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WEAK NONTRIVIAL SOLUTIONS FOR DISCRETE NONLINEAR PROBLEMS OF KIRCHHOFF TYPE WITH VARIABLE EXPONENTS IN A TWO DIMENSIONAL HILBERT SPACE

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Abstract. In this paper, we deal with the existence results for weak solutions to some problems of Kirchhoff type. The originality of this work lies in the generalization of discrete nonlinear problem of Kirchhoff type in a two dimensional Hilbert space with variable exponents. The proof is mainly based on the critical point theory. We then show that the energy functional associated to our problem is weakly lower semi-continuous, coercive and bounded from below.

Keywords: Discrete Kirchhoff type problem; Dirichlet boundary; critical point theory; weak solution; Hilbert space.

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1 Introduction

As a reminder, differences equations have been a very active field of investigation in recent years. We refer to the recent results ([1]-[3], [5], [7]-[12]) and the other references therein. Through these references, we can say that in different research areas, such as calculus of variations and elasticity, computer science, mechanical engineering, control systems, non-Newtonian fluids, image processing and many others, the dynamics of some important problems are naturally modelled by nonlinear difference equations.
For example, in [10], the authors proved the existence of weak solutions for discrete nonlinear

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system of Kirchhoff type in one dimensional Hilbert space with suitable norms.
So, our goal is to contribute to the extension of the study of difference equations in two dimension. These models are of independent interest since their mathematical structure has a different nature. The main obstacles to this extension is related to the definition of forward difference operator Δ, of spaces, of norms and the computation of the Gateaux derivative. In two dimension, there are several ways to overcome this problem, see Du and Zhou [4], Ibrango *et al.* [6]. One of them is to use the definition due to Ibrango *et al.* [6], where the authors considered the following equation

$$-\Delta\big(a(k-1,h-1,\Delta u(k-1,h-1))\big) = f(k,h,u(k,h)), \ \ (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2], \quad (1.1)$$

where

$$\Delta u(k,h) = u(k+1,h+1) - u(k,h) \quad (1.2)$$

is the forward difference operator, and for $n,m\in\mathbb{N}$, with $n\leq m$, the set $\mathbb{N}[n,m]$ is the discrete interval $\{n,n+1,\ldots,m\}$.

A particular case is studied in [4], where the authors considered the following p-Laplacian problem:

$$\Delta_1(\phi_p(\Delta_1 u(i-1,j))) + \Delta_2(\phi_p(\Delta_2 u(i,j-1))) + \lambda f((i,j),u(i,j)) = 0, \quad (1.3)$$

for any $(i,j)\in\mathbb{N}[1,m]\times\mathbb{N}[1,n]$, with

$$\Delta_1 u(i,j) = u(i+1,j) - u(i,j) \quad \text{and} \quad \Delta_2 u(i,j) = u(i,j+1) - u(i,j).$$

In the literature, to our best knowledge, there were no such existence results for a Kirchhoff anisotropic or isotropic problem in two dimension which is nevertheless discrete variants of the anisotropic or isotropic partial differential equations. So, we investigate the existence of solutions for the following nonlinear discrete Dirichlet boundary value problem:

$$\begin{cases} -M(A(k-1,h-1,\Delta u(k-1,h-1)))\,\Delta a(k-1,h-1,\Delta u(k-1,h-1)) \\ \qquad\qquad\qquad = f(k,h,u(k,h)) \ \text{ for } (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2], \\ \qquad u(k,h) = 0 \qquad\qquad\quad \text{for } (h,k)\in\Gamma, \end{cases} \quad (1.4)$$

where

$$\Gamma = (\{0,T_1+1\}\times\mathbb{N}[0,T_2+1]) \cup (\mathbb{N}[0,T_1+1]\times\{0,T_2+1\})$$

is the boundary of the domain $\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1]$, Δ is defined in (1.2), and

$$M:\mathbb{R}\longrightarrow\mathbb{R}, \quad a,A:\mathbb{N}[0,T_1]\times\mathbb{N}[0,T_2]\times\mathbb{R}\longrightarrow\mathbb{R}, \quad f:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}\longrightarrow\mathbb{R}$$

are functions that will be defined in the second section through assumptions.

We use a critical point theory in order to establish some existence results of weak solutions. For this, we need to transform the problem of the existence of solutions into the problem of the existence of a minimizer for some associated energy functional.

In this paper we consider the same boundary conditions as in [6], but the expression

$$M\left(A(k-1,h-1,\Delta u(k-1,h-1))\right)$$

that appears in this paper makes the work much more general. Indeed, if we take $M(t)=1$ in the problem (1.4), we obtain the problem studied in [6].

The outline of the paper is as follows. In Section 2, we will recall briefly some basic definitions and preliminaries facts which will be used throughout the following sections. The main existence result is stated and proved in Section 3. In the last section of this paper, we give a conclusion.

2 Preliminaries

We consider the function space

$$H = \{u : \mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1] \longrightarrow \mathbb{R} \quad \text{such that} \quad u(k, h) = 0, \quad \forall\ (h, k) \in \Gamma\}.$$

The space H is a $(T_1 \times T_2)$-dimensional Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} u(k,h)v(k,h)$$

and the associated norm defined by

$$\|u\| = \left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^2\right)^{1/2}.$$

However, we introduce another norm on the space H, namely

$$|u|_m = \left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^m\right)^{1/m}, \ \forall\, m \geq 2.$$

In this paper, the function p is defined such that:

$$p : \mathbb{N}[0, T_1] \times \mathbb{N}[0, T_2] \longrightarrow (1, +\infty). \tag{2.1}$$

For the data a, M and f we impose the following conditions.

$$a(k, h, .) : \mathbb{R} \longrightarrow \mathbb{R} \ \text{ is continuous}, \ \forall\ (k, h) \in \mathbb{N}[0, T_1] \times \mathbb{N}[0, T_2] \tag{2.2}$$

and there exists a mapping $A : \mathbb{N}[0, T_1] \times \mathbb{N}[0, T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$a(k, h, \xi) = \frac{\partial}{\partial \xi} A(k, h, \xi) \quad \text{and} \quad A(k, h, 0) = 0, \quad \forall (k, h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2]. \tag{2.3}$$

We also assume that there exists a positive constant c_1 such that

$$|a(k, h, \xi)| \leq c_1 \left(1 + |\xi|^{p(k,h)-1}\right). \tag{2.4}$$

The following relations hold true for all $(k, h) \in \mathbb{N}[0, T_1] \times \mathbb{N}[0, T_2]$:

$$(a(k, h, \xi) - a(k, h, \eta))\,(\xi - \eta) > 0, \ \forall\, \xi, \eta \in \mathbb{R} \ \text{ with } \xi \neq \eta \tag{2.5}$$

and

$$|\xi|^{p(k,h)} \leq a(k, h, \xi)\xi \leq p(k, h)A(k, h, \xi), \ \forall\, \xi \in \mathbb{R}. \tag{2.6}$$

We also assume that the function $M : (0, +\infty) \longrightarrow (0, +\infty)$ is continuous, non decreasing and there exist two positive real numbers $c_2 \leq c_3$ and $\alpha \geq 1$ with

$$c_2 t^{\alpha-1} \leq M(t) \leq c_3 t^{\alpha-1}, \quad \text{for} \quad t > 0. \tag{2.7}$$

Example 2.1
Let the following functions

$$A(k,h,\xi) = \frac{1}{p(k,h)}\big((1+|\xi|^2)^{p(k,h)/2} - 1\big),$$

where

$$a(k,h,\xi) = \big(1+|\xi|^2\big)^{(p(k,h)-2)/2}\xi,\ \ \forall\,(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2],\ \xi\in\mathbb{R}$$

and

$$M(t) = a + bt, \quad f(k,h,t) = 0,\ t\in\mathbb{R}.$$

The conditions on the functions A, a, M and f are checked.

We will use the following notations:

$$p^- = \min_{k\in\mathbb{N}[0,T_1]}\left(\min_{h\in\mathbb{N}[0,T_2]} p(k,h)\right) \quad \text{and} \quad p^+ = \max_{k\in\mathbb{N}[0,T_1]}\left(\max_{h\in\mathbb{N}[0,T_2]} p(k,h)\right). \tag{2.8}$$

For each couple $(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$, the function $f(k,h,.):\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and there exists a constant $c_4>0$ and $r:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow[2,+\infty)$ such that

$$|f\,(k,h,x)\,| \le c_4\left(1+|x|^{r(k,h)-1}\right), \tag{2.9}$$

where $2\le r(k,h)<p^-$ for all $(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$.
In what follows, we let

$$r^- = \min_{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]} r(k,h) \qquad \text{and} \qquad r^+ = \max_{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]} r(k,h).$$

We denote

$$F(k,h,x) = \int_0^x f(k,h,s)\,\mathrm{d}s \ \text{ for }\ (k,h,x)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}$$

and we deduce that there exists a constant $c_5>0$ such that

$$|F(k,h,x)| \le c_5\left(1+|x|^{r(k,h)}\right). \tag{2.10}$$

We employ the following assumption to show that the weak solution is nontrivial: there exists constants $c_6, c_7>0$ and $\mu>\alpha p^+$ such that

$$F(k,\xi) \ge c_6|\xi|^\mu - c_7. \tag{2.11}$$

We need the following auxiliary result throughout our paper.

Lemma 2.2 *(see [6])*
For any function $u\in H$ with $\|u\|>1$, there exist constants $c_8, c_9>0$ such that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \ge c_8\|u\|^{p^-} - c_9. \tag{2.12}$$

3 Main Results

In this section we study the existence of nontrivial weak solution that we state in the following theorem. By a weak solution for problem (1.4) we understand a function $u \in H$ such that

$$\begin{cases} M\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1,h-1,\Delta u(k-1,h-1)\right)\right) \\ \times \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1) \\ = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f\left(k,h,u(k,h)\right)v(k,h) \end{cases} \tag{3.1}$$

for any $v \in H$.

Theorem 3.1 *Assume that (2.2)-(2.1), (2.9)-(2.10) are satisfied. Then there is at least one nontrivial weak solution for problem (1.4).*

To study the boundary value Problem (1.4), we define the following energy functional $J : H \longrightarrow \mathbb{R}$

$$J(u) = \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1,h-1,\Delta u(k-1,h-1)\right)\right) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F\left(k,h,u(k,h)\right) \tag{3.2}$$

where $\widehat{M} : \mathbb{R} \longrightarrow \mathbb{R}$ is a primitive of M, namely, $\widehat{M}(x) = \int_0^x M(s)\,\mathrm{d}s$.

This energy functional is vastly different from the energy functions defined before this work. Thus we indicate its properties. We can see that the functional J is of class $C^1(H,\mathbb{R})$, so well defined and Gateaux differentiable.
For any $u \in H$, let us put $J(u) = I(u) - L(u)$ where

$$I(u) = \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1,h-1,\Delta u(k-1,h-1)\right)\right)$$

and

$$L(u) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F\left(k,h,u(k,h)\right).$$

For $v \in H$, we use

$$\langle J'(u), v\rangle = \lim_{h\to 0^+}\frac{I(u+hv)-I(u)}{h} - \lim_{h\to 0^+}\frac{L(u+hv)-L(u)}{h}$$

to have the Gateaux derivative of J' at u as

$$\begin{aligned}\langle J'(u), v\rangle &= M\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1, h-1, \Delta u(k-1, h-1)\right)\right)\\ &\times \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1, h-1, \Delta u(k-1, h-1))\Delta v(k-1, h-1)\\ &- \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f\left(k, h, u(k,h)\right) v(k,h) \qquad (3.3)\end{aligned}$$

for all $v \in H$.

The objective is to show that the functional J admits at least one minimum and this minimum is a weak solution of problem (1.4). The following results prove Theorem 3.1.

Proposition 3.2 *The functional J is coercive and bounded from below.*

Proof. Writing $S = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1, h-1, \Delta u(k-1, h-1)\right)$, we have

$$\begin{aligned}J(u) &= \widehat{M}\left(S\right) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F\left(k, h, u(k,h)\right)\\ &= \int_0^S M(x)\,\mathrm{d}x - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\int_0^{u(k,h)} f(k,h,x)\,\mathrm{d}x\\ &\geq \frac{c_2}{\alpha} S^\alpha - c_5 \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(1 + |u(k,h)|^{r(k,h)}\right)\\ &\geq \frac{c_2}{\alpha\left(p^+\right)^\alpha}\left[\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1, h-1)|^{p(k-1,h-1)}\right]^\alpha\\ &\quad -c_5\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{r(k,h)} - c_5 T_1 T_2. \qquad (3.4)\end{aligned}$$

For the coerciveness, we take $\|u\| > 1$. According to the Lemma 2.2 we get

$$J(u) \geq \frac{c_2}{\alpha\left(p^+\right)^\alpha}\left[c_8\,\|u\|^{p^-} - c_9\right]^\alpha - c_2\,c_{10}\left(\|u\|^{r^+} + \|u\|^{r^-}\right) - c_5\,T_1\,T_2. \qquad (3.5)$$

Hence, since $p^- > 1$, the functional J is coercive.
For the boundedness, for all u in H we have

$$\begin{aligned}J(u) &\geq -\left[c_2\,c_{10}\left(\|u\|^{r^+} + \|u\|^{r^-}\right) + c_5\,T_1\,T_2\right]\\ &\geq -\infty.\end{aligned}$$

□

Proposition 3.3 *The functional J is weakly lower semi-continuous.*

Proof. According to the assumptions (2.3), (2.5), (2.6) and the fact that the function M is continuous, non decreasing, we obtain that the energy functional J is convex and continuous. Then J is weakly lower semi-continuous. □

Proof. (of Theorem 3.1)
Since the functional J is weakly lower semi-continuous, coercive and bounded from below, then there exists $\bar{u} \in H$ such that $J(\bar{u}) \leq J(u)$ for all $u \in H$.

Using the relation between critical points of J and problem (1.4), we deduce that $\bar{u}$ is a weak solution of problem (1.4). Indeed, relation (3.3) implies that the weak solutions of (1.4) coincide with the critical points of the energy functional J.

Now, let us prove that the solution $\bar{u}$ is nontrivial. For $u \in H$ with $u \neq 0$ and $t > 1$, we have

$$\begin{aligned}
J(tu) &= \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1,h-1,t\,\Delta u(k-1,h-1)\right)\right) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F\left(k,h,tu(k,h)\right) \\
&\leq \frac{c_3}{\alpha-1}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A\left(k-1,h-1,t\,\Delta u(k-1,h-1)\right)\right)^{\alpha} \\
&\quad - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(c_6\,|tu(k,h)|^{\mu} - c_7\right) \\
&\leq \frac{c_3}{\alpha-1}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} c_{11}\Big(1+|t\,\Delta u(k-1,h-1)|^{p(k-1,h-1)}\Big)\right)^{\alpha} \\
&\quad - c_6\,t^{\mu}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\mu} + c_7\,T_1\,T_2 \\
&\leq \frac{c_3\,(c_{11})^{\alpha}}{\alpha-1}\Big((T_1+1)(T_2+1) + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} t^{p(k-1,h-1)}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\Big)^{\alpha} \\
&\quad - c_6\,t^{\mu}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\mu} + c_7\,T_1\,T_2.
\end{aligned}$$

Then, there exist positive constants c_{12}, c_{13} such that

$$\begin{aligned}
J(tu) &\leq c_{12}\,t^{\alpha p^+}\left(\frac{1}{t^{p^+}}(T_1+1)(T_2+1) + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^{\alpha} \\
&\quad - c_6\,t^{\mu}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\mu} + c_{13}.
\end{aligned}$$

Consequently, since $\mu > \alpha p^+$, for sufficiently large $t > 1$, we assert that $J(tu) < 0$.
Therefore there exists $v \in H$ with $v \neq 0$ such that

$$J(v) < 0,$$

which shows that $\bar{u} \neq 0$ necessarily.
We conclude that there exists $\bar{u} \in H$ with $\bar{u} \neq 0$ who is a critical point of J. □

4 Conclusion

In this paper, we study the existence of nontrivial weak solutions for discrete nonlinear problems in two dimensional Hilbert space. The minimization technique allows us to show that the energy functional admits at least one nontrivial critical point which is a weak solution of the associated problem.

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