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STABILITY FOR SHEAR BEAM MODEL AND NEW FACTS RELATED TO THE CLASSICAL TIMOSHENKO SYSTEM WITH VARIABLE DELAY

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Abstract. In this paper we study a Timoshenko type beam model with a variable delay. It is mainly about, on the one hand, a study of the existence and uniqueness of the solution and on the other hand, to present a study of exponential stability of the obtained solution. The introduction of the variable delay term is the added value brought by this work.

Keywords: Timoshenko system, Exponential stability, Faedo Galerkin Method, Time delay.

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1 Introduction

Here we consider the following Shear beam model and new facts related to the classical Timoshenko system with a variable delay

$$\begin{cases} \rho_1\varphi_{tt}(x,t)-\kappa(\varphi_x(x,t)+\psi(x,t))_x+\mu_1\varphi_t(x,t)+\mu_2\varphi_t(x,t-\tau(t))=0 & \text{in } (0,L)\times(0,+\infty), \\ -b\,\psi_{xx}(x,t)+\kappa(\varphi_x(x,t)+\psi(x,t))=0 & \text{in } (0,L)\times(0,+\infty), \\ \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x) & \text{in } (0,L), \\ \psi_x(0,t)=\psi_x(L,t)=\varphi(0,t)=\varphi(L,t)=0 & \text{in } (0,+\infty). \end{cases} \tag{1.1}$$

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The functions φ and ψ describe the transverse displacement of the beam and the rotation angle of a filament of the beam, respectively. $\rho_1, \kappa, \mu_1, \mu_2$ and b are positive constants and the function $\tau(t)$ satisfies the conditions

$$\forall t > 0: \ 0 < \tau_0 < \tau(t) < M, \tag{1.2}$$

$$\forall t > 0: \ \tau'(t) \leq d < 1, \tag{1.3}$$

$$\frac{\mu_2}{\sqrt{1-d}} < \zeta < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{1.4}$$

where M, d and ζ are positive constants.

Recently Júnior, Ramos and Freitas [1] obtained the existence and the exponential decay of the energy for the dissipative system given by

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x(x,t)+\psi(x,t))_x + \mu\,\varphi_t(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ -b\,\psi_{xx}(x,t) + \kappa(\varphi_x(x,t)+\psi(x,t)) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x) & \text{in } (0,L), \\ \psi_x(0,t)=\psi_x(L,t)=\varphi(0,t)=\varphi(L,t)=0 & \text{in } (0,+\infty), \end{cases} \tag{1.5}$$

using the Faedo-Galerkin method and a Lyapunov function.

With reference to their idea [1] we study the existence and exponential decay of problem (1.1). To our knowledge, this is a new contribution to the study of problem (1.5) because these results are an extension of those of Júnior, Ramos and Freitas by adding the delay term in the first equation.

The paper is organized as follows. In Section 2 we will prove the decrease of energy. In Section 3 we will establish the well posedness of problem (1.1) using the Faedo-Galerkin method. And finally in Section 4 we will give the exponential stability of the problem using a Lyapunov function.

2 Dissipative nature of the system

Let us set

$$\forall x \in (0,L), \forall \rho \in (0,1), \ \forall t > 0, \ z(x,\rho,t) = \varphi_t(x, t-\rho\,\tau(t)). \tag{2.1}$$

Problem (1.1) is now equivalent to

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x(x,t)+\psi(x,t))_x + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ -b\,\psi_{xx}(x,t) + \kappa(\varphi_x(x,t)+\psi(x,t)) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \tau(t)z_t(x,\rho,t) + (1-\rho\,\tau'(t))z_\rho(x,\rho,t) = 0 & \text{in } (0,L)\times(0,1)\times(0,+\infty), \\ \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x) & \text{in } (0,L), \\ z(x,\rho,0) = f_0(x,-\rho\,\tau(t)) & \text{in } (0,L)\times(0,1), \\ z(x,0,t) = \varphi_t(x,t) & \forall\, t \in (0,+\infty), \\ \psi_x(0,t)=\psi_x(L,t)=\varphi(0,t)=\varphi(L,t)=0 & \text{in } (0,+\infty). \end{cases} \tag{2.2}$$

Let $(\varphi, \varphi_t, \psi, z)$ be a solution of (2.2), the corresponding energy is given by

$$E(t) = \frac{\rho_1}{2}\int_0^L |\varphi_t|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_x|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^L |\varphi_x+\psi|^2\,\mathrm{d}x + \frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\ \mathrm{d}\rho\,\mathrm{d}x.$$

Theorem 2.1 *Under the assumption* (1.3)-(1.4) *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E(t) \le 0.$$

Proof. Multiply the first equation of (2.2) by φ_t and integrate on $(0, L)$ we have

$$\rho_1 \int_0^L \varphi_{tt}(x,t)\varphi_t(x,t)\,\mathrm{d}x - \kappa \int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]_x \varphi_t(x,t)\,\mathrm{d}x$$
$$+ \mu_1 \int_0^L \varphi_t(x,t)\varphi_t(x,t)\,\mathrm{d}x + \mu_2 \int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x = 0.$$

This leads to

$$\frac{\rho_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x - \kappa\left[(\varphi_x + \psi)\varphi_t(x,t)\right]_0^L + \kappa\int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]\varphi_{tx}(x,t)\,\mathrm{d}x$$
$$+ \mu_1 \int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \mu_2 \int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x = 0.$$

Since φ is zero at 0 and L we have

$$\frac{\rho_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \kappa\int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]\varphi_{tx}(x,t)\,\mathrm{d}x$$
$$+ \mu_1 \int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \mu_2 \int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x = 0. \tag{2.3}$$

Multiplying now the second equation of (2.2) by ψ_t and integrating on $(0, L)$ and after some calculations we have

$$-b\left[\psi_x\psi_t(x,t)\right]_0^L + b\int_0^L \psi_x(x,t)\psi_{xt}(x,t)\,\mathrm{d}x + \kappa\int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]\psi_t(x,t)\,\mathrm{d}x = 0.$$

Using the fact that $\varphi_x(0) = \varphi_x(L) = 0$, we have

$$\frac{b}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\psi_x(x,t)|^2\,\mathrm{d}x + \kappa\int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]\psi_t(x,t)\,\mathrm{d}x = 0. \tag{2.4}$$

Multiplying the third equation of (2.2) by ζz and integrating on $(0, L) \times (0, 1)$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\ \mathrm{d}\rho\,\mathrm{d}x\right] &= \frac{\zeta}{2}\tau'(t)\int_0^L\int_0^1 |z|^2\ \mathrm{d}\rho\,\mathrm{d}x + \zeta\,\tau(t)\int_0^L\int_0^1 zz_t\ \mathrm{d}\rho\,\mathrm{d}x \\
&= \frac{\zeta}{2}\tau'(t)\int_0^L\int_0^1 |z|^2\ \mathrm{d}\rho\,\mathrm{d}x - \zeta\int_0^L\int_0^1 \left[1 - \rho\,\tau'(t)\right] zz_\rho\ \mathrm{d}\rho\,\mathrm{d}x \\
&= -\frac{\zeta}{2}\int_0^L\int_0^1 \frac{\partial}{\partial\rho}\Big[\left[1 - \rho\,\tau'(t)\right]|z|^2\Big]\ \mathrm{d}\rho\,\mathrm{d}x \\
&= -\frac{\zeta}{2}\int_0^L \Big[\left[1 - \rho\,\tau'(t)\right]|z|^2\Big]_0^1\mathrm{d}x.
\end{aligned}$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\,\mathrm{d}\rho\,\mathrm{d}x\right] = -\frac{\zeta}{2}\int_0^L \left[1-\tau'(t)\right]|z(x,1,t)|^2\,\mathrm{d}x + \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x. \tag{2.5}$$

By adding the equations (2.3), (2.4) and (2.5) we obtain

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \kappa\int_0^L [\varphi_x(x,t)+\psi(x,t)]\,\varphi_{tx}(x,t)\,\mathrm{d}x + \mu_1\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x\\
&+\mu_2\int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x + \frac{b}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\psi_x(x,t)|^2\,\mathrm{d}x + \kappa\int_0^L [\varphi_x(x,t)+\psi(x,t)]\,\psi_t(x,t)\,\mathrm{d}x\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\,\mathrm{d}\rho\,\mathrm{d}x\right] + \frac{\zeta}{2}\int_0^L \left[1-\tau'(t)\right]|z(x,1,t)|^2\,\mathrm{d}x - \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x = 0,
\end{aligned}$$

which means that

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \mu_1\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \kappa\int_0^L [\varphi_x(x,t)+\psi(x,t)]\,[\varphi_x(x,t)+\psi(x,t)]_t\,\mathrm{d}x\\
&+\mu_2\int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x + \frac{b}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\psi_x(x,t)|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\,\mathrm{d}\rho\,\mathrm{d}x\right]\\
&+\frac{\zeta}{2}\int_0^L \left[1-\tau'(t)\right]|z(x,1,t)|^2\,\mathrm{d}x - \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x = 0.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\rho_1}{2}\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_x(x,t)|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2\,\mathrm{d}x\right.\\
&\qquad\left.+\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\,\mathrm{d}\rho\,\mathrm{d}x\right\} + \mu_1\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x + \mu_2\int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x\\
&\qquad\qquad+\frac{\zeta}{2}\int_0^L \left[1-\tau'(t)\right]|z(x,1,t)|^2\,\mathrm{d}x - \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x = 0,
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) = &-\mu_1\int_0^L |\varphi_t(x,t)|^2\,\mathrm{d}x - \mu_2\int_0^L z(x,1,t)\varphi_t(x,t)\,\mathrm{d}x\\
&-\frac{\zeta}{2}\int_0^L \left[1-\tau'(t)\right]|z(x,1,t)|^2\,\mathrm{d}x + \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x.
\end{aligned}$$

Since $\varphi_t(x,t) = z(x,0,t)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) = &-\mu_1\int_0^L |z(x,0,t)|^2\,\mathrm{d}x - \mu_2\int_0^L z(x,1,t)z(x,0,t)\,\mathrm{d}x\\
&-\frac{\zeta}{2}\int_0^L \left[1-\tau'(t)\right]|z(x,1,t)|^2\,\mathrm{d}x + \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x.
\end{aligned}\tag{2.6}$$

Now using the inequality of Young, we have

$$-\mu_2\int_0^L z(x,1,t)z(x,0,t)\,\mathrm{d}x \le \frac{\mu_2\sqrt{1-d}}{2}\int_0^L |z(x,1,t)|^2\,\mathrm{d}x + \frac{\mu_2}{2\sqrt{1-d}}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x.$$

Then, equation (2.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le -\mu_1\int_0^L |z(x,0,t)|^2\,\mathrm{d}x - \frac{\zeta}{2}\int_0^L \big[1-\tau'(t)\big]\,|z(x,1,t)|^2\,\mathrm{d}x + \frac{\zeta}{2}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x + \frac{\mu_2\sqrt{1-d}}{2}\int_0^L |z(x,1,t)|^2\,\mathrm{d}x + \frac{\mu_2}{2\sqrt{1-d}}\int_0^L |z(x,0,t)|^2\,\mathrm{d}x.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le \frac{1}{2}\Big[\zeta\Big(\tau'(t)-1\Big)+\mu_2\sqrt{1-d}\Big]\int_0^L |z(x,1,t)|^2\,\mathrm{d}x + \frac{1}{2}\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^L |z(x,0,t)|^2\,\mathrm{d}x. \tag{2.7}$$

From (1.3) we have $\tau'(t)-1\le d-1$, so (2.7) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le \frac{1}{2}\Big[\zeta(d-1)+\mu_2\sqrt{1-d}\Big]\int_0^L |z(x,1,t)|^2\,\mathrm{d}x + \frac{1}{2}\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^L |z(x,0,t)|^2\,\mathrm{d}x.$$

This means that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le \frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^L |z(x,1,t)|^2\,\mathrm{d}x + \frac{1}{2}\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^L |z(x,0,t)|^2\,\mathrm{d}x. \tag{2.8}$$

According to the assumptions (1.3) and (1.4), we obtain

$$\frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]<0$$

and

$$\frac{1}{2}\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]<0.$$

Hence $\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le 0$ and the system (2.2) is dissipative. □

3 Existence and uniqueness

Let us introduce the following spaces:

$$L^2_*(0,L)=\left\{u\in L^2(0,L),\int_0^L u(x)\,\mathrm{d}x=0\right\},$$

$$H^1_*(0,L)=H^1(0,L)\cap L^2_*(0,L),$$

and

$$H^2_*(0,L)=H^2(0,L)\cap H^1_*(0,L).$$

Denote by $\mathcal{H}$ and $\mathcal{H}_1$ the Hilbert spaces as below

$$\mathcal{H} = H^1(0,L) \times L^2(0,L) \times H^1_*(0,L) \times L^2((0,L) \times (0,1))$$

and

$$\mathcal{H}_1 = (H^2(0,L) \cap H^1_0(0,L))^2 \times H^2_*(0,L) \times H^1((0,L) \times (0,1)).$$

We equipped $\mathcal{H}$ with the norm

$$\left\|(u,v,w,z)^{\mathsf{T}}\right\|^2_{\mathcal{H}} = \frac{\rho_1}{2}\int_0^L |v|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |w_x|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^L |u_x+w|^2\,\mathrm{d}x + \frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z|^2\,\mathrm{d}\rho\,\mathrm{d}x,$$

where ζ is a positive constant verifying (1.4).

Here, we can use now the definition of weak solution in article [1]. We state

Definition 3.1 *Given an initial data $U_0 = (\varphi_0, \varphi_1, \psi_0, z_0) \in \mathcal{H}$, a function $U = (\varphi, \varphi_t, \psi, z) \in C([0,T];\mathcal{H})$ is said to be a weak solution of (2.2) if for almost everywhere $t \in [0,T]$,*

$$\begin{cases} \rho_1 \dfrac{\mathrm{d}}{\mathrm{d}t}(\varphi_t, u) + \kappa(\varphi_x + \psi, u_x) + \mu_0(\varphi_t, u) + \mu_1(z(\cdot,1,t), u) = 0, \\ b(\psi_x, v_x) + \kappa(\varphi_x + \psi, v) = 0, \\ \tau(t)(z_t(\rho,t), w) + (1 - \rho\,\tau'(t))(z_\rho(\rho,t), w) = 0, \end{cases} \tag{3.1}$$

for all $u \in H^1_0(0,L)$, $v \in H^2_*(0,L)$, $w \in L^2((0,L) \times (0,1)$ and $(\varphi(0), \varphi_t(0), \psi(0), z(0)) = (\varphi_0, \varphi_1, \psi_0, z_0)$.

Theorem 3.1 *Assume that* (1.2)-(1.4) *hold. Then for any data $U_0 = (\varphi_0, \varphi_1, \psi_0, f_0)$ belongs to $\mathcal{H}$, problem* (2.2) *has one and only one weak solution $U = (\varphi, \varphi_t, \psi, z)$ verifying:*

$$\begin{cases} \varphi \in L^\infty\left(0, T, H^1_0(0,L)\right), \\ \varphi_t \in L^\infty\left(0, T, L^2(0,L)\right), \\ \psi \in L^\infty\left(0, T, H^1_*(0,L)\right), \\ z \in L^\infty\left(0, T, L^2((0,L) \times (0,1))\right). \end{cases} \tag{3.2}$$

Moreover, if $U_0 = (\varphi_0, \varphi_1, \psi_0, f_0)$ belongs to $\mathcal{H}_1$, then problem (2.2) *has one and only one strong solution $U = (\varphi, \varphi_t, \psi, z)$ which satisfies*

$$\begin{cases} \varphi \in L^\infty\left(0, T, H^2(0,L) \cap H^1_0(0,L)\right), \\ \varphi_t \in L^\infty\left(0, T, H^1_0(0,L)\right), \\ \psi \in L^\infty\left(0, T, H^2_*(0,L)\right), \\ z \in L^\infty\left(0, T, H^1((0,L) \times (0,1))\right). \end{cases} \tag{3.3}$$

Proof. The Faedo-Galerkin method will be the key to prove the existence of a global solution.

Step 1 *Let us consider initial data $(\varphi_0, \varphi_1, \psi_0, f_0) \in \mathcal{H}$. Let $\{u^k\}, k \in \mathbb{N}^\star$ and $\{v^k\}, k \in \mathbb{N}^\star$ be bases formed by eigenfunctions of $-\partial_{xx}$. These bases can be considered orthogonal in $H^2(0,L) \cap H^1_0(0,L)$ and $H^2_*(0,L)$ respectively and both orthogonormal [6, 7] in $L^2(0,L)$. We also define the sequence $\{w^k\}, k \in \mathbb{N}^\star$ in the following way [3, 4] $w^k(x,0) = u^k(x)$, then we extend $w^k(x,0)$ by $w^k(x,\rho)$ on $L^2((0,L) \times (0,1))$.*

Approximation spaces H_n, V_n and W_n of finite dimensions are given by $H_n = span\{u^1, u^2, \cdots, u^n\}$, $V_n = span\{v^1, v^2, \cdots, v^n\}$ and $W_n = span\{w^1, w^2, \cdots, w^n\}, n \in \mathbb{N}^\star$.

We will find an approximate solution of the form

$$\varphi^n(t,x) = \sum_{j=1}^{n} a^{jn}(t)u^j(x), \quad \psi^n(t,x) = \sum_{j=1}^{n} b^{jn}(t)v^j(x), \quad z^n(x,t,\rho) = \sum_{j=1}^{n} c^{jn}(t)w^j(x,\rho),$$

to the following approximate problem

$$\begin{cases} \rho_1 \displaystyle\int_0^L \varphi_{tt}^n(x,t)u\,\mathrm{d}x - \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right]_x u\,\mathrm{d}x \\ \qquad\qquad + \mu_1 \displaystyle\int_0^L \varphi_t^n(x,t)u\,\mathrm{d}x + \mu_2 \int_0^L z^n(x,1,t)\,u\,\mathrm{d}x = 0, \\ \qquad - b \displaystyle\int_0^L \psi_{xx}^n(x,t)\,v\,\mathrm{d}x + \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right] v\,\mathrm{d}x = 0, \\ \zeta\,\tau(t) \displaystyle\int_0^L \int_0^1 z_t^n(x,\rho,t)\,w\,\mathrm{d}\rho\ \mathrm{d}x + \zeta \int_0^L \int_0^1 (1-\rho\,\tau'(t)) z_\rho^n(x,\rho,t)\,w\,\mathrm{d}\rho\ \mathrm{d}x = 0, \end{cases} \tag{3.4}$$

for all $u \in H_n, v \in V_n, w \in W_n$, with initial conditions such that

$$(\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0) \tag{3.5}$$

strongly in $\mathcal{H}$. a^{jn}, b^{jn} and $c^{jn}, 1 \le j \le n$ form the temporal weighting coefficients.

According to the standard theory of ordinary differential equations, the finite dimensional problem (3.4)-(3.5) *has a solution $(a^{jn}, b^{jn}, c^{jn}), 1 \le j \le n$ defined on $[0, t_n)$ with $0 < t_n < T$ for every $n \in \mathbb{N}^*$. Then the a priori estimates that follow imply that in fact $t_n = T$.*

Step 2 *Replacing u by φ_t^n, v by ψ_t^n and w by z^n in* (3.4)*, we obtain*

$$\begin{cases} \dfrac{\rho_1}{2} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^L |\varphi_t^n(x,t)|^2\,\mathrm{d}x + \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right] \varphi_{tx}^n(x,t)\,\mathrm{d}x \\ \qquad + \mu_1 \displaystyle\int_0^L |\varphi_t^n(x,t)|^2\,\mathrm{d}x + \mu_2 \int_0^L z^n(x,1,t)\varphi_t^n(x,t)\,\mathrm{d}x = 0, \\ \dfrac{b}{2} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^L |\psi_x^n(x,t)|^2\,\mathrm{d}x + \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right] \psi_t^n(x,t)\,\mathrm{d}x = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \Big[\dfrac{\zeta}{2}\tau(t) \displaystyle\int_0^1 |z^n|^2\,\mathrm{d}\rho\,\mathrm{d}x\Big] + \dfrac{\zeta}{2} \int_0^L \left[1 - \tau'(t)\right] |z(x,1,t)|^2\,\mathrm{d}x \\ \qquad\qquad - \dfrac{\zeta}{2} \displaystyle\int_0^L |z^n(x,0,t)|^2\,\mathrm{d}x = 0. \end{cases} \tag{3.6}$$

By making the same transformations as in the session of the dissipative character we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E^n(t) \le \frac{1-d}{2}\Big[-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big] \int_0^L |z^n(x,1,t)|^2\,\mathrm{d}x \\ + \frac{1}{2}\Big[\zeta - (2\mu_1 - \frac{\mu_2}{\sqrt{1-d}})\Big] \int_0^L |z^n(x,0,t)|^2\,\mathrm{d}x, \end{aligned} \tag{3.7}$$

where

$$E^n(t) = \frac{\rho_1}{2}\int_0^L |\varphi_t^n|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_x^n|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^L |\varphi_x^n + \psi^n|^2\,\mathrm{d}x + \frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z^n|^2\,\mathrm{d}\rho\,\mathrm{d}x.$$

Thus integrating (3.7) from 0 to $t < t_n$, we obtain from our choice of initial data that for all $t \in [0, t_n]$ and for every $n \in \mathbb{N}$,

$$\begin{aligned} E^n(t) - E^n(0) \le \frac{1-d}{2}\Big[-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L |z^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ + \frac{1}{2}\Big[\zeta - (2\mu_1 - \frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L |z^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

which means

$$\begin{aligned} E^n(t) - \frac{1-d}{2}\Big[-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L |z^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ - \frac{1}{2}\Big[\zeta - (2\mu_1 - \frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L |z^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s \le E^n(0), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} E^n(0) &= \frac{\rho_1}{2}\int_0^L |\varphi_t^n(0)|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_x^n(0)|^2\,\mathrm{d}x \\ &+ \frac{\kappa}{2}\int_0^L |\varphi_x^n(0) + \psi^n(0)|^2\,\mathrm{d}x + \frac{\zeta}{2}\tau(0)\int_0^L\int_0^1 |z^n(0)|^2\,\mathrm{d}\rho\,\mathrm{d}x. \end{aligned}$$

As $(\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0)$ strongly in $\mathcal{H}$, we can deduce that each of the sequences $\{\varphi^n(0)\}$, $\{\varphi_t^n(0)\}$, $\{\psi^n(0)\}$ and $\{z^n(0)\}$ is bounded.

Thus there exists a positive constant C_1 such that $E^n(0) \le C_1$. Hence

$$\begin{aligned} E^n(t) - \frac{1-d}{2}\Big[-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L |z^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ - \frac{1}{2}\Big[\zeta - (2\mu_1 - \frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L |z^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s \le C_1, \end{aligned} \tag{3.9}$$

which means that

$$\begin{aligned} &\frac{\rho_1}{2}\int_0^L |\varphi_t^n|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_x^n|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^L |\varphi_x^n + \psi|^2\,\mathrm{d}x + \frac{\zeta}{2}\tau(t)\int_0^L\int_0^1 |z^n|^2\,\mathrm{d}\rho\,\mathrm{d}x \\ &- \frac{1-d}{2}\Big[-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L |z^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &- \frac{1}{2}\Big[\zeta - (2\mu_1 - \frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L |z^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s \le C_1. \end{aligned} \tag{3.10}$$

As the constant C_1 does not depend on n we can therefore take $t_n = T$.

Step 3 *Let us differentiate the first equation of* (3.4) *with respect to t and then replacing u by φ^n_{tt}, we obtain*

$$\rho_1 \int_0^L \varphi^n_{ttt}(x,t)\varphi^n_{tt}(x,t)\,\mathrm{d}x - \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]_x \varphi^n_{tt}(x,t)\,\mathrm{d}x$$
$$+ \mu_1 \int_0^L \varphi^n_{tt}(x,t)\varphi^n_{tt}(x,t)\,\mathrm{d}x + \mu_2 \int_0^L z^n_t(x,1,t)\varphi^n_{tt}(x,t)\,\mathrm{d}x = 0.$$

By integrating by parts, we obtain

$$\frac{\rho_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\varphi^n_{tt}(x,t)|^2\,\mathrm{d}x - \kappa\left[(\varphi^n_{xt} + \psi^n_t)\varphi^n_{tt}(x,t)\right]_0^L + \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]\varphi^n_{xtt}(x,t)\,\mathrm{d}x$$
$$+ \mu_1 \int_0^L |\varphi^n_{tt}(x,t)|^2\,\mathrm{d}x + \mu_2 \int_0^L z^n_t(x,1,t)\varphi^n_{tt}(x,t)\,\mathrm{d}x = 0.$$

As φ is null at 0 *and at L we have*

$$\frac{\rho_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\varphi^n_{tt}(x,t)|^2\,\mathrm{d}x + \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]\varphi^n_{xtt}(x,t)\,\mathrm{d}x + \mu_1 \int_0^L |\varphi^n_{tt}(x,t)|^2\,\mathrm{d}x$$
$$+ \mu_2 \int_0^L z^n_t(x,1,t)\varphi^n_{tt}(x,t)\,\mathrm{d}x = 0. \tag{3.11}$$

Let us differentiate the second equation of (3.4) *with respect to t and then replacing v by ψ^n_{tt}, we obtain*

$$-b \int_0^L \psi^n_{xxt}(x,t)\psi^n_{tt}(x,t)\,\mathrm{d}x + \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]\psi^n_{tt}(x,t)\,\mathrm{d}x = 0.$$

Integrating by parts, we obtain

$$-b\left[\psi^n_{xt}\psi^n_{tt}(x,t)\right]_0^L + b \int_0^L \psi^n_{xt}(x,t)\psi^n_{xtt}(x,t)\,\mathrm{d}x + \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]\psi^n_{tt}(x,t)\,\mathrm{d}x = 0.$$

Since ψ_x is zero at the edge we have

$$b \int_0^L \psi^n_{xt}(x,t)\psi^n_{xtt}(x,t)\,\mathrm{d}x + \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]\psi^n_{tt}(x,t)\,\mathrm{d}x = 0,$$

which can be written as

$$\frac{b}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |\psi^n_{xt}(x,t)|^2\,\mathrm{d}x + \kappa \int_0^L \left[\varphi^n_{xt}(x,t) + \psi^n_t(x,t)\right]\psi^n_{tt}(x,t)\,\mathrm{d}x = 0. \tag{3.12}$$

Summing (3.11) *and* (3.12) *we obtain*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\rho_1 \int_0^L |\varphi^n_{tt}(x,t)|^2\,\mathrm{d}x + k \int_0^L |\varphi^n_{xt}(x,t) + \psi^n_t(x,t)|^2\,\mathrm{d}x + b \int_0^L |\psi^n_{xt}(x,t)|^2\,\mathrm{d}x\right\}$$
$$+ \mu_2 \int_0^L z^n_t(x,1,t)\varphi^n_{tt}(x,t)\,\mathrm{d}x + \mu_1 \int_0^L |\varphi^n_{tt}(x,t)|^2\,\mathrm{d}x = 0,$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t} G^n(t) + \mu_2 \int_0^L z_t^n(x,1,t)\varphi_{tt}^n(x,t)\,\mathrm{d}x + \mu_1 \int_0^L |\varphi_{tt}^n(x,t)|^2\,\mathrm{d}x = 0 \tag{3.13}$$

with

$$G^n(t) = \frac{\rho_1}{2}\int_0^L |\varphi_{tt}^n(x,t)|^2\,\mathrm{d}x + \frac{k}{2}\int_0^L |\varphi_{xt}^n(x,t) + \psi_t^n(x,t)|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_{xt}^n(x,t)|^2\,\mathrm{d}x.$$

By integrating (3.13) *from* 0 *to* t *we have*

$$G^n(t) + \mu_2 \int_0^t\int_0^L z_s^n(x,1,s)\varphi_{ss}^n(x,s)\ \mathrm{d}x\,\mathrm{d}s + \mu_1 \int_0^t\int_0^L |\varphi_{ss}^n(x,s)|^2\ \mathrm{d}x\,\mathrm{d}s = G(0),$$

where

$$G^n(0) = \frac{\rho_1}{2}\int_0^L |\varphi_{tt}^n(x,0)|^2\,\mathrm{d}x + \frac{k}{2}\int_0^L |\varphi_{xt}^n(x,0) + \psi_t^n(x,0)|^2\,\mathrm{d}x + \frac{b}{2}\int_0^L |\psi_{xt}^n(x,0)|^2\,\mathrm{d}x.$$

As $(\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0)$ *strongly in* $\mathcal{H}$*, then there exists a positive constant* C_2 *such that* $G^n(0) \leq C_2$*. Hence*

$$G^n(t) + \mu_2 \int_0^t\int_0^L z_s^n(x,1,s)\varphi_{ss}^n(x,s)\ \mathrm{d}x\,\mathrm{d}s + \mu_1 \int_0^t\int_0^L |\varphi_{ss}^n(x,s)|^2\ \mathrm{d}x\,\mathrm{d}s \leq C_2. \tag{3.14}$$

Step 4 ***Passage to Limit.*** *From the relations* (3.10) *and* (3.14) *we have*
$\{\varphi^n\}$ *is bounded in* $L^\infty\left(0,T,H_0^1(0,L)\right)$, $\{\varphi_t^n\}$ *is bounded in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\varphi_{tt}^n\}$ *is bounded in* $L^\infty\left(0,T,L^2(0,L)\right)$, $\left(\psi^n\right)$ *is bounded in* $L^\infty\left(0,T,H_*^1(0,L)\right)$,
$\left(\psi_t^n\right)$ *is bounded in* $L^\infty\left(0,T,L^2(0,L)\right)$, $\{z^n\}$ *is bounded in* $L^\infty\left(0,T,L^2((0,L)\times(0,1))\right)$.

So by [5, Theorem 1.3.4], we can extract subsequences $\{\varphi^n\}$, $\{\psi^n\}$ *and* $\{z^n\}$ *such as*

$$\begin{aligned}
&\{\varphi^n\} \to \varphi \ \text{ weakly star in } L^\infty\left(0,T,H_0^1(0,L)\right),\\
&\{\varphi_t^n\} \to \varphi_t \ \text{ weakly star in } L^\infty\left(0,T,L^2(0,L)\right),\\
&\{\varphi_{tt}^n\} \to \varphi_{tt} \ \text{ weakly star in } L^\infty\left(0,T,L^2(0,L)\right),\\
&\{\psi^n\} \to \psi \ \text{ weakly star in } L^\infty\left(0,T,H_*^1(0,L)\right),\\
&\{\psi_t^n\} \to \psi_t \ \text{ weakly star in } L^\infty\left(0,T,L^2(0,L)\right),\\
&\{z^n\} \to z \ \text{ weakly star in } L^\infty\left(0,T,L^2((0,L)\times(0,1))\right).
\end{aligned}$$

Moreover, from (3.10) *we have* $\{\varphi^n\}$ *is bounded in* $L^2\left(0,T,H_0^1(0,L)\right)$, $\{\varphi_t^n\}$ *is bounded in* $L^2\left(0,T,L^2(0,L)\right)$. *And since* $H_0^1(0,L)$ *is compactly injected into* $L^2(0,L)$ *(see [2,5]) we have by the Aubin-Lions theorem [8] that* $\{\varphi^n\} \to \varphi$ *strongly in* $L^\infty\left(0,T,L^2(0,L)\right)$.

We also show that $\{\varphi_t^n\} \to \varphi_t$ *strongly in* $L^\infty\left(0,T,L^2(0,L)\right)$, $\{\psi^n\} \to \psi$ *strongly in* $L^\infty\left(0,T,H_0^1(0,L)\right)$. *Then we can pass to limit the approximate problem* (3.4)-(3.5) *in order to get a weak solution of problem* (2.2)*. And we use density arguments to get a global weak solution satisfying*

$$\begin{cases}
\varphi \in L^\infty\left(0,T,H_0^1(0,L)\right),\\
\varphi_t \in L^\infty\left(0,T,L^2(0,L)\right),\\
\psi \in L^\infty\left(0,T,H_\star^1(0,L)\right),\\
z \in L^\infty\left(0,T,L^2((0,L)\times(0,1))\right).
\end{cases} \tag{3.15}$$

Step 5 *Suppose that the initial data in the approximate problem* (3.4) *satisfies* $(\varphi_0,\varphi_1,\psi_0,f_0)\in\mathcal{H}_1$ *and*

$$(\varphi_0^n,\varphi_1^n,\psi_0^n,z_0^n)\to(\varphi_0,\varphi_1,\psi_0,f_0) \tag{3.16}$$

strongly in $\mathcal{H}_1$. *Replacing* u *by* $-\varphi_{xxt}^n$, v *by* $-\psi_{xxt}^n$ *and* w *by* z_{xx}^n *in* (3.4)*, we arrive at*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F^n(t)\le{}&\frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^L|z_x^n(x,1,t)|^2\,\mathrm{d}x\\&+\frac12\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^L|z_x^n(x,0,t)|^2\,\mathrm{d}x,\end{aligned}\tag{3.17}$$

where

$$F^n(t)=\frac{\rho_1}{2}\int_0^L|\varphi_{tx}^n|^2\,\mathrm{d}x+\frac{b}{2}\int_0^L|\psi_{xx}^n|^2\,\mathrm{d}x+\frac{\kappa}{2}\int_0^L|\varphi_{xx}^n+\psi_x^n|^2\,\mathrm{d}x+\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1|z_x^n|^2\,\mathrm{d}\rho\,\mathrm{d}x.$$

Thus integrating (3.17) *from* 0 *to* t*, we obtain from our choice of initial data that,*

$$\begin{aligned}F^n(t)-F^n(0)\le{}&\frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L|z_x^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s\\&+\frac12\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L|z_x^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

which means

$$\begin{aligned}&F^n(t)-\frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L|z_x^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s\\&-\frac12\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L|z_x^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s\le F^n(0),\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}F^n(0)={}&\frac{\rho_1}{2}\int_0^L|\varphi_{tx}^n(0)|^2\,\mathrm{d}x+\frac{b}{2}\int_0^L|\psi_{xx}^n(0)|^2\,\mathrm{d}x\\&+\frac{\kappa}{2}\int_0^L|\varphi_{xx}^n(0)+\psi_x^n(0)|^2\,\mathrm{d}x+\frac{\zeta}{2}\tau(0)\int_0^L\int_0^1|z_x^n(0)|^2\,\mathrm{d}\rho\,\mathrm{d}x.\end{aligned}$$

As $(\varphi^n(0),\varphi_t^n(0),\psi^n(0),z^n(0))=(\varphi_0^n,\varphi_1^n,\psi_0^n,z_0^n)\to(\varphi_0,\varphi_1,\psi_0,f_0)$ *strongly in* $\mathcal{H}_1$*, then there exists a positive constant* C_3 *such that* $F^n(0)\le C_3$. *Hence*

$$\begin{aligned}&F^n(t)-\frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L|z_x^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s\\&-\frac12\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L|z_x^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s\le C_3,\end{aligned}\tag{3.19}$$

which means that

$$\begin{aligned}&\frac{\rho_1}{2}\int_0^L|\varphi_{tx}^n|^2\,\mathrm{d}x+\frac{b}{2}\int_0^L|\psi_{xx}^n|^2\,\mathrm{d}x+\frac{\kappa}{2}\int_0^L|\varphi_{xx}^n+\psi_x|^2\,\mathrm{d}x+\frac{\zeta}{2}\tau(t)\int_0^L\int_0^1|z_x^n|^2\,\mathrm{d}\rho\,\mathrm{d}x\\&-\frac{1-d}{2}\Big[-\zeta+\frac{\mu_2}{\sqrt{1-d}}\Big]\int_0^t\int_0^L|z_x^n(x,1,s)|^2\,\mathrm{d}x\,\mathrm{d}s\\&-\frac12\Big[\zeta-(2\mu_1-\frac{\mu_2}{\sqrt{1-d}})\Big]\int_0^t\int_0^L|z_x^n(x,0,s)|^2\,\mathrm{d}x\,\mathrm{d}s\le C_3,\end{aligned}\tag{3.20}$$

where C_3 is a positive constant independent of t and n but depending on initial data. Then we can conclude that

$$\begin{aligned}
\{\varphi^n\} &\quad \text{is bounded in} \quad L^\infty\left(0,T,H^2(0,L)\cap H_0^1(0,L)\right),\\
\{\varphi_t^n\} &\quad \text{is bounded in} \quad L^\infty\left(0,T,H_0^1(0,L)\right),\\
\{\psi^n\} &\quad \text{is bounded in} \quad L^\infty\left(0,T,H_\star^2(0,L)\right),\\
\{z^n\} &\quad \text{is bounded in} \quad L^\infty(0,T,H^1((0,L)\times(0,1))).
\end{aligned}$$

This implies that

$$\begin{aligned}
\{\varphi^n\}\to\varphi &\quad \text{weakly star in} \quad L^\infty\left(0,T,H^2(0,L)\cap H_0^1(0,L)\right),\\
\{\varphi_t^n\}\to\varphi_t &\quad \text{weakly star in} \quad L^\infty\left(0,T,H_0^1(0,L)\right),\\
\{\psi^n\}\to\psi &\quad \text{weakly star in} \quad L^\infty\left(0,T,H_\star^2(0,L)\right),\\
\{z^n\}\to z &\quad \text{weakly star in} \quad L^\infty\left(0,T,H^1((0,L)\times(0,1))\right).
\end{aligned}$$

From above limits we conclude that $(\varphi,\varphi_t,\psi,z)$ is a strong solution satisfying

$$\begin{cases}
\varphi\in L^\infty\left(0,T,H^2(0,L)\cap H_0^1(0,L)\right),\\
\varphi_t\in L^\infty\left(0,T,H_0^1(0,L)\right),\\
\psi\in L^\infty\left(0,T,H_\star^2(0,L)\right),\\
z\in L^\infty\left(0,T,H^1((0,L)\times(0,1))\right).
\end{cases}\tag{3.21}$$

Step 6 ***Continuous dependence***
Let $U(t)=(\varphi,\varphi_t,\psi,z)$ and $V(t)=(\varphi',\varphi'_t,\psi',z')$ be the stronger weak solutions of problem (2.2) *corresponding to initial data $U(0)=(\varphi_0,\varphi_1,\psi_0,z_0)$, $V(0)=(\varphi'_0,\varphi'_1,\psi'_0,z'_0)\in\mathcal{H}_1$. Then $(\Phi,\Phi_t,\Psi,Z)=U(t)-V(t)$ is a solution of the system*

$$\begin{cases}
\rho_1\Phi_{tt}(x,t)-\kappa(\Phi_x(x,t)+\Psi(x,t))_x+\mu_1\Phi_t(x,t)+\mu_2 Z(x,1,t)=0 \ \text{in}\ (0,L)\times(0,+\infty), & (3.22)\\
-b\,\Phi_{xx}(x,t)+\kappa(\Phi_x(x,t)+\Psi(x,t))=0 \ \text{in}\ (0,L)\times(0,+\infty), & (3.23)\\
\tau(t)Z_t(x,\rho,t)+(1-\rho\,\tau'(t))Z_\rho(x,\rho,t)=0 \ \text{in}\ (0,L)\times(0,1)\times(0,+\infty),
\end{cases}$$

with initial data $\Big(\Phi(0),\Phi_t(0),\Psi(0),Z(0)\Big)=U(0)-V(0)$.

Multiplying (3.22) *by Φ_t and* (3.23) *by Ψ_t and integrating, we obtain*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{G}(t)+\mu_2\int_0^L\Phi_t(x,t)Z(x,1,t)\,\mathrm{d}x+\mu_1\int_0^L|\Phi_t(x,t)|^2\,\mathrm{d}x=0$$

with

$$\mathcal{G}(t)=\frac{\rho_1}{2}\int_0^L|\Phi_t(x,t)|^2\,\mathrm{d}x+\frac{k}{2}\int_0^L|\Phi_x(x,t)+\Psi(x,t)|^2\,\mathrm{d}x+\frac{b}{2}\int_0^L|\Psi_x(x,t)|^2\,\mathrm{d}x.$$

Applying Young's inequality, we obtain the existence of a constant M_1 such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{G}(t) &\le M_1\int_0^L|\Phi_t(x,t)|^2\,\mathrm{d}x\\
&\le M_1\Big[\int_0^L|\Phi_t(x,t)|^2\,\mathrm{d}x+\int_0^L|\Phi_x(x,t)+\Psi(x,t)|^2\,\mathrm{d}x+\int_0^L|\Psi_x(x,t)|^2\,\mathrm{d}x\Big].
\end{aligned}\tag{3.24}$$

By integrating (3.24) *from* 0 *to* t *we get*

$$\mathcal{G}(t) \leq \mathcal{G}(0) + M_1 \int_0^t \Big[\int_0^L |\Phi_\lambda(x,\lambda)|^2 \, \mathrm{d}x + \int_0^L |\Phi_x(x,\lambda) + \Psi(x,\lambda)|^2 \, \mathrm{d}x + \int_0^L |\Psi_x(x,\lambda)|^2 \, \mathrm{d}x \Big] \, \mathrm{d}\lambda. \tag{3.25}$$

On the other hand we know that for $M_2 = \min\left\{\frac{\rho_1}{2}, \frac{k}{2}, \frac{b}{2}\right\}$, *we have*

$$\mathcal{G}(\lambda) \geq M_2 \Big[\int_0^L |\Phi_\lambda(x,\lambda)|^2 \, \mathrm{d}x + \int_0^L |\Phi_x(x,\lambda) + \Psi(x,\lambda)|^2 \, \mathrm{d}x + \int_0^L |\Psi_x(x,\lambda)|^2 \, \mathrm{d}x \Big]. \tag{3.26}$$

From (3.25) *and* (3.26) *we have*

$$\mathcal{G}(t) \leq \mathcal{G}(0) + \frac{M_1}{M_2} \int_0^t \mathcal{G}(\lambda) \, \mathrm{d}\lambda. \tag{3.27}$$

Applying Gronwall's inequality we have

$$\mathcal{G}(t) \leq \mathcal{G}(0) \, e^{Rt} \tag{3.28}$$

with the positive constant $R = \frac{M_1}{M_2}$. *So we obtain the continuous dependence of solution on the initial data. In particular the solution is unique.*

□

4 Exponential stability

Theorem 4.1 *Assume that* (1.2)-(1.4) *hold. Then there are two positive constants* K_1 *and* K_2 *such that*

$$\forall t \geq 0 : \; E(t) \leq K_1 E(0) e^{-K_2 t}.$$

Proof. Let

$$\mathcal{F}(t) = 2c\rho_1 \int_0^L \varphi_t \varphi \, \mathrm{d}x + \mu_1 c \int_0^L |\varphi|^2 \, \mathrm{d}x,$$

where c is a constant whose conditions will be specified later.

Multiplying the first equation of (2.2) by $2c\varphi$ and integrating from 0 to L, we obtain

$$2c\rho_1 \int_0^L \varphi_{tt}(x,t)\varphi(x,t) \, \mathrm{d}x - 2c\kappa \int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]_x \varphi(x,t) \, \mathrm{d}x + 2c\mu_1 \int_0^L \varphi_t(x,t)\varphi(x,t) \, \mathrm{d}x + 2c\mu_2 \int_0^L z(x,1,t)\varphi(x,t) \, \mathrm{d}x = 0.$$

As $\varphi_{tt}\varphi = \dfrac{\partial}{\partial t}(\varphi_t\varphi) - |\varphi_t|^2$ we have

$$2c\rho_1 \int_0^L \frac{\partial}{\partial t}[\varphi_t(x,t)\varphi(x,t)]\,\mathrm{d}x - 2c\rho_1 \int_0^L |\varphi_t|^2\,\mathrm{d}x + 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]\,\varphi_x(x,t)\,\mathrm{d}x$$
$$+ c\mu_1 \int_0^L \frac{\partial}{\partial t}|\varphi(x,t)|^2\,\mathrm{d}x + 2c\mu_2 \int_0^L z(x,1,t)\varphi(x,t)\,\mathrm{d}x = 0,$$

which means that

$$\frac{\partial}{\partial t}\int_0^L \left[2c\rho_1\varphi_t(x,t)\varphi(x,t) + c\mu_1|\varphi(x,t)|^2\right]\mathrm{d}x - 2c\rho_1\int_0^L |\varphi_t|^2\,\mathrm{d}x$$
$$+ 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]\,\varphi_x(x,t)\,\mathrm{d}x + 2c\mu_2 \int_0^L z(x,1,t)\varphi(x,t)\,\mathrm{d}x = 0. \tag{4.1}$$

Multiplying the second equation of (2.2) by $2c\psi$ and integrating over 0 to L, we obtain

$$-2cb \int_0^L \psi_{xx}(x,t)\psi\,\mathrm{d}x + 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]\,\psi\,\mathrm{d}x = 0.$$

By integrating by parts, we have

$$-2cb\,[\psi_x\psi]_0^L + 2cb\int_0^L |\psi_x|^2\,\mathrm{d}x + 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]\,\psi\,\mathrm{d}x = 0.$$

Since ψ_x is zero at 0 and L, we have

$$2cb\int_0^L |\psi_x|^2\,\mathrm{d}x + 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]\,\psi\,\mathrm{d}x = 0. \tag{4.2}$$

By summing (4.1)-(4.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{F}(t) = 2c\rho_1\int_0^L |\varphi_t|^2\,\mathrm{d}x - 2c\kappa\int_0^L |\varphi_x(x,t) + \psi(x,t)|^2\,\mathrm{d}x$$
$$- 2c\mu_2\int_0^L z(x,1,t)\varphi(x,t)\,\mathrm{d}x - 2cb\int_0^L |\psi_x|^2\,\mathrm{d}x.$$

By applying the inequality of Young at $-2c\mu_2 \displaystyle\int_0^L z(x,1,t)\varphi(x,t)\,\mathrm{d}x$, for all $\varepsilon > 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{F}(t) &\le 2c\rho_1\int_0^L |\varphi_t|^2\,\mathrm{d}x - 2c\kappa\int_0^L |\varphi_x(x,t) + \psi(x,t)|^2\,\mathrm{d}x \\ &\quad + \frac{(2c\mu_2)^2}{4\varepsilon}\int_0^L |z(x,1,t)|^2\,\mathrm{d}x + \varepsilon\int_0^L |\varphi(x,t)|^2\,\mathrm{d}x - 2cb\int_0^L |\psi_x|^2\,\mathrm{d}x \\ &\le 2c\rho_1\int_0^L |\varphi_t|^2\,\mathrm{d}x - 2c\kappa\int_0^L |\varphi_x(x,t) + \psi(x,t)|^2\,\mathrm{d}x + \varepsilon\int_0^L |\varphi(x,t)|^2\,\mathrm{d}x \\ &\quad + \frac{(c\mu_2)^2}{\varepsilon}\int_0^L |z(x,1,t)|^2\,\mathrm{d}x - 2cb\int_0^L |\psi_x|^2\,\mathrm{d}x.\end{aligned}$$

By applying the inequality of Poincaré at $\varepsilon \int_0^L |\varphi(x,t)|^2 \,\mathrm{d}x$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 \,\mathrm{d}x - 2c\kappa \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + \varepsilon L^2 \int_0^L |\varphi_x(x,t)|^2 \,\mathrm{d}x + \frac{(c\mu_2)^2}{\varepsilon} \int_0^L |z(x,1,t)|^2 \,\mathrm{d}x - 2cb \int_0^L |\psi_x|^2 \,\mathrm{d}x. \tag{4.3}$$

We know that

$$\int_0^L |\varphi_x(x,t)|^2 \,\mathrm{d}x \leq 2\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + 2\int_0^L |\psi(x,t)|^2 \,\mathrm{d}x.$$

By Poincaré's inequality we have

$$\int_0^L |\varphi_x(x,t)|^2 \,\mathrm{d}x \leq 2\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + 2L^2\int_0^L |\psi_x(x,t)|^2 \,\mathrm{d}x. \tag{4.4}$$

Multiplying (4.4) by εL^2 we have

$$\begin{aligned}\varepsilon L^2 \int_0^L |\varphi_x(x,t)|^2 \,\mathrm{d}x &\leq 2\varepsilon L^2 \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + 2\varepsilon L^4 \int_0^L |\psi_x(x,t)|^2 \,\mathrm{d}x \\ &= \frac{2\varepsilon L^2}{k}\cdot k \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + \frac{2\varepsilon L^4}{b}\cdot b \int_0^L |\psi_x(x,t)|^2 \,\mathrm{d}x.\end{aligned}$$

If we put $c = \max\left\{\frac{2\varepsilon L^2}{k}, \frac{2\varepsilon L^4}{b}\right\}$ we obtain

$$\varepsilon L^2 \int_0^L |\varphi_x(x,t)|^2 \,\mathrm{d}x \leq ck \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + cb\int_0^L |\psi_x(x,t)|^2 \,\mathrm{d}x. \tag{4.5}$$

From (4.3) and (4.5), for all $\varepsilon > 0$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 \,\mathrm{d}x - 2c\kappa \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + ck \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + cb \int_0^L |\psi_x(x,t)|^2 \,\mathrm{d}x + \frac{(c\mu_2)^2}{\varepsilon} \int_0^L |z(x,1,t)|^2 \,\mathrm{d}x - 2cb \int_0^L |\psi_x|^2 \,\mathrm{d}x,$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 \,\mathrm{d}x - c\kappa \int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + \frac{(c\mu_2)^2}{\varepsilon} \int_0^L |z(x,1,t)|^2 \,\mathrm{d}x - cb \int_0^L |\psi_x|^2 \,\mathrm{d}x. \tag{4.6}$$

Let's put

$$I(t) = c\zeta e^{2M}\tau(t) \int_0^L \int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho \,\mathrm{d}x.$$

We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t) &= c\zeta e^{2M}\tau'(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x \\
&+ c\zeta e^{2M}\tau(t)\Big[-2\int_0^L\int_0^1 \rho\tau'(t)e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + 2\int_0^L\int_0^1 e^{-2\tau(t)\rho} z z_t \,\mathrm{d}\rho\,\mathrm{d}x\Big] \\
&= c\zeta e^{2M}\tau'(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x - 2c\zeta e^{2M}\int_0^L\int_0^1 \rho\tau(t)\tau'(t)e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x \\
&+ 2c\zeta e^{2M}\int_0^L\int_0^1 e^{-2\tau(t)\rho}\Big(\tau(t)z_t\Big) z \,\mathrm{d}\rho\,\mathrm{d}x.
\end{aligned}$$

From the third equation of (2.2) we have $\tau(t)z_t(\rho,t) = -(1-\rho\tau'(t))z_\rho(\rho,t)$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t) &= c\zeta e^{2M}\tau'(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x - 2c\zeta e^{2M}\int_0^L\int_0^1 \rho\tau(t)\tau'(t)e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x \\
&- 2c\zeta e^{2M}\int_0^L\int_0^1 e^{-2\tau(t)\rho}\Big(1-\rho\tau'(t)\Big) z z_\rho \,\mathrm{d}\rho\,\mathrm{d}x \\
&= -2c\zeta e^{2M}\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L\int_0^1 \frac{\partial}{d\rho}\Big[e^{-2\tau(t)\rho}\Big(\rho\tau'(t)-1\Big)|z|^2\Big] \,\mathrm{d}\rho\,\mathrm{d}x \\
&= -2c\zeta e^{2M}\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L \Big[e^{-2\tau(t)\rho}\Big(\rho\tau'(t)-1\Big)|z|^2\Big]_0^1 \mathrm{d}x \\
&= -2c\zeta e^{2M}\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L e^{-2\tau(t)}\Big(\tau'(t)-1\Big)|z(x,1,t)|^2 \,\mathrm{d}x \\
&+ c\zeta e^{2M}\int_0^L e^0|z(x,0,t)|^2 \,\mathrm{d}x.
\end{aligned}$$

From (1.3) we have $\tau'(t)-1<0$. This leads to

$$c\zeta e^{2M}\int_0^L e^{-2\tau(t)}\Big(\tau'(t)-1\Big)|z(x,1,t)|^2 \,\mathrm{d}x \le 0.$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \le -c\zeta e^{2M}\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L |z(x,0,t)|^2 \,\mathrm{d}x.$$

By (1.2) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t) &\le -c\zeta e^{2M}\tau(t)\int_0^L\int_0^1 e^{-2M}|z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L |z(x,0,t)|^2 \,\mathrm{d}x \\
&\le -c\zeta\tau(t)\int_0^L\int_0^1 |z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L |z(x,0,t)|^2 \,\mathrm{d}x. \qquad (4.7)
\end{aligned}$$

Summing (4.6) and (4.7) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(I(t)+\mathcal{F}(t)) \le 2c\rho_1\int_0^L |\varphi_t|^2 \,\mathrm{d}x - c\kappa\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 \,\mathrm{d}x + \frac{(c\mu_2)^2}{\varepsilon}\int_0^L |z(x,1,t)|^2 \,\mathrm{d}x \\
- cb\int_0^L |\psi_x|^2 \,\mathrm{d}x - c\zeta\tau(t)\int_0^L\int_0^1 |z|^2 \,\mathrm{d}\rho\,\mathrm{d}x + c\zeta e^{2M}\int_0^L |z(x,0,t)|^2 \,\mathrm{d}x.
\end{aligned}$$

Thereby

$$\frac{\mathrm{d}}{\mathrm{d}t}(I(t)+\mathcal{F}(t)) \leq -2cE(t) + (c\zeta e^{2M} + 3c\rho_1)\int_0^L |z(x,0,t)|^2\,\mathrm{d}x + \frac{(c\mu_2)^2}{\varepsilon}\int_0^L |z(x,1,t)|^2\,\mathrm{d}x. \quad (4.8)$$

Take

$$\forall t > 0: \ \mathcal{L}(t) = NE(t) + I(t) + \mathcal{F}(t),$$

where N is a constant whose conditions we will specify later. We obtain from (2.8) and (4.8)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L}(t) \leq -2E(t) + \Big[\frac{N}{2}\Big(\zeta - 2\mu_1 + \frac{\mu_2}{\sqrt{1-d}}\Big) + c\zeta e^{2M} + 3c\rho_1\Big]\int_0^L |z(x,0,t)|^2\,\mathrm{d}x$$
$$+ \Big[N\frac{1-d}{2}\Big(-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big) + \frac{(c\mu_2)^2}{\varepsilon}\Big]\int_0^L |z(x,1,t)|^2\,\mathrm{d}x.$$

If we take

$$N \geq \max\left\{\frac{-c\zeta e^{2M} - 3c\rho_1}{\zeta - 2\mu_1 + \frac{\mu_2}{\sqrt{1-d}}}, \frac{-c^2\mu_2^2}{(1-d)\Big(-\zeta + \frac{\mu_2}{\sqrt{1-d}}\Big)}\right\},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L}(t) \leq -2\,c\,E(t). \quad (4.9)$$

Moreover, it is easy to see the existence of two constants α and β such that

$$\forall t \geq 0: \ \alpha\,E(t) \leq \mathcal{L}(t) \leq \beta\,E(t). \quad (4.10)$$

By (4.9) and (4.10) we obtain

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-2\,c\,E(t)}{\mathcal{L}(t)} \leq \frac{-2\,c\,E(t)}{\beta\,E(t)},$$

which means that

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-2\,c}{\beta}. \quad (4.11)$$

Integrating (4.11) from 0 to t we have

$$\ln\mathcal{L}(t) - \ln\mathcal{L}(0) \leq \frac{-2\,c}{\beta}\,t,$$

which means that

$$\mathcal{L}(t) \leq \mathcal{L}(0)\,e^{\frac{-2\,c}{\beta}\,t}. \quad (4.12)$$

By (4.10) we obtain

$$\alpha\,E(t) \leq \mathcal{L}(t)$$

and

$$\mathcal{L}(0) \leq \beta\,E(0).$$

Thus (4.12) can be written

$$E(t) \leq \frac{\beta E(0)}{\alpha}\,e^{\frac{-2\,c\,t}{\beta}}.$$

We obtain by taking $K_1 = \frac{\beta}{\alpha}$ and $K_2 = \frac{2c}{\beta}$

$$\forall t \geq 0: \ E(t) \leq K_1\,E(0)\,e^{-K_2\,t}.$$

The proof of the theorem is thus completed. □

5 Conclusion

In this article, we focus on the stability of a Timoshenko system with a variable delay. The well-posedness of the system is established in the second Section using the Faedo-Galerkin method. An exponential stability result is then established using a Lyapunov function.

As an outlook, it would be interesting to repeat the study, this time introducing the delay term into the second equation of problem (1.1).

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