X}$ *is an* (ω, c)*–asymptotically periodic function, then the function*

$$F(t) := \int_0^t E_\alpha(t - \xi)f(\xi)\, \mathrm{d}\xi$$

belongs to $AP_{\omega,c}(\mathbb{X})$.

In what follows, we will need the following results which you can refer to [13]:
Let $h : \mathbb{R}_+ \to [1; \infty)$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}_+$ and $h(t) \to \infty$ as $t \to \infty$. Initially we set $C_h(\mathbb{R}_+, \mathbb{X})$ for the space consisting of continuous function $u : \mathbb{R}_+ \to \mathbb{X}$ such that $\|u\|_h = \sup_{t \in \mathbb{R}_+} \frac{\|u\|_{a\omega,c}}{h(t)}$, endowed with the norm $\|u\|_h = \sup_{t \in \mathbb{R}_+} \frac{\|u\|_{a\omega,c}}{h(t)}$. It turns out to be a Banach space. We also denote

$$C_h^0(\mathbb{R}_+, \mathbb{X}) := \Big\{u \in C_h(\mathbb{R}_+, \mathbb{X}) : \lim_{t \to +\infty} \frac{\|u\|_{a\omega,c}}{h(t)} = 0\Big\}.$$

Lemma 2.19 *([18]): A subset* $R \subseteq C_h^0(\mathbb{R}_+, \mathbb{X})$ *is a relatively compact set if it verifies the following conditions:*

(1) The set $R_b = \{u_{[0,b]} : u \in R\}$ *is relatively compact in* $C([0, b], \mathbb{X}), \forall b \in \mathbb{R}_+$.

(2) $\lim_{t \to +\infty} \frac{\|u\|_{a\omega,c}}{h(t)} = 0$, *uniformly for* $u \in R$.

3 Main results

In this section, we mainly deal with the existence of (ω, c)–asymptotically periodic mild solutions to the following semilinear fractional differential equations:

$$D_t^{\alpha} u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0, \tag{3.1}$$

$$u(0) = u_0\,, \tag{3.2}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type $\widetilde{\omega} < 0$ on a complex Banach space $\mathbb{X}, u_0 \in \mathbb{X}$, $f \in AP_{\omega,c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$ such that $f(t, x) := g(t, x) + h(t, x)$ where $g \in P_{\omega,c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$ and $h \in C_{0,c}(\mathbb{X})$. $D_t^{\alpha}(\cdot)$ $(1 < \alpha < 2)$ stands for the Riemann–Liouville fractional derivative.

Definition 3.1 *([18]): Assume that A generates an integral solution operator E_α. A continuous function $u : \mathbb{R}_+ \to \mathbb{X}$ satisfying the integral equation:*

$$u(t) = E_\alpha(t)u_0 + \int_0^t E_\alpha(t-s) f(s, u(s))\ \mathrm{d}s, \quad t \geq 0,$$

is called a mild solution on $\mathbb{R}_+$ to equations (3.1)–(3.2)

In the sequel, we assume that:

(H1) $h_\tau(z) = c^\wedge(-t)h(c^\wedge(t)z)$ is uniformly continuous for z in any bounded subset of $\mathbb{X}$ uniformly for $t \geq d$ and $h_\tau(z) \to 0$ as $t \to \infty$ uniformly in z.

(H2) $f \in AP_{\omega,c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$ and there exists a constant $L_f > 0$ such that:

$$\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|,\ for\ all\ t \in \mathbb{R}_+,\ x, y\ \in \mathbb{X}.$$

(H3) $(E_\alpha(t))_{t \geq 0} \subset B(\mathbb{X})$ is a strongly continuous family of linear operators.

Theorem 3.2 *Under assumptions (H1)–(H3), if we assume that $|c| \geq 1$, then there exists a unique (ω, c)– asymptotically periodic mild solution to equations* (3.1)–(3.2)*, provided that there is a constant $\gamma := \frac{CM(\frac{\pi}{\alpha})}{|\widetilde{\omega}|^{\frac{1}{\alpha}} \sin(\frac{\pi}{\alpha})} L_f < 1$.*

Proof. Consider the operator $\Lambda : AP_{\omega,c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X}) \to AP_{\omega,c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$ such that

$$(\Lambda u)(t) := E_\alpha(t)u_0 + \int_0^t E_\alpha(t-s) f(s, u(s))\ \mathrm{d}s, \quad t \geq 0. \tag{3.3}$$

According to Theorem 2.15, we clearly have that $f(t, u(t)) \in AP_{\omega,c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$. Moreover, by Theorem 2.18, we have $\int_0^t E_\alpha(t-s) f(s, u(s))\ \mathrm{d}s \in AP_{\omega,c}(\mathbb{X})$. Now, from Lemma 2.2 we note that

$$\||c|^\wedge(-t) E_\alpha(t) u_0\| \leq \frac{CM|c|^{-t/\omega}}{1 + |\widetilde{\omega}| t^\alpha} \|u_0\|. \tag{3.4}$$

Since

$$\lim_{t\to\infty} \frac{CM|c|^{-t/\omega}}{1+|\widetilde{\omega}|t^{\alpha}}\|u_0\| = 0,$$

then $E_\alpha(t)u_0 \in C_{0,c}(\mathbb{X}) \subseteq AP_{\omega,c}(\mathbb{X})$ and hence $E_\alpha(t)u_0 \in AP_{\omega,c}(\mathbb{X})$. Therefore, by Lemma 2.14, we have $E_\alpha(t)u_0 + \int_0^t E_\alpha(t-s)f(s,u(s))\,\mathrm{d}s \in AP_{\omega,c}(\mathbb{X})$. Finally Λ is well defined.
Let $u, v \in AP_{\omega,c}(\mathbb{X})$. Then

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\|_{a\omega,c} &= \sup_{t\geq 0}\Big\{\||c|^{-t/\omega}\int_0^t E_\alpha(t-s)\Big[f(s,u(s)) - f(s,v(s))\Big]\,\mathrm{d}s\|\Big\} \\
&= \sup_{t\geq 0}\Big\{\|\int_0^t |c|^{-(t-s)/\omega}E_\alpha(t-s)|c|^{-s/\omega}\Big[f(s,u(s)) - f(s,v(s))\Big]\,\mathrm{d}s\|\Big\} \\
&\leq L_f \sup_{t\geq 0}\Big\{\int_0^t |c|^{-(t-s)/\omega}\|E_\alpha(t-s)\|_{B(\mathbb{X})} \times \||c|^{-s/\omega}(u(s)-v(s))\|\,\mathrm{d}s\Big\} \\
&\leq CML_f \sup_{t\geq 0}\Big\{\int_0^t |c|^{-(t-s)/\omega}\frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s\Big\}\|u-v\|_{a\omega,c}.
\end{aligned}$$

On the one hand, if $|c| \geq 1$ and $0 \leq s \leq t$, then $|c|^{-(t-s)/\omega} \leq 1$.
On the other hand, according to [7], we get

$$\int_0^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s = \frac{\frac{\pi}{\alpha}}{|\widetilde{\omega}|^{\frac{1}{\alpha}}\sin(\frac{\pi}{\alpha})}.$$

Hence

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\|_{a\omega,c} &\leq \frac{CM(\frac{\pi}{\alpha})}{|\widetilde{\omega}|^{\frac{1}{\alpha}}\sin(\frac{\pi}{\alpha})}L_f\|u-v\|_{a\omega,c} \\
&\leq \gamma\|u-v\|_{a\omega,c}
\end{aligned}$$

Therefore when $\gamma < 1$, we deduce by the Banach contraction principle that Λ has a unique mild solution $u \in AP_{\omega,c}(\mathbb{X})$. Finally, Problem (3.1)–(3.2) has a unique (ω,c)–asymptotically periodic mild solution. □

Let us formulate the following assumptions:

(H4) $f(t,u)$ is uniformly continuous on any bounded subset $\Omega \in \mathbb{X}$ uniformly in $t \in \mathbb{R}_+$ and for every bounded subset $\Omega \in \mathbb{X}$, $\{f(.,u) : u \in \Omega\}$ is bounded in $AP_{\omega,c}(\Omega,\mathbb{X})$.

(H5) There exists a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ and $u \in \mathbb{X}$, $\|f(t,u)\|_{a\omega,c} \leq \psi(\|u\|_{a\omega,c})$.

(H6) f is satisfying Theorem 2.15.

Now we establish an existence theorem of (ω,c)–asymptotically periodic mild solution to equations (3.1)–(3.2) without a Lipschitz condition.

Theorem 3.3 *Assume that $f \in AP_{\omega,c}(\mathbb{R}_+ \times \mathbb{X},\mathbb{X})$ with $|c| \geq 1$, satisfying (H3)–(H6), and the following additional conditions:*

(1) For each $r > 0$*,*

$$\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(rh(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\ \mathrm{d}s\Big\} < \infty$$

that is

$$\lim_{t\to\infty}\frac{1}{h(t)}\Big(\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(rh(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\ \mathrm{d}s\Big\}\Big) = 0,$$

where h *is the function given in lemma 2.19 and we set*

$$\varrho(r) = CM\Big\|\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(rh(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\ \mathrm{d}s\Big\}\Big\|_h.$$

(2) For each $\varepsilon > 0$*, there is* $\delta > 0$ *such that for every* $u, v \in C_h^0(\mathbb{X})$, $\|u-v\|_h < \delta$ *implies that for all* $t \in \mathbb{R}_+$*,*

$$\sup_{t\geq 0}\Big\{\int_0^t \frac{\|f(s,u(s)) - f(s,v(s))\|_{a\omega,c}}{1+|\widetilde{\omega}|(t-s)^\alpha}\ \mathrm{d}s\Big\} \leq \frac{\varepsilon}{CM},$$

(3) For each $\alpha, \beta \in \mathbb{R}_+$ *and* $r > 0$*, the set* $\Big\{f(s, h(s)u) : \alpha \leq s \leq \beta,\ u \in C_h^0(\mathbb{X}),\ \|u\|_h \leq r\Big\}$ *is relatively compact in* $\mathbb{X}$*.*

(4) $\lim_{\xi\to\infty} \frac{\xi}{\varrho(\xi)} > 1$.

Then, equations (3.1)–(3.2) *admit one mild solution in* $AP_{\omega,c}(\mathbb{X})$*.*

Proof. We define the nonlinear operator $\Gamma : C_h^0(\mathbb{X}) \to C_h^0(\mathbb{X})$ by

$$(\Gamma u)(t) := E_\alpha(t)u_0 + \int_0^t E_\alpha(t-s) f(s,u(s))\ \mathrm{d}s\ t \geq 0.$$

We will show that Γ has a fixed point in $AP_{\omega,c}(\Omega, \mathbb{X})$ by the following steps:

(1) For $u \in C_h^0(\mathbb{X})$, we have $\|u\|_h < \infty$ and

$$\begin{aligned}
\frac{\|\Gamma u\|_{a\omega,c}}{h(t)} &= \frac{1}{h(t)}\sup_{t\geq 0}\Big\{\||c|^{-t/\omega}E_\alpha(t)u_0 + |c|^{-t/\omega}\int_0^t E_\alpha(t-s)f(s,u(s))\ \mathrm{d}s\|\Big\} \\
&\leq \frac{1}{h(t)}\Big[\|E_\alpha(t)\|_{a\omega,c}\|u_0\| + \sup_{t\geq 0}\Big\{\int_0^t \|E_\alpha(t-s)\|_{B(\mathbb{X})} \times \|f(s,u(s))\|\ \mathrm{d}s\Big\}\Big] \\
&\leq \frac{1}{h(t)}\Big[\frac{CM\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{CM}{1+|\widetilde{\omega}|(t-s)^\alpha}\psi(\|u\|_{a\omega,c})\ \mathrm{d}s\Big\}\Big]
\end{aligned}$$

Since

$$\|u\|_h = \sup_{t\geq 0}\frac{\|u\|_{a\omega,c}}{h(t)}$$

then

$$\frac{\|\Gamma u\|_{a\omega,c}}{h(t)} \leq \frac{1}{h(t)}CM\Big[\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(\|u\|_h.h(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\ \mathrm{d}s\Big\}\Big]$$

It follows from condition 1. that Γ is well defined.

(2) For each $\varepsilon > 0$, there is $\delta > 0$ such that for every $u, v \in C_h^0(\mathbb{X})$, $\|u - v\|_h < \delta$, we have

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\|_{a\omega,c} &= \|\int_0^t E_\alpha(t-s)(f(s,u(s)) - f(s,v(s)))\,\mathrm{d}s\|_{a\omega,c} \\
&\leq \sup_{t\geq 0}\Big\{\int_0^t \||c|^{-(t-s)/\omega}E_\alpha(t-s)\| \cdot \||c|^{-s/\omega}(f(s,u(s)) - f(s,v(s))\|\,\mathrm{d}s\Big\} \\
&\leq \sup_{t\geq 0}\Big\{\int_0^t \|E_\alpha(t-s)\|_{a\omega,c} \cdot \|f(s,u(s)) - f(s,v(s))\|_{a\omega,c}\,\mathrm{d}s\Big\} \\
&\leq CM\sup_{t\geq 0}\Big\{\int_0^t \frac{\|f(s,u(s)) - f(s,v(s))\|_{a\omega,c}}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s\Big\}
\end{aligned}$$

Using condition 2., we get

$$\|\Gamma u(t) - \Gamma v(t)\|_{a\omega,c} \leq \varepsilon,$$

which shows that Γ is continuous.

(3) Next we show that Γ is completely continuous. We set $B_r(\mathbb{X})$ for the closed unit ball with centre at 0 and radius r in the space $\mathbb{X}$. $\vartheta = \Gamma(B_r(C_h^0(\mathbb{R}_+, \mathbb{X})))$ and $\zeta = \Gamma(u)$ for $u \in (B_r(C_h^0(\mathbb{R}_+, \mathbb{X})))$. First, we will prove that $\vartheta_b(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in [0, b]$. In fact, by the continuity of $E_\alpha(.)$ and condition 3. of f, we infer that the set $Z = \{E_\alpha(s)f(\tau, h(\tau)u) : 0 \leq s, \tau \leq t, u \in C_h^0(\mathbb{R}_+, \mathbb{X})), \|u\|_h \leq r\}$ is relatively compact. On the other hand, we can get $\vartheta_b(t) \in E_\alpha(t)u_0 + t.\overline{c_0(Z)}$, where $\overline{c_0(Z)}$ denotes the convex hull of Z, which establishes our assertion.
Second, we show that the set ϑ_b is equicontinuous. In fact, we can decompose

$$\begin{aligned}
\zeta(t+s) - \zeta(t) &= \Big[E_\alpha(t+s) - E_\alpha(t)\Big]u_0 + \int_0^{t+s} E_\alpha(t+s-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \\
&\quad - \int_0^t E_\alpha(t-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \\
&= \Big[E_\alpha(t+s) - E_\alpha(t)\Big]u_0 + \int_t^{t+s} E_\alpha(t+s-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \\
&\quad + \int_0^t \Big[E_\alpha(t+s-\xi) - E_\alpha(t-\xi)\Big]f(\xi,u(\xi))\,\mathrm{d}\xi \\
&= \Big[E_\alpha(t+s) - E_\alpha(t)\Big]u_0 + \int_t^{t+s} E_\alpha(t+s-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \\
&\quad + \int_0^t \Big[E_\alpha(\xi+s) - E_\alpha(\xi)\Big]f(t-\xi,u(t-\xi))\,\mathrm{d}\xi.
\end{aligned}$$

Then from $(H5)$ and above decomposition of $\zeta(t+s) - \zeta(t)$, it follows that the set ϑ_b is equicontinuous. Finally, applying condition 1, we have

$$\frac{\|\zeta(t)\|_{a\omega,c}}{h(t)} \leq \frac{CM}{h(t)}\Big[\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(rh(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s\Big\}\Big] \;\to\; 0 \; as\; t \;\to\; \infty.$$

and this convergence is independent of $u \in B_r(C_h^0(\mathbb{R}_+, \mathbb{X}))$. Hence, by Lemma 2.19, ϑ is a relatively compact set in $C_h^0(\mathbb{R}_+, \mathbb{X})$.

(4) Let $u^\lambda(.)$ be a solution of equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $\lambda \in (0,1)$. we have,

$$\begin{aligned}\|u^\lambda\|_{a\omega,c} &= \|\lambda\Gamma(u^\lambda)\|_{a\omega,c}\\ &\leq \|\Gamma(u^\lambda)\|_{a\omega,c}\\ &\leq CM \times \Big(\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(\|u^\lambda\|_h h(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s\Big\}\Big).\end{aligned}$$

Since

$$\varrho(\|u^\lambda\|_h) = CM\Big\|\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \sup_{t\geq 0}\Big\{\int_0^t \frac{\psi(\|u^\lambda\|_h h(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s\Big\}\Big\|_h,$$

then

$$\|u^\lambda\|_{a\omega,c} \leq \varrho(\|u^\lambda\|_h)h(t).$$

Therefore

$$\|u^\lambda\|_h \leq \varrho(\|u^\lambda\|_h)$$

hence

$$\frac{\|u^\lambda\|_h}{\varrho(\|u^\lambda\|_h)} \leq 1.$$

and by condition 4, we see that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(5) It follows from$(H4)$ and Theorem 2.15 that $t \mapsto f(t,u(t))$ belong to $AP_{\omega,c}(\Omega,\mathbb{X})$ when $u \in AP_{\omega,c}(\Omega,\mathbb{X})$. Moreover, from Theorem 2.18 and Theorem 3.2, we can deduce that $\Gamma(AP_{\omega,c}(\Omega,\mathbb{X})) \subset AP_{\omega,c}(\Omega,\mathbb{X})$. We note that $AP_{\omega,c}(\Omega,\mathbb{X})$ is a closed subspace of $C_h^0(\mathbb{R}_+,\mathbb{X})$, consequently, we can consider

$$\Gamma : \overline{AP_{\omega,c}(\Omega,\mathbb{X})} \to \overline{AP_{\omega,c}(\Omega,\mathbb{X})}.$$

By assumptions (1)–(3) of Theorem 3.3, we deduce that this map is completely continuous. Applying the well–know Leray–Schauder alternative theorem (see [8]), we infer that Γ has a fixed point $u \in AP_{\omega,c}(\mathbb{X})$ which is the (ω,c)–asymptotically periodic mild solution to equations. (3.1)–(3.2).

□

From Theorem 3.3, we can obtain the following interesting corollary.

Corollary 3.4 *Let $f : \mathbb{R}_+ \times \mathbb{X} \to \mathbb{X}$ be a function satisfying assumption $(H4)$ and the following Hölder–type condition:*

$$\|f(t,u) - f(t,v)\|_{a\omega,c} \leq \rho\|u-v\|^{\vartheta}_{a\omega,c}\ ,\ 0 < \vartheta < 1$$

for all $t \in \mathbb{R}_+$ and $u, v \in \mathbb{X}$ where $\rho > 0$ is a constant. Moreover, assume the following conditions are satisfied:

(a) $\sup\limits_{t\in\mathbb{R}_+} CM\Big(\frac{\|u_0\|}{1+|\widetilde{\omega}|t^\alpha} + \int_0^t \frac{(h(s))^\vartheta}{1+|\widetilde{\omega}|(t-s)^\alpha}\,\mathrm{d}s\Big) = \eta < \infty,$

(b) For each $\alpha, \beta \in \mathbb{R}_+$ *and* $r > 0$*, the set* $\Big\{ f(s, h(s)u) : \alpha \leq s \leq \beta,\ u \in C_h^0(\mathbb{R}_+, \mathbb{X}),$ $\|u\|_h \leq r \Big\}$ *is relatively compact in* $\mathbb{X}$*,*

(c) $\lim_{\xi \to \infty} \frac{\xi}{\varrho(\xi)} > 1.$

Then equations (3.1)–(3.2) *admit at least one* (ω, c)*–asymptotically periodic mild solution.*

Proof. Let $\eta_1 = \rho$ and we take $\psi(\xi) = \eta_1 \xi^{\vartheta}$. Then, condition$(H5)$ is satisfied. It follows from (a), that the function f satisfies (1) in Theorem 3.3. Note that for each $\varepsilon > 0$ there is $0 < \delta^{\vartheta} < \frac{\varepsilon}{\eta \times \eta_1}$ such that for every $u, v \in C_h^0(\mathbb{R}_+, \mathbb{X})$, $\|u - v\|_h \leq \delta$ implies that

$$\sup_{t \in \mathbb{R}_+} \left\{ \int_0^t \frac{\|f(s, u(s)) - f(s, v(s))\|_{a\omega, c}}{1 + |\widetilde{\omega}|(t-s)^{\alpha}} \, \mathrm{d}s \right\} \leq \frac{\varepsilon}{CM}$$

for all $t \in \mathbb{R}_+$. The assumption (3) in Theorem 3.3 can be easily verified by the definition of ψ. So, from Theorem 3.3 we can conclude that equations (3.1)–(3.2) admit at least one (ω, c)-asymptotically periodic mild solution. □

Acknowledgements. The authors would like to express their sincere gratitude to the referees and editors for their careful reading and valuable suggestions.

References

[1] L. Abadias, E. Alvarez, R. Grau, (ω, c)*–Periodic Mild Solutions to Non–Autonomous Abstract Differential Equations*, Mathematics, Vol. 9, no. 5 (2021), pp. 474–489.

[2] M. Agaoglou, M. Feckan, A. P. Panagiotidou, *Existence and uniqueness of* (ω, c)*–periodic solutions of semilinear evolution equations*, International Journal of Dynamical Systems and Differential Equations, Vol. 10, no. 2 (2020), pp. 149–166.

[3] E. Alvarez, A. Gomez, M. Pinto, (ω, c)*–periodic functions and mild solutions to abstract fractional integro–differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, no. 16 (2018), pp. 1–8.

[4] E. Alvarez, M. Pinto, S. Castillo, (ω, c)*–asymptotically periodic functions, first–order Cauchy problem, and Lasota–Wazewska model with unbounded oscillating production of red cells*, Mathematicals Methods in the Applied Sciences, Vol. 43, no. 1 (2020), pp. 305–319.

[5] D. Brindle, G. M. N'Guérékata, *S–asymptotically* ω*-periodic mild solutions to fractional differential equations*, Electronic Journal of Differential Equations, Vol. 2020, no. 30 (2020), pp. 1–12.

[6] W. Dimbour, G. Mophou, G. M. N'Guérékata, S*–asymptotically periodic solutions for partial differential equations with finite delay*, Electronic Journal of Differential Equations, Vol. 2011, no. 117 (2011), pp. 1–12.

[7] R. G. Foko Tiomela, G.M. N'Guérékata, (ω, c)–*asymptotically periodic solutions to some fractional integro-differential equations*, Journal of Fractional Calculus and Applications, Vol. 13, no. 2 (2022), pp. 100–115.

[8] A. Granas, J. Dugundji, *Fixed point theory*, New York, Springer, 2003.

[9] M. F. Hasler, G. M. N'Guérékata, *Bloch–periodic functions and some applications*, Nonlinear Studies, Vol. 21, no. 1 (2014), pp. 21–30.

[10] H. R. Henriquez, M. Pierri, P. Taboas, *On* S*–asymptotically* ω*–periodic functions on Banach spaces and applications*, Journal of Mathematical Analysis and Applications, Vol. 343, no. 2 (2008), pp. 1119–1130.

[11] M. Kéré , G. M. N'Guérékata, E. R. Oueama, *An existence result of* (ω, c)*–almost periodic mild solutions to some fractional differential equations*, PanAmerican Mathematical Journal, Vol. 31, no. 3 (2021), pp. 11–20.

[12] M. T. Khalladi, M. Kostic, M. Pinto, A. Rahmani, D. Velinov, C*–almost periodic type functions and applications*, Nonautonomous Dynamical Systems, Vol. 7, no. 1 (2020), pp. 176–193.

[13] J. Larrouy, G. M. N'Guérékata, (ω, c)*–periodic and asymptotically* (ω, c)*–periodic mild solutions to fractional Cauchy problems*, Applicable Analysis, Vol. 102, no. 3 (2023), pp. 958–976.

[14] M. Li, J. R. Wang, M. Feckan, (ω, c)*–periodic solutions for impulsive differential systems*, Communication in Mathematical Analysis, Vol. 74, no. 2 (2018), pp. 35–46.

[15] G. Mophou, G. M. N'Guérékata, *An existence result of* (ω, c)*–periodic mild solutions to some fractional differential equations*, Nonlinear Studies, Vol. 27, no. 1 (2020), pp. 167–175.

[16] I. Podlubny, *Fractional differential equations*, Elsevier Science (1998).

[17] J. R. Wang, L. Ren and Y. Zhou, (ω, c)*–periodic solutions for time varying impulsive differential equations*, Advances in Difference Equations, Vol. 2019, no. 259 (2019), pp. 1–9.

[18] J. K. Zao, Y. K. Chang, G. M. N'Guérékata, *Asymptotic Behavior of Mild Solutions to Similinear Fractional Differential Equations*, Journal of Optimization Theory and Applications, Vol. 156, no. 1 (2013), pp. 106–114.