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ON GLOBAL EXISTENCE RESULTS FOR SOME NON-AUTONOMOUS EVOLUTION SYSTEMS OF VOLTERRA TYPE IN FRÉCHET SPACES

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Abstract. In this paper, we discuss the global existence of mild solutions for some non-autonomous deterministic and stochastic integro-differential systems with time-dependent delay in Fréchet spaces. We develop several existence results that are built on the notion of two-parameter resolvent operators in the sense of Grimmer. Examples are provided to show how the general conclusions reached in this paper are made.

Keywords: C_0-semigroup, mild solution, nonlinear alternative of Avramescu, partial functional integro-differential equation with delay, resolvent operator.

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1 Introduction

The study of integro-differential equations and inclusions has grown in importance in recent years, because they can model many complex situations and phenomena in science and engineering. Some papers that discuss the existence of mild solutions for these types of systems are [4, 35] and others. Moreover, many models that describe reality need to include a delay term, since some of the processes that affect the system dynamics depend on the past state. However, integro-differential equations or inclusions may not capture the randomness that is present in some scenarios. To address this issue stochastic models are suggested. Both delay and stochastic effects are common in various real-world systems. Stochastic integro-differential equations are a generalization of ordinary differential equations and can be used to describe different models of real systems and problems in areas such as population dynamics, finance, physics, biology, economics, and more; see [21, 9, 36]. Many studies on deterministic or stochastic (integro)-differential equations focused on a bounded interval, where they used different fixed point theorems in Banach spaces to show the existence of mild solutions for such equations. However, only a few studies looked at such equations on an unbounded interval and Fréchet spaces (see [30] for more details).

Fréchet spaces are a setting where deterministic evolution equations have many fascinating and significant results on the existence, uniqueness, and asymptotic behaviour of their solutions. Abbas *et al.* studied the existence and uniqueness of mild solutions for various types of fractional differential equations in Fréchet spaces in [1, 2, 3]. The existence and stability of solutions for many kinds of functional integral equations in Fréchet spaces are discussed in [16, 8]. Recently, the papers [19, 36] have explored the existence as well as uniqueness of mild solutions, and the controllability for some types of autonomous integro-differential equations with state-dependent nonlocal conditions in Fréchet spaces. We will extend some of these results concerning mild solutions to the following class of nonautonomous retarded integro-differential equations of Volterra type in a Banach space $\mathbf{X}$:

$$\begin{cases} x'(t) = \mathbb{A}(t)x(t) + \displaystyle\int_0^t \Upsilon(t,s)x(s)\,\mathrm{d}s + f_1(t,x_t) + f_2(t,x_t) \ \text{ for } t \geq 0, \\ x_0 = \varphi \ \text{ a history function.} \end{cases} \tag{1.1}$$

Here, $\mathbb{A}(t)$ and $\Upsilon(t,s)$ are closed linear operators with fixed domain $\mathsf{Dom}(\Upsilon) \supset \mathsf{Dom}(A)$. The nonlinearities $f_1, f_2 \colon [0,+\infty) \times \mathcal{F} \to \mathbf{X}$ are given functions. Moreover, $\mathcal{F} := \mathcal{C}([-r,+\infty);\mathbf{X})$ is a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. For $x \in \mathcal{F}$ and $t \geq 0$, x_t denotes the history function defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

Chang *et al.* [11] studied the global uniqueness of solutions for some stochastic integro-differential equations driven by a Wiener process in Fréchet spaces. Yang and Lu [36] examined the existence and controllability of fractional stochastic neutral functional integro-differential systems with state-dependent delay in Fréchet spaces. Anguraj and Ramkumar [5] showed the global existence and uniqueness results for some stochastic integro-differential equation with Poisson jumps in Fréchet spaces. Shu and Ndambomve [32] explored the existence and uniqueness of cylindrical mild solutions and exact cylindrical controllability results for some cylindrical stochastic integro-differential equations in Fréchet spaces. Salim *et al.* [31] obtained the existence and controllability results for some second-order functional differential equations with random effects in Fréchet spaces. All these papers deal with autonomous stochastic evolution systems. The study of nonautonomous stochastic retarded integro-differential system with fractional Brownian motion in Fréchet spaces is still missing in the literature. Therefore, the second aim of this paper is to investigate the existence and uniqueness of mild solutions for the first-order stochastic evolution equations of the abstract form:

$$\begin{cases} \mathrm{d}x(t)=\left[\mathbb{A}(t)x(t)+\displaystyle\int_0^t \Upsilon(t,s)x(s)\,\mathrm{d}s\right]\mathrm{d}t+h_1(t,x_t)\mathrm{dW}(t)+h_2(t)\mathrm{dW}^H(t) \text{ for } t\geq 0, \\ x_0=\varphi \text{ a history function,} \end{cases} \tag{1.2}$$

where $\{W^H(t) : t \geq 0\}$ is a cylindrical fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ and values in a separable Hilbert space K_0. Moreover, $\{\mathsf{W}(t) : t \geq 0\}$ is a cylindrical Wiener process on a separable Hilbert space K independent of W^H. The functions h_1 and h_2 are subject to some additional conditions.

The main difficulty to study the stochastic evolution equation (1.2) is due the fact that the fBm is neither a Markov process nor a semimartingale, unless $H = \frac{1}{2}$. Thus, the usual stochastic calculus cannot be applied. There are essentially two different methods to define stochastic integrals with respect to fBm. One, by Ciesielski, Kerkyacharian and Roynette [12] and Zähle [37], is a path-wise approach that uses the Hölder continuity of the sample paths. The other, by Dereusefond and Üstünel [15], is the stochastic calculus of variations (Malliavin calculus) for the fBm.

This paper is a continuation and generalization of [5, 32, 19, 11, 36] for nonautonomous integro-differential equations in Fréchet spaces. We will use the resolvent operator theory of Grimmer to study equation (1.1) and equation (1.2). This theory is related to abstract linear first order partial integro-differential equations in a similar way as semigroup theory of bounded linear operators is related to first order partial differential equations, and both are appealing for their clarity and simplicity [19, 25]. Then, we will extend the study of global mild solutions for first-order evolution equations to the autonomous deterministic equation (1.1) and the first-order stochastic evolution integro-differential equation with delay (1.2) in Fréchet spaces. Using Avramescu's fixed point theorem and Grimmer's resolvent operators, we will obtain the first result under the condition that the nonlinear terms of equation (1.1) satisfy some mixed Carathéodory–Lipschitz conditions with a compact resolvent operator. Secondly, using Granas–Frigon's fixed point result, under some mixed Carathéodory–Lipschitz conditions on nonlinearities, we will prove the global existence of a mild solution of equation (1.2). This result is a generalization of several important results in the literature. Let us underline that we do not assume the compactness of the resolvent operator and the semigroup of operators.

This paper is organized as follows. Section 2 introduces the necessary definitions and background information for our main results. Section 3 establishes two global existence theorems for mild solutions of equation (1.1) and equation (1.2). In the last section, we present three examples to illustrate the application of our theoretical results.

2 Preliminaries

2.1 Resolvent operator

Let U and V be two Banach spaces with norms $\|\cdot\|_{\mathsf{U}}$ and $\|\cdot\|_{\mathsf{V}}$, respectively. We denote by $\mathcal{L}(\mathsf{U},\mathsf{V})$ the set of all linear bounded operators from U into V which is equipped with the usual operator norm $\|\cdot\|$. Moreover, we set $\mathcal{L}(\mathsf{U}) = \mathcal{L}(\mathsf{U},\mathsf{U})$. Let us consider the following Cauchy problem

$$\begin{cases} x'(t) = \mathbb{A}(t)x(t) + \displaystyle\int_0^t \Upsilon(t,s)x(s)\,\mathrm{d}s \text{ for } t \geq 0, \\ x(0) = x_0 \in \mathbf{X}. \end{cases} \tag{2.1}$$

Also, let Y be the Banach space $\mathsf{Dom}(A(0))$ endowed with the graph norm of $A(0)$.

We recall that a collection of linear operators $(\mathbb{A}(t))_{t\geq 0}$ is called stable if $(\mathbb{A}(t))_{t\geq 0}$ is a collection of infinitesimal generators of C_0-semigroups and there exist constants $N_0 \geq 1$ and α_0 such that

$$\left\|(A(s_k) - \lambda \operatorname{Id}_{\mathbf{X}})^{-1}(A(s_{k-1}) - \lambda \operatorname{Id}_{\mathbf{X}})^{-1} \cdots (A(s_1) - \lambda \operatorname{Id}_{\mathbf{X}})^{-1}\right\| \leq \frac{N_0}{(\lambda - \alpha_0)^k}$$

for all $\lambda > \alpha_0$ and $0 \leq s_1 \leq s_2 \leq \ldots \leq s_k < \infty$, where $k = 1, 2, 3, \ldots$ (see [33, Page 93] for more details).

Definition 2.1 ([25]) *A family $\{R(t,s) : 0 \leq s \leq t < \infty\}$ of bounded linear operators on $\mathbf{X}$ is called resolvent operator for equation* (2.1) *if it satisfies the following properties*:

(i) *$R(t,s)$ is strongly continuous on $0 \leq s \leq t$, $R(s,s) = \operatorname{Id}_{\mathbf{X}}$ (the identity map of $\mathbf{X}$) and $\|R(t,s)\| \leq Ne^{\delta(t-s)}$ for some constants $N \geq 1$ and $\delta \in \mathbb{R}$,*

(ii) *$R(t,s)\mathsf{Y} \subset \mathsf{Y}$ and $R(t,s)$ is strongly continuous in s and t on Y,*

(iii) *for every $\mathsf{y} \in \mathsf{Y}$, $R(t,s)\mathsf{y}$ is strongly continuously differentiable in s and t and*

$$\begin{cases} \dfrac{\partial R}{\partial t}(t,s)\mathsf{y} = \mathbb{A}(t)R(t,s)\mathsf{y} + \displaystyle\int_s^t \Upsilon(t,r)R(r,s)\mathsf{y}\,\mathrm{d}r, \\ -\dfrac{\partial R}{\partial s}(t,s)\mathsf{y} = -R(t,s)\mathbb{A}(s)y - \displaystyle\int_s^t R(t,r)\Upsilon(r,s)\mathsf{y}\,\mathrm{d}r, \end{cases} \tag{2.2}$$

with $\left(\dfrac{\partial R}{\partial t}\right)(t,s)\mathsf{y}$ and $\left(\dfrac{\partial R}{\partial s}\right)(t,s)\mathsf{y}$ strongly continuous on $0 \leq s \leq t < \infty$.

From [25] there exists at most one resolvent operator. However, to ensure the existence of the resolvent operator for (2.1) we need to make some assumptions. By $\Gamma(t)$ we denote the operator defined by $(\Gamma(t)\mathsf{y})(s) = \Upsilon(t+s,t)\mathsf{y}$ for $\mathsf{y} \in \mathsf{Y}$ and $t, s \geq 0$. Let $BUC([0,+\infty), \mathbf{X})$ be the space of $\mathbf{X}$-valued bounded uniformly continuous functions defined on $[0,+\infty)$. We say that the condition $(\mathbf{R})$ is satisfied if the following requirements hold.

(i) The family $(\mathbb{A}(t))_{t\geq 0}$ is stable and $\mathbb{A}(t)\mathsf{y}$ is strongly continuously differentiable on $[0,+\infty)$ for $\mathsf{y} \in \mathsf{Y}$. In addition, $\Gamma(t)\mathsf{y}$ is strongly continuously differentiable on $[0,+\infty)$ for $\mathsf{y} \in \mathsf{Y}$.

(ii) $\Gamma(t)$ is continuous from $[0,+\infty)$ into $\mathfrak{F}$, where $\mathfrak{F}$ is a subspace of $BUC([0,+\infty), \mathbf{X})$ and $\mathfrak{F}$ is a Banach space with a norm stronger than the supremum norm on $BUC([0,+\infty), \mathbf{X})$.

(iii) $\Gamma(t)\colon \mathsf{Y} \to \mathsf{Dom}(D_s)$ for all $t \geq 0$, where D_s is the generator of C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathfrak{F}$ defined by $[T(t)\mathfrak{f}](s) = \mathfrak{f}(t+s)$ for $\mathfrak{f} \in \mathfrak{F}, t \geq 0$.

(iv) $D_s\Gamma(t)$ is continuous from $[0,+\infty)$ into $\mathcal{L}(\mathsf{Y}, \mathfrak{F})$.

Theorem 2.2 ([25]) *Under condition $(\mathbf{R})$ equation* (2.1) *has a resolvent operator. Moreover, if $\mathbb{A}(t) \equiv A$ and $\Upsilon(t,s) \equiv \Upsilon(t-s)$, then the resolvent operator is a one parameter family.*

The motivation to study mild solutions (see below) of equation (1.1) using the eventual resolvent operator comes from the following lemma.

Lemma 2.3 *Under condition* $(\mathbf{R})$*, if* $x_0 \in \mathsf{Dom}(A)$ *and* $g \in C^1([0,T],\mathbf{X})$*, then the following Volterra integral equation*

$$\begin{cases} x'(t) = \mathbb{A}(t)x(t) + \displaystyle\int_0^t \Upsilon(t,s)x(s)\,\mathrm{d}s + g(t) & \textit{for } t \geq 0, \\ x(0) = x_0 \in \mathbf{X}, \end{cases}$$

has a classical solution given by

$$x(t) = R(t,0)x_0 + \int_0^t R(t,s)g(s)\,\mathrm{d}s.$$

We conclude this section by recalling the following useful result.

Lemma 2.4 ([20]) *Let condition* $(\mathbf{R})$ *be satisfied. For any* $\mathtt{T}$ *there exists a constant* $C = C(\mathtt{T})$ *such that* $\|R(t,t-\epsilon)R(t-\epsilon,s) - R(t,s)\| \leq C\epsilon$ *for all* $0 \leq s \leq t \leq \mathtt{T}$ *and* $0 < \epsilon \leq t$.

Remark 2.5 *If* n *is a positive integer and* $0 \leq s \leq t \leq n$*, then there exists a positive constant* $N_n \geq 1$ *such that* $\sup_{0\leq s\leq t\leq n} \|R(t,s)\| \leq N_n$.

2.2 Measure of non-compactness

Next, we introduce the Kuratowski measure of non-compactness β defined for each bounded subset B in a Banach space $\mathbf{X}$ by

$$\beta(B) = \inf\big\{d > 0 : B \text{ can be covered by a finite number of sets of diameter less than } d\big\}.$$

Some basic properties of β are given in the following lemma.

Lemma 2.6 ([7]) *Let* $\mathbf{X}$ *be a Banach space and let* $B, C \subset \mathbf{X}$ *be bounded. Then,*

(a) $\beta(B) = 0$ *if and only if* B *is relatively compact,*

(b) $\beta(B) = \beta(\mathrm{co}(B))$*, where* $\mathrm{co}(B)$ *is the closed convex hull of* B*,*

(c) $\beta(B) \leq \beta(C)$*, when* $B \subset C$*,*

(d) $\beta(B + C) \leq \beta(B) + \beta(C)$*,*

(e) $\beta(B \cup C) \leq \max\{\beta(B), \beta(C)\}$*,*

(f) $\beta(B(0,r)) = 2r$*, where* $B(0,r) = \{u \in \mathbf{X} : |u| \leq r\}$*, if* $\mathbf{X}$ *is of infinite dimension.*

Lemma 2.7 ([26]) *Let* $(g_n)_{n\in\mathbb{N}}$ *be a sequence of functions in* $L^1([0,T],\mathbf{X})$ *satisfying the condition* $\sup_{n\geq 1} \|g_n(t)\| \leq g(t)$ *for almost every* $t \in [0,T]$*, where* $g \in L^1([0,T],[0,+\infty))$*. Then, the function* $t \mapsto \beta(\{g_n(t) : n \geq 1\})$ *is integrable on* $[0,T]$ *and satisfies the following inequality*

$$\beta\left(\left\{\int_0^T g_n(s)\,\mathrm{d}s : n \geq 1\right\}\right) \leq 2\int_0^T \beta(\{g_n(s) : n \geq 1\})\,\mathrm{d}s.$$

We recall some properties of Fréchet spaces.

Definition 2.8 *A Fréchet space is a vector space with a countable separate family of semi-norms* $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ *that is complete for the distance* d *defined by*

$$d(x,y)=\sum_{n\in\mathbb{N}}2^{-n}\frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

In addition, $(x_n)_{n>0}$ *is Cauchy for* d *if and only if for all* $k\in\mathbb{N}$ *we have* $\sup_{m,n\geq p}\|x_m-x_n\|_k\to 0$*, when* $p\to\infty$*.*

We assume that the family of semi-norms $\{(\|\cdot\|_n)\}_{n\in\mathbb{N}}$ is non-decreasing, that is, $\|x\|_1\leq\|x\|_2\leq\|x\|_3\leq\dots$ for every x in the Fréchet space. Let $\mathcal{F}$ be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ and let $\mathcal{G}\subset\mathcal{F}$. We say that $\mathcal{G}$ is bounded if for every $n\in\mathbb{N}$ there exists $\delta_n>0$ such that $\|y\|_n\leq\delta_n$ for all $y\in\mathcal{G}$. In other words, $\mathcal{G}$ is bounded if there exists a sequence of positive numbers $(\delta_n)_{n\in\mathbb{N}}$ such that

$$\bigcap_{n=0}^{\infty}\big\{x\in\mathbf{X}:\|x\|_n<\delta_n\big\}\supset\mathcal{G}.$$

To $\mathcal{F}$ we associate a sequence of Banach spaces $\{(\mathcal{F}_n,\|\cdot\|_n)\}$ as follows. For every $n\in\mathbb{N}$ we consider the equivalence relation $=_{/n}$ defined by: $x=_{/n}y$ if and only if $\|x-y\|_n=0$ for all $x,y\in\mathcal{F}$. We denote by $\mathcal{F}_n:=(\mathcal{F}/=_{/n},\|\cdot\|_n)$ the quotient space, and by $(\mathcal{F}_n,\|\cdot\|_n)$ the completion of $\mathcal{F}_n$ with respect to $\|\cdot\|_n$. (The norm on $\mathcal{F}_n$ induced by $\|\cdot\|_n$ and its extension to $\mathcal{F}_n$ are still denoted by $\|\cdot\|_n$.) The construction defines a continuous map $\mu_n\colon\mathcal{F}\to\mathcal{F}_n$. For each $\mathcal{G}\subset\mathcal{F}$ and each $n\in\mathbb{N}$ we set $\mathcal{G}_n=\mu_n(\mathcal{G})$, and by $\overline{\mathcal{G}}_n$, $\mathrm{Int}(\mathcal{G}_n)$ and $\partial_n(\mathcal{G}_n)$ we denote, respectively, the closure, the interior, and the boundary of $\mathcal{G}_n$ with respect to $\|\cdot\|_n$ in $\mathcal{F}_n$. To every $\mathcal{G}\subset\mathcal{F}$ we associate a sequence $\{\mathcal{G}_n\}$ of subset $\mathcal{G}_n\subset\mathcal{F}_n$ as follows. For every $x\in\mathcal{F}$, we denote by $[x]_n$ the equivalence class of x in $\mathcal{F}_n$ and we define $\mathcal{G}_n=\{[x]_n:x\in\mathcal{G}\}$.

Example 2.9 (The space $C(U)$) *Let* $U\subset\mathbb{R}^d$ *be open. Define compact subsets of* U *by*

$$K_n:=\{x\in\mathbb{R}^d:|x|\leq n,\ d(x,U^c)\geq\tfrac{1}{n}\},\ n\in\mathbb{N}.$$

Then, $K_n\subset\mathrm{Int}(K_{n+1})$ *and* $K_n\uparrow U$*. Now, define semi-norms* $\|\cdot\|_n$ *on* $C(U)$ *by*

$$\|g\|_n=\sup\big\{|f(x)|:x\in K_n\big\}.$$

Since the sets $\{x:d(x,U^c)>\frac{1}{n}\}$ *form an increasing open cover of* U*, every compact set is contained in some* K_n*. Thus, convergence in the locally convex topology generated by the semi-norms* $\|\cdot\|_n$ *is uniform convergence on compact subsets of* U*, also called local uniform convergence. As each space* $C(K_n)$ *is complete,* $C(U)$ *is a Fréchet space.*

We recall the definition of a contraction in Fréchet space.

Definition 2.10 ([23]) *A function* $J\colon\mathcal{F}\to\mathcal{F}$ *is said to be a contraction if for each* $n\in\mathbb{N}$ *there exists* $k_n\in(0,1)$ *such that* $\|J(x)-J(y)\|_n\leq k_n\|x-y\|_n$ *for all* $x,y\in\mathcal{F}$*.*

Now, we recall the key tools we will use in this work: nonlinear alternatives of Avramescu and Granas–Frigon.

Lemma 2.11 (Nonlinear alternative of Avramescu, [6]) *Let $\mathcal{F}$ be a Fréchet space and let $\mathbf{S}_1, \mathbf{S}_2\colon \mathcal{F} \to \mathcal{F}$ be two operators. Moreover, assume that $\mathbf{S}_1$ is a contraction and $\mathbf{S}_2$ is compact. Then, one of the following statements holds*:

(a) *the operator $\mathbf{S}_1 + \mathbf{S}_2$ has a fixed point, or*

(b) *the set $\Lambda = \{x \in \mathcal{F} : x = \lambda \mathbf{S}_1\left(\frac{x}{\lambda}\right) + \lambda \mathbf{S}_2(x), 0 < \lambda < 1\}$ is unbounded.*

Lemma 2.12 (Nonlinear alternative of Granas–Frigon, [23]) *Let $\mathcal{F}$ be a Fréchet space and let $\Pi \subset \mathcal{F}$ be a closed subset. If $\mathcal{S}\colon \Pi \to \mathcal{F}$ is a contraction such that $\mathcal{S}(\Pi)$ is bounded, then one of the following statements hold*:

(A1) *$\mathcal{S}$ has a unique fixed point, or*

(A2) *there exists $k \in [0, 1)$, $n \in \mathbb{N}$ and $z \in \partial_n \Pi$ such that $\|z - k\mathcal{S}(z)\|_n = 0$.*

3 Main results and proofs

Accordingly, we suppose that the linear part of equation (1.1) and equation (1.2), that is equation (2.1), has a resolvent operator $(R(t, s))_{0\le s\le t}$.

3.1 Existence of solutions for equation (1.1)

This section proves the existence of mild solutions of equation (1.1) when the corresponding resolvent operator is compact and the nonlinear terms satisfy some mixed Carathéodory–Lipschitz conditions. The proof uses the nonlinear alternative of Avramescu.

Definition 3.1 *Let $\varphi \in \mathcal{F}$. A continuous function $x\colon [-r, +\infty) \to \mathbf{X}$ is a mild solution of equation* (1.1)*, if x satisfies the following equation*

$$x(t) = \begin{cases} R(t,0)\varphi(0) + \displaystyle\int_0^t R(t,s)[f_1(s,x_s) + f_2(s,x_s)]\,\mathrm{d}s & \text{for } t \ge 0, \\ \varphi(t) & \text{for } t \in [-r, 0]. \end{cases}$$

Theorem 3.2 *Let $\varphi \in \mathcal{F}$. Moreover, assume that the following conditions are satisfied.*

(A1) *The resolvent operator $(R(t, s))_{0\le s\le t}$ of equation* (2.1) *is compact,* i.e., *$R(t, s)$ is compact for all $0 \le s < t$.*

(A2) *The functions $f_1, f_2\colon [0, +\infty) \times \mathcal{F} \to \mathbf{X}$ are measurable with respect to the first argument and continuous with respect to the second argument and*

(a) *there exist a function* $\nu \in L^1_{loc}([0,+\infty);[0,+\infty))$ *and a continuous non-decreasing function* $q\colon [0,+\infty) \to [0,+\infty)$ *such that* $\|f_1(t,\phi)\| \leq \nu(t)q(\|\phi\|)$ *for every* $t \in [0,+\infty)$ *and* $\phi \in \mathcal{F}$,

(b) *there exists a function* $\mu \in L^1([0,+\infty);[0,+\infty))$ *with* $\|\mu\|_{L^1} < 1/M$ *such that* $\|f_2(t,\phi_1) - f_2(t,\phi_2)\| \leq \mu(t)\|\phi_1 - \phi_2\|$ *for every* $t \in [0,+\infty)$ *and each* $\phi_1, \phi_2 \in \mathcal{F}$.

Then, equation (1.1) *has at least one mild solution on* $[0,+\infty)$ *provided that*

$$\int_{c_n}^{+\infty} \frac{\mathrm{d}s}{s+q(s)} > N_n \int_0^n \alpha(s)\,\mathrm{d}s \quad \textit{for each } n > 0, \tag{3.1}$$

where $c_n = N_n\big(\|\varphi(0)\| + \int_0^n f_2(s,0)\,\mathrm{d}s\big)$ *and* N_n *is the constant appearing in Remark* 2.5.

Proof. Let $\alpha > 1$. For every $n \in N$ in $\mathcal{C}([-r,+\infty],\mathbf{X})$ we define the following semi-norms

$$\|x\|_n := \sup\left\{e^{-\alpha\ell(t)} \sup_{-r\leq s\leq t} \|x(s)\| : \, t \in [-r,n]\right\},$$

where $\ell(t) := \int_0^t N_n\mu(s)\,\mathrm{d}s$. Then, $\mathcal{C}([-r,+\infty),\mathbf{X})$ is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $\alpha(t) = \max\{\nu(s),\mu(s)\}$. Define $\mathbf{S}\colon \mathcal{F} \to \mathcal{F}$ by setting

$$(\mathbf{S}x)(t) = \begin{cases} R(t,0)\varphi(0) + \displaystyle\int_0^t R(t,s)[f_1(s,x_s) + f_2(s,x_s)]\,\mathrm{d}s & \text{for } t \geq 0, \\ \varphi(t) & \text{for } t \in [-r,0] \end{cases}$$

for $x \in \mathcal{F}$. We will show that $\mathbf{S}$ has a fixed point which is clearly a mild solution of equation (1.1). To this end we will use the nonlinear alternative of Avramescu. We start with defining $\mathbf{S}_1, \mathbf{S}_2\colon \mathcal{F} \to \mathcal{F}$ by

$$(\mathbf{S}_1x)(t) = \begin{cases} \displaystyle\int_0^t R(t,s)f_2(s,x_s)\,\mathrm{d}s & \text{for } t \geq 0, \\ 0 & \text{for } t \in [-r,0] \end{cases}$$

and

$$(\mathbf{S}_2x)(t) = \begin{cases} R(t,0)\varphi(0) + \displaystyle\int_0^t R(t,s)f_1(s,x_s)\,\mathrm{d}s & \text{for } t \geq 0, \\ \varphi(t) & \text{for } t \in [-r,0]. \end{cases}$$

The rest of the proof is divided into five steps.

Step 1. We claim that the operator $\mathbf{S}_1$ is a contraction for all $n \in \mathbb{N}$. Using the assumption on h, for all $x, y \in \mathcal{F}$, each $t \in [0,n]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \sup_{0\leq\tau\leq t} \|(\mathbf{S}_1x)(\tau) - (\mathbf{S}_1y)(\tau)\| &\leq \int_0^t \|R(t,s)\|\|f_2(s,x_s) - f_2(s,y_s)\|\,\mathrm{d}s \\ &\leq \int_0^t N_n\mu(s)\|x_s - y_s\|\,\mathrm{d}s \\ &\leq \int_0^t N_n\mu(s)e^{\alpha\ell(s)}e^{-\alpha\ell(s)} \sup_{-r\leq p\leq s} \|x(p) - y(p)\|\,\mathrm{d}s \\ &\leq \left(\int_0^t N_n\mu(s)e^{\alpha\ell(s)}\,\mathrm{d}s\right)\|x - y\|_n \end{aligned}$$

$$\leq \frac{e^{\alpha\ell(t)}}{\alpha}\|x-y\|_n.$$

It follows that

$$\|\mathbf{S}_1(x)-\mathbf{S}_1(y)\|_n \leq (1/\alpha)\|x-y\|_n.$$

Hence, the operator $\mathbf{S}_1$ is a contraction on $\mathcal{F}_n$ for all $n \in \mathbb{N}$.

Step 2. We claim that $\mathbf{S}_2$ is continuous. Let $(x^k)_{k>0}$ be a sequence in $\mathcal{F}$ such that $x^k \longrightarrow x$ in $\mathcal{F}$. Then, $\|x^k - x\|_n \longrightarrow 0$ for all $n \in \mathbb{N}$. For all $0 \leq \tau \leq t \leq n$ we have

$$e^{-\alpha\ell(t)} \sup_{0\leq\tau\leq t} \|(\mathbf{S}_2x^k)(\tau)-(\mathbf{S}_2x)(\tau)\| \leq N_n \int_0^t \|f_1(s,x_s^k)-f_1(s,x_s)\| \,\mathrm{d}s.$$

Thus,

$$\|\mathbf{S}_2x^k - \mathbf{S}_2x\|_n \leq N_n \int_0^n \|f_1(s,x_s^k)-f_1(s,x_s)\| \,\mathrm{d}s. \tag{3.2}$$

Since f_1 is continuous with respect to the second argument, $f_1(s,x_s^k) \longrightarrow f_1(s,x_s)$. Moreover, $\|f_1(s,x_s^k)\| \leq \nu(s)q\left(x_s^k\right)$ implies that the sequence of functions under the above integral is bounded by $\nu \in L^1_{loc}([0,+\infty);[0,+\infty))$. Hence, in light of (3.2) and the dominated convergence theorem $\|\mathbf{S}_2x^k - \mathbf{S}_2x\|_n \longrightarrow 0$, when $k \to +\infty$.

Step 3. Let $\mathcal{G}$ be a bounded subset of $\mathcal{F}$. We recall that a subset of a Fréchet space is bounded if it is bounded with respect to all continuous semi-norms on $\mathcal{F}$. Let $n > 0$. Then, there exists $\delta_n > 0$ such that $\|x\|_n \leq \delta_n$ for $x \in \mathcal{G}$.

We claim that $\mathbf{S}_2(\mathcal{G}_n)$ is equicontinuous on $\mathcal{F}_n$ for all $n \in \mathbb{N}$. Let $t_0 = 0 < t \leq n$. Then,

$$\begin{aligned}
&\|(\mathbf{S}_2x)(t)-(\mathbf{S}_2x)(0)\| \\
&\quad\leq \|R(t,0)\varphi(0)-\varphi(0)\| + \int_0^t \|R(t,s)f_1(s,x_s)\| \,\mathrm{d}s \\
&\quad\leq \|R(t,0)\varphi(0)-\varphi(0)\| + \int_0^t \|R(t,s)\|\|f_1(s,x_s)\| \,\mathrm{d}s \\
&\quad\leq \|R(t,0)\varphi(0)-\varphi(0)\| + N_n\int_0^t q(\|x_s\|)\nu(s) \,\mathrm{d}s \\
&\quad\leq \|R(t,0)\varphi(0)-\varphi(0)\| + N_n\int_0^t q\Big(e^{\alpha\ell(s)}e^{-\alpha\ell(s)} \sup_{-r\leq p\leq s}\|x(p)\|\Big)\nu(s) \,\mathrm{d}s \\
&\quad\leq \|R(t,0)\varphi(0)-\varphi(0)\| + N_n\int_0^t q\big(e^{\alpha\ell(s)}\delta_n\big)\nu(s) \,\mathrm{d}s.
\end{aligned}$$

Using the strong continuity of the resolvent operator $R(t,0)$, we deduce that $\lim_{t\to t_0}\|(\mathbf{S}_2x)(t) - (\mathbf{S}_2x)(0)\| = 0$ uniformly in $x \in \mathcal{G}$. Consequently, $\mathbf{S}_2(\mathcal{G}_n)$ is equicontinuous at $t_0 = 0$.

For $0 < t_0 < t \leq n$ we have

$$\begin{aligned}
&\|(\mathbf{S}_2x)(t)-(\mathbf{S}_2x)(t_0)\| \\
&\quad\leq \|R(t,0)\varphi(0)-R(t_0,0)\varphi(0)\| + \int_{t_0}^t \|R(t,s)\|\|f_1(s,x_s)\| \,\mathrm{d}s \\
&\qquad+ \int_0^t \|R(t,s)-R(t_0,s)\|\|f_1(s,x_s)\| \,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le \|R(t,0)\varphi(0) - R(t_0,0)\varphi(0)\| + N_n \int_{t_0}^{t} \nu(s) q(e^{\alpha\ell(s)} e^{-\alpha\ell(s)} \|x_s\|)\,\mathrm{d}s \\
&\quad + \int_0^{t_0} \|R(t,s) - R(t_0,s)\| \nu(s) q\left(e^{\alpha\ell(s)} e^{-\alpha\ell(s)} \|x_s\|\right) \mathrm{d}s \\
&\le \|(R(t,0) - R(t_0,0))\varphi(0)\| + N_n \int_{t_0}^{t} \nu(s) q\big(e^{\alpha\ell(s)} \delta_n\big)\,\mathrm{d}s \\
&\quad + \int_0^{t_0} \|R(t,s) - R(t_0,s)\| q\big(e^{\alpha\ell(s)} \delta_n\big) \nu(s)\,\mathrm{d}s.
\end{aligned}
$$

Since $\|R(t,s) - R(t_0,s)\| \longrightarrow 0$ as $t \longrightarrow t_0$ for almost all $s \neq t_0$,

$$
\|R(t,s) - R(t_0,s)\| \nu(s) q\big(e^{\alpha\ell(s)} \delta_n\big) \le 2N_n q\big(e^{\alpha\ell(n)} \delta_n\big) \nu(s)
$$

and the later function belongs to $L^1([0,t_0])$, by dominated convergence theorem, we deduce that

$$
\int_0^{t_0} \|R(t,s) - R(t_0,s)\| \nu(s)\,\mathrm{d}s \longrightarrow 0, \text{ as } t \longrightarrow t_0.
$$

Consequently, $\lim_{t\to t_0} \|(\mathbf{S}_2 x)(t) - (\mathbf{S}_2 x)(t_0)\| = 0$ uniformly in $x \in \mathcal{G}$.

For $t < t_0$, this leads to the same result when $t \longrightarrow t_0$, which means that $\mathbf{S}_2(\mathcal{G}_n)$ is equicontinuous at $t \in [0,n]$.

Step 4. Now, we claim that $\mathbf{S}_2(\mathcal{G}_n) = \{\mathbf{S}_2 x : [x]_n \in \mathcal{G}_n\}$ is relatively compact. We will prove that $(\mathbf{S}_2(\mathcal{G}_n))(t) = \{(\mathbf{S}_2 x)(t) : [x] \in \mathcal{G}_n\}$ is relatively compact in $\mathbf{X}$ for every $t \in [0,n]$.

Note that $(\mathbf{S}_2\mathcal{G})(0) = \{\varphi(0)\}$. This means that $(\mathbf{S}_2\mathcal{G})(t)$ is compact for $t = 0$. For $0 < t \le n$ we have

$$
(\mathbf{S}_2\mathcal{G})(t) = \left\{ R(t,0)\varphi(0) + \int_0^t R(t-s) f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n \right\}.
$$

Combining Lemma 2.6 and the compactness of the resolvent operator, we need only to show that the set

$$
\left\{ \int_0^t R(t-s) f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n \right\}
$$

is relatively compact in $\mathbf{X}$. Now, let $0 < t \le n$ and $t > \epsilon > 0$. Observe that

$$
\begin{aligned}
(\mathbf{S}_2 x)(t) &= R(t,0)\varphi(0) + \int_0^t R(t,s) f_1(s,x_s)\,\mathrm{d}s \\
&= R(t,0)\varphi(0) + \int_0^{t-\epsilon} R(t,s) f_1(s,x_s)\,\mathrm{d}s + \int_{t-\epsilon}^{t} R(t,s) f_1(s,x_s)\,\mathrm{d}s \\
&= R(t,0)\varphi(0) + \int_0^{t-\epsilon} [R(t,s) - R(t,t-\epsilon)R(t-\epsilon,s)] f_1(s,x_s)\,\mathrm{d}s \\
&\quad + \int_0^{t-\epsilon} R(t,t-\epsilon)R(t-\epsilon,s) f_1(s,x_s)\,\mathrm{d}s + \int_{t-\epsilon}^{t} R(t,s) f_1(s,x_s)\,\mathrm{d}s \\
&= R(t,0)\varphi(0) + \int_0^{t-\epsilon} [R(t,s) - R(t,t-\epsilon)R(t-\epsilon,s)] f_1(s,x_s)\,\mathrm{d}s \\
&\quad + R(t,t-\epsilon) \int_0^{t-\epsilon} R(t-\epsilon,s) f_1(s,x_s)\,\mathrm{d}s + \int_{t-\epsilon}^{t} R(t,s) f_1(s,x_s)\,\mathrm{d}s,
\end{aligned}
$$

where we have used boundedness of the resolvent operator. From Lemma 2.4 there exists $C > 0$ depending on n such that

$$\begin{aligned}
&\left\| \int_0^{t-\epsilon} [R(t,s) - R(t,t-\epsilon)R(t-\epsilon,s)] f_1(s,x_s)\,\mathrm{d}s \right\| \\
&\quad \le C(n)\epsilon \int_0^{t-\epsilon} \|f_1(s,x_s)\|\,\mathrm{d}s \le C(n)\epsilon \int_0^{t-\epsilon} q(\|x_s\|)\nu(s)\,\mathrm{d}s \qquad (3.3)\\
&\quad \le C(n)\epsilon q(e^{\alpha\ell(n)}\delta_n) \int_0^n \nu(s)\,\mathrm{d}s \le C(n) n q(e^{\alpha\ell(n)}\delta_n) \int_0^n \nu(s)\,\mathrm{d}s;
\end{aligned}$$

here, we have used the above estimates.

Observe that, in light of Lemma 2.6 and Lemma 2.7 as well as (3.3), we have

$$\begin{aligned}
&\beta\left(\left\{ \int_0^{t-\epsilon} [R(t,s) - R(t,t-\epsilon)R(t-\epsilon,s)] f(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n \right\}\right) \\
&\qquad\qquad \le 2C(n)\epsilon q(e^{\alpha\ell(n)}\delta_n) \int_0^n \nu(s)\,\mathrm{d}s.
\end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\beta\left(\left\{ \int_0^{t-\epsilon} [R(t,s) - R(t,t-\epsilon)R(t-\epsilon,s)] f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n \right\}\right) = 0.$$

Observe that $\left\{\int_0^{t-\epsilon} R(t-\epsilon,s) f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n\right\}$ is uniformly bounded. Then, by the compactness of $R(t,t-\epsilon)$, the set $\left\{R(t,t-\epsilon)\int_0^{t-\epsilon} R(t-\epsilon,s) f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n\right\}$ is relatively compact in $\mathbf{X}$. By applying Lemma 2.6, we get that

$$\beta\left(\left\{ R(t,t-\epsilon) \int_0^{t-\epsilon} R(t-\epsilon,s) f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n \right\}\right) = 0.$$

Using similar arguments as before, we obtain

$$\left\| \int_{t-\epsilon}^t R(t,s) f_1(s,x_s)\,\mathrm{d}s \right\| \le M q(e^{\alpha\ell(n)}\delta_n) \int_{t-\epsilon}^t \nu(s)\,\mathrm{d}s.$$

The measure of noncompactness of a centred ball yields the following estimate

$$\beta\left(\left\{ \int_{t-\epsilon}^t R(t,s) f_1(s,x_s)\,\mathrm{d}s : [x]_n \in \mathcal{G}_n \right\}\right) \le 2M q(e^{\alpha\ell(n)}\delta_n) \int_{t-\epsilon}^t \nu(s)\,\mathrm{d}s.$$

Consequently, $\mathbf{S}_2(\mathcal{G}_n)$ is relatively compact in $\mathbf{X}$ for all $t \in [0,n]$. This proves our claim.

Combining the Steps 3–4, together with the Arzela–Ascoli's theorem, we conclude that $\mathbf{S}_2$ is compact on each Banach space $\mathcal{F}_n$ for all $n \in \mathbb{N}$. Hence, $\mathbf{S}_2$ is compact on $\mathcal{F}$.

Step 5. We claim that the set

$$\Lambda := \left\{ x \in \mathcal{F} : x = \lambda \mathbf{S}_2(x) + \lambda \mathbf{S}_1\left(\tfrac{x}{\lambda}\right),\ 0 < \lambda < 1 \right\}$$

is bounded. Let $x \in \Lambda$ and $n \in \mathbb{N}$. Then, for each $t \in [0,n]$ we have

$$x(t) = \lambda R(t,0)\varphi(0) + \lambda \int_0^t R(t,s) f_2\left(s, \frac{x_s}{\lambda}\right) \mathrm{d}s + \lambda \int_0^t R(t,s) f_1(s,x_s)\,\mathrm{d}s.$$

Thus,

$$\left\|\frac{x(t)}{\lambda}\right\| \leq \|R(t,0)\varphi(0)\| + \int_0^t \left\|R(t,s)f_2\left(s,\frac{x_s}{\lambda}\right) - R(t,s)f_2(s,0) + R(t,s)f_2(s,0)\right\| \mathrm{d}s + \int_0^t \|R(t,s)f_1(s,x_s)\| \mathrm{d}s.$$

With the growth condition on f_1 and f_2 and the Lipschitz condition on h, we get that

$$\left\|\frac{x(t)}{\lambda}\right\| \leq N_n\|\varphi(0)\| + N_n\int_0^t \nu(s)q(\|x_s\|)\mathrm{d}s + N_n\int_0^t \mu(s)\left\|\frac{x_s}{\lambda}\right\| \mathrm{d}s + N_n\int_0^t \|f_2(s,0)\| \mathrm{d}s.$$

Moreover,

$$\begin{aligned}
&\left\|\frac{x(t)}{\lambda}\right\| \\
&\leq N_n\|\varphi(0)\| + N_n\int_0^t \|f_2(s,0)\| \mathrm{d}s + N_n\int_0^t \nu(s)q\left(\lambda\left\|\frac{x_s}{\lambda}\right\|\right) \mathrm{d}s + N_n\int_0^t \mu(s)\left\|\frac{x_s}{\lambda}\right\| \mathrm{d}s \\
&\leq N_n\|\varphi(0)\| + N_n\int_0^n \|f_2(s,0)\|\mathrm{d}s + N_n\int_0^t \nu(s)q\left(\left\|\frac{x_s}{\lambda}\right\|\right) \mathrm{d}s + N_n\int_0^t \mu(s)\left\|\frac{x_s}{\lambda}\right\| \mathrm{d}s \\
&\leq c_n + N_n\int_0^t \nu(s)q\left(\left\|\frac{x_s}{\lambda}\right\|\right) \mathrm{d}s + N_n\int_0^t \mu(s)\left\|\frac{x_s}{\lambda}\right\| \mathrm{d}s;
\end{aligned}$$

here, we have used the fact that q is non-decreasing and $0<\lambda<1$. We recall that $c_n = N_n\|\varphi(0)\| + N_n\int_0^n \|f_2(s,0)\| \mathrm{d}s$. Let $\mathtt{m}$ be the function defined by

$$\mathtt{m}(t) := \sup\left\{\left\|\frac{x(s)}{\lambda}\right\| : 0 \leq s \leq t\right\} \quad \text{for } t \in [0,n].$$

Let $t_1 \in [0,t]$ be such that $\mathtt{m}(t) = \|x(t_1)/\lambda\|$. Then, using the above inequality we get that $\mathtt{m}(t) \leq z(t)$ for $t \in [0,n]$, where z is the function defined by

$$z(t) := c_n + N_n\int_0^t \nu(s)q(\mathtt{m}(s))\,\mathrm{d}s + N_n\int_0^t \mu(s)\mathtt{m}(s)\,\mathrm{d}s.$$

Note that z satisfies

$$z(0) = c_n \quad \text{and} \quad z'(t) = N_n\nu(t)q(\mathtt{m}(t)) + N_n\mu(t)\mathtt{m}(t).$$

Since q is increasing, we obtain

$$z'(t) \leq N_n\nu(t)q(z(t)) + N_n\mu(t)z(t) \leq N_n\alpha(t)(z(t)+q(z(t))),$$

where $\alpha(t) = \max\{\mu(t),\nu(t)\}$. From the above inequality it follows that

$$\int_0^t \frac{z'(s)}{z(s)+q(z(s))}\,\mathrm{d}s \leq \int_0^t N_n\alpha(s)\,\mathrm{d}s,$$

which gives

$$\int_{c_n}^{z(t)} \frac{\mathrm{d}s}{s+q(s)} \leq \int_0^t N_n\alpha(s)\,\mathrm{d}s \leq \int_0^n N_n\alpha(s)\,\mathrm{d}s < \int_{c_n}^{+\infty} \frac{\mathrm{d}s}{s+q(s)};$$

here, we have used inequality (3.1). We deduce that there exists $\delta_n > 0$ such that $\sigma(t) \leq \delta_n$ for all $t \in [0,n]$. Therefore, there exists δ_n only depending on M, ν, μ and q such that $\|x(t)\| \leq \lambda\delta_n$ for all $t \in [0,n]$. Hence, $\|x\|_n \leq \delta_n$ uniformly in $x \in \Lambda$. We conclude that Λ is bounded in $\mathcal{F}$.

The nonlinear alternative of Avramescu implies that the operator $\mathbf{S} := \mathbf{S}_1 + \mathbf{S}_2$ has a fixed point, which is a mild solution of equation (1.1). □

3.2 Existence of mild solutions for equation (1.2)

In this section, we study the existence of mild solutions for the stochastic partial integro-differential equation (1.2) with dependent delay driven by fBm. The outcomes are obtained in the situation when the associated resolvent operator is not necessarily compact and the nonlinear terms satisfy some mixed Carathéodory–Lipschitz conditions. The proof is based on the nonlinear alternative of Granas–Frigon.

We recall some basic definitions, notations and lemmas which are used throughout this section. By H, K and K_0 we denote real separable Hilbert spaces. Moreover, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space satisfying the usual conditions, meaning that the filtration is a right-continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. The notation $\mathcal{L}(\mathsf{K},\mathsf{H})$ stands for the space of all bounded linear operators from K to H. We also write $\mathcal{L}(\mathsf{K})$ whenever $\mathsf{K} = \mathsf{H}$. For convenience, we use the same symbol $\|\cdot\|$ to denote the norms in H, K and $\mathcal{L}(\mathsf{K},\mathsf{H})$, and $(\cdot,\cdot)$ to denote the inner product of H and K when there is no confusion. We denote by $L^2(\Omega,\mathsf{H})$ the space of all H-valued random variables ϑ such that $\mathbb{E}\|\vartheta\|^2 = \int_\Omega \|\vartheta\|^2 \,\mathrm{d}\mathbb{P} < \infty$. For $\vartheta \in L^2(\Omega,\mathsf{H})$ let $\|\vartheta\|_{L^2(\Omega,\mathsf{H})} = \big(\int_\Omega \|\vartheta\|^2 \,\mathrm{d}\mathbb{P}\big)^{\frac{1}{2}}$. Then, $L^2(\Omega,\mathsf{H})$, equipped with the norm $\|\cdot\|_{L^2(\Omega,\mathsf{H})}$, is a Hilbert space. Let $\{e_k : k \in \mathbb{N}\}$ be a complete orthonormal basis of K. Suppose that $\{\mathsf{W}(t) : t \geq 0\}$ is a K-valued trace class Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,b]}, \mathbb{P})$ with a finite trace nuclear covariance $Q \geq 0$. Set $\mathsf{Tr}(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty$, where the numbers λ_k are such that $Qe_k = \lambda_k e_k$ for $k \in \mathbb{N}$. Let $\{\mathsf{w}_k(t) : k \in \mathbb{N}\}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathsf{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\mathsf{w}_k(t) e_k.$$

We refer the reader to [14, Proposition 4.1] for further details on series representation for $\mathsf{W}(t)$.

In addition, we recall the definition of a H-valued stochastic integral with respect to the K-valued Q-Wiener process W. For $\Psi \in \mathcal{L}(\mathsf{K},\mathsf{H})$ we define

$$\|\Psi\|_2^2 = \mathsf{Tr}(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\,\Psi e_n\|^2,$$

where Ψ^* is the adjoint of the operator Ψ^*. If $\|\Psi\|_2^2 < \infty$, then Ψ is called a Hilbert–Schmidt operator. Let $L_Q(\mathsf{K},\mathsf{H})$ denote the space of all Q-Hilbert–Schmidt operators Ψ. It is known that $L_Q(\mathsf{K},\mathsf{H})$ is a separable Hilbert space when endowed with the norm $\|\cdot\|_2^2$. Let $\Psi\colon [0,b] \to L_Q(\mathsf{K},\mathsf{H})$ be a predictable $\mathcal{F}_t$-adapted process such that

$$\mathbb{E}\int_0^t \|\Psi(s)\|_2^2 \,\mathrm{d}s < \infty \quad \text{for } t \in [0,b].$$

Then, we define the H-valued stochastic integral

$$\int_0^t \Psi(s)\,\mathrm{d}\mathsf{W}(s)$$

which will be a continuous square integrable martingale. For more details on stochastic integrals we refer to [14, 24]. Let $a \leq b$. We denote by $\mathcal{C}([a,b], L^2(\Omega,\mathsf{H}))$ the space of all continuous,

$\mathcal{F}_t$-adapted, measurable processes from $[a,b]$ to $L^2(\Omega,\mathsf{H})$ satisfying $\sup_{t\in[a,b]}\mathbb{E}\|\vartheta(t)\|^2<\infty$. It is easy to see that $\mathcal{C}([a,b],L^2(\Omega,\mathsf{H}))$ is a Banach space equipped with the norm

$$\|\vartheta\| = \left(\sup_{t\in[a,b]}\mathbb{E}\|\vartheta(t)\|^2\right)^{\frac{1}{2}}.$$

Here, is an useful inequality that will be used to prove our main results.

Lemma 3.3 ([13]) *If* $G\colon [0,b]\times\mathsf{H}\to L_Q(\mathsf{K},\mathsf{H})$ *is a continuous function and* $\vartheta\in\mathcal{C}([0,b],L^2(\Omega,\mathsf{H}))$*, then*

$$\mathbb{E}\left\|\int_0^b G(t,\vartheta(t))\,\mathrm{d}\mathsf{W}(t)\right\|^2 \le \mathsf{Tr}(Q)\int_0^b \mathbb{E}\|G(t,\vartheta(t))\|^2\,\mathrm{d}t.$$

Analogically, let $\Theta\in\mathcal{L}(\mathsf{K}_0)$ be defined by $\Theta e_n=\lambda_n e_n$ with finite trace $\mathrm{tr}(\Theta)=\sum_{n=1}^{\infty}\lambda_n<\infty$, where $\lambda_n\ge 0$ for $n\in\mathbb{N}$ and $\{e_n : n\in\mathbb{N}\}$ is a complete orthonormal basis in K_0. Let β_n^H be a sequence of real, independent fBm. We define the infinite dimensional fBm on K_0 with covariance Q as

$$\mathrm{W}^H(t)=\mathrm{W}^H_\Theta(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\,e_n\,\beta_n^H(t).$$

Let $L_2^0=\mathcal{L}_2^0(\mathsf{K}_0,\mathsf{H})$ be the space of all Θ-Hilbert–Schmidt operators from K_0 to H. Now, let $b>0$ and consider the function $\zeta\colon[0,b]\to L_2^0$. The Wiener integral of ζ with respect to W^H is defined by

$$\int_0^t \zeta(s)\,\mathrm{d}\mathrm{W}^H(s)=\sum_{n=0}^{\infty}\int_0^t\sqrt{\lambda_n}\,\zeta(s)\,e_n\,\mathrm{d}\beta_n^H(s)=\sum_{n=0}^{\infty}\int_0^t\sqrt{\lambda_n}\,K^*(\zeta\,e_n)(s)\ \mathrm{d}\beta_n(s), \tag{3.4}$$

where K_t^* is an isometry between H and $L^2([0,b])$ (for more details see the paper [29]).

Lemma 3.4 ([10]) *If* $\zeta\colon[0,b]\to L_2^0$ *satisfies* $\int_0^b\|\zeta(s)\|_{L_2^0}\,\mathrm{d}s<\infty$*, the above-mentioned sum* (3.4) *is well-defined as a* H*-valued random variable, and we have*

$$\mathbb{E}\left\|\int_0^t\zeta(s)\,\mathrm{d}\mathrm{W}^H(s)\right\|^2\le 2Ht^{2H-1}\int_0^t\|\zeta(s)\|^2_{L_2^0}\ \mathrm{d}s.$$

Throughout this work we make the following assumptions. For each $t\ge 0$ by $\mathcal{F}_t$ we denote the σ-field generated by the random variables $\{\mathrm{W}^H(s),\mathrm{W}(s):0\le s\le t\}$ and the $\mathbb{P}$-null sets. In addition to the natural filtration $\{\mathcal{F}_t:t\ge 0\}$, we consider bigger filtration $\{\mathcal{G}_t:t\ge 0\}$ such that

(1) $\{\mathcal{G}_t\}$ is right-continuous and $\mathcal{G}_0$ contains the $\mathbb{P}$-null sets,

(2) W^H is $\mathcal{G}_0$-measurable and W is a $\mathcal{G}_t$-Brownian motion.

Let $C_{\mathcal{G}_t}([0,+\infty),\mathsf{H})$ denote the space of all continuous and $\mathcal{G}_t$-adapted measurable processes from $[0,+\infty)$ into H. We consider

$$B_{+\infty}=\left\{x\colon[-r,+\infty)\to\mathsf{H}: x|_{[0,+\infty)}\in C_{\mathcal{G}_t}([0,+\infty),\mathsf{H}) \text{ and } x_0\in L_2^0(\Omega,\mathsf{H})\right\},$$

where $L_2^0(\Omega,\mathsf{H})=\{x\in L^2(\Omega,\mathsf{H}): x \text{ is } \mathcal{G}_0\text{-measurable}\}$, as well as $\mathcal{B}=\mathcal{C}([-r,0];\mathsf{H})$ equipped with the norm $\|\phi\|_{\mathcal{B}}=\sup_{t\in[-r,0]}\|\phi(t)\|$.

Definition 3.5 *A function $\vartheta\colon [0,+\infty)\times \mathsf{H}\to L_Q(\mathsf{K},\mathsf{H})$ is said to be an L^2-Carathéodory function if it satisfies the following conditions*:

(i) *for each $t\in[0,+\infty)$ the function $\vartheta(t,\cdot)\colon \mathsf{H}\to L_Q(\mathsf{K},\mathsf{H})$ is continuous,*

(ii) *for each $x\in\mathsf{H}$ the function $\vartheta(\cdot,x)\colon [0,+\infty)\to L_Q(\mathsf{K},\mathsf{H})$ is $\mathcal{F}_t$-measurable,*

(iii) *for every positive integer δ there exists $h_\delta\in L^l_{1oc}([0,+\infty),[0,+\infty))$ such that $\mathbb{E}\|\vartheta(t,x)\|^2\le h_\delta(t)$ for all $\mathbb{E}\|x\|^2\le\delta$ and almost all $t\in[0,+\infty)$.*

Definition 3.6 *Let $\varphi\in\mathcal{B}$. An $\mathcal{G}_t$-adapted stochastic process $x\colon[-r,+\infty)\to\mathsf{H}$ is called mild solution of equation* (1.2) *if $x_0\in L^0_2(\Omega,\mathsf{H})$ and it satisfies the following equation*

$$x(t)=\begin{cases} R(t,0)\varphi(0)+\displaystyle\int_0^t R(t,s)h_1(s,x_s)\,\mathrm{dW}(s) & \\ \qquad\qquad\qquad +\displaystyle\int_0^t R(t,s)h_2(s)\,\mathrm{dW}^H(s) & \text{for } t\ge 0,\\ \varphi(t) & \text{for } t\in[-r,0]. \end{cases} \tag{3.5}$$

Theorem 3.7 *Let $\varphi\in\mathcal{B}$. Moreover, assume that the following conditions are satisfied.*

(B1) *The function $h_1\colon[0,+\infty)\times\mathcal{B}\longrightarrow L_Q(\mathsf{K},\mathsf{H})$ is L^2-Carathéodory and*

(a) *there exist a function $\beta\in L^1_{loc}([0,+\infty);[0,+\infty))$ and a continuous non-decreasing function $\upsilon\colon[0,+\infty)\to[0,+\infty)$ such that $\mathbb{E}\|h_1(t,\phi)\|^2\le\beta(t)\upsilon(\mathbb{E}\|\phi\|^2)$ for every $t\in[0,+\infty)$ and $\phi\in\mathcal{B}$,*

(b) *for every $R>0$ there exists a function $L_R\in L^1([0,+\infty);[0,+\infty))$ such that*

$$\mathbb{E}\|h_1(t,\phi_1)-h_1(t,\phi_2)\|^2\le L_R(t)\mathbb{E}\|\phi_1-\phi_2\|^2$$

for every $t\in[0,+\infty)$ and each $\phi_1,\phi_2\in\mathcal{B}$ with $\mathbb{E}\|\phi_1\|^2\le R$ and $\mathbb{E}\|\phi_2\|^2\le R$.

(B2) *The nonlinear function $h_2\colon[0,+\infty)\to L^0_2$ satisfies $\int_0^t\|h_2(s)\|^2_{L^0_2}<\infty$, and for every $t\in[0,\infty)$ there exists a constant C_θ such that $\|h_2(s)\|^2_{L^0_2}\le C_\theta$ uniformly on $[0,+\infty)$.*

Then, equation (1.2) *has a unique mild solution on $[0,+\infty)$ provided that*

$$\int_{c_n}^{+\infty}\frac{\mathrm{d}s}{\upsilon(s)}<4\mathsf{Tr}(Q)M^2\int_0^n\beta(s)\,\mathrm{d}s\quad\text{for each } n>0, \tag{3.6}$$

where $c_n:=2(1+M)\mathbb{E}\|\varphi\|^2+8M^2n^{2H-1}\int_0^n\|h_2(s)\|^2_{L^0_2}\,\mathrm{d}s$ and $M:=\sup_{0<s<t}\|R(t,s)\|<+\infty$.

Proof. Let $\alpha>1$. For every $n\in N$ we define in $\mathcal{C}([-r,+\infty),\mathsf{H})$ semi-norms by

$$\|x\|_n:=\sup\left\{e^{-\alpha\ell(t)}\mathbb{E}\sup_{-r\le s\le t}\|x(s)\|^2: t\in[-r,n]\right\},$$

where $\ell(t) := \int_0^t \mathsf{Tr}(Q)M^2 L_R(s)\,\mathrm{d}s$. Endowed with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$, $\mathcal{C}([-r,+\infty),\mathsf{H})$ is a Fréchet space. Let $\alpha(t) = \max\{\nu(s),\mu(s)\}$. Define $\bar{\mathbf{S}}\colon B_{+\infty} \to B_{+\infty}$ as

$$(\bar{\mathbf{S}}x)(t) = \begin{cases} R(t,0)\varphi(0) + \displaystyle\int_0^t R(t,s)h_1(s,x_s)\,\mathrm{d}W(s) & \\ \qquad\qquad + \displaystyle\int_0^t R(t,s)h_2(s)\,\mathrm{d}W^H(s) & \text{for } t \geq 0, \\ \varphi(t) & \text{for } t \in [-r,0]. \end{cases}$$

To show that $\bar{\mathbf{S}}$ has a fixed point (and thus, equation (1.2) has a mild solution) we will use the nonlinear alternative of Granas–Frigon. For $\varphi \in \mathcal{B}$ we define the function $y\colon [-r,+\infty) \to \mathsf{H}$ by

$$y(t) = \begin{cases} R(t,0)\varphi(0) & \text{for } t \geq 0, \\ \varphi(t) & \text{for } t \in [-r,0]. \end{cases}$$

Then, $y_0 = \varphi$. For each function $z \in B_{+\infty}$ let $x(t) = z(t) + y(t)$. It is obvious that x satisfies (3.5) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t R(t,s)h_1(s,z_s+y_s)\,\mathrm{dW}(s) + \int_0^t R(t,s)h_2(s)\,\mathrm{dW}^H(s) \quad \text{for} \quad t \geq 0.$$

Let $B_{+\infty}^0 = \{z \in B_{+\infty} : z_0 = 0\}$ be endowed with the semi-norm defined by

$$\|z\|_n := \sup\left\{ e^{-\alpha\ell(t)}\mathbb{E}\sup_{0\leq s\leq t}\|z(s)\|^2 : t \in [0,n]\right\}.$$

Define $\mathbf{S}\colon B_{+\infty}^0 \to B_{+\infty}^0$ by

$$\mathbf{S}(z)(t) = \int_0^t R(t,s)h_1(s,z_s+y_s)\,\mathrm{dW}(s) + \int_0^t R(t,s)h_2(s)\,\mathrm{dW}^H(s) \quad \text{for} \quad t \geq 0.$$

Obviously, the operator $\bar{\mathbf{S}}$ has a fixed point if and only if $\mathbf{S}$ has. So, it is enough to prove that $\mathbf{S}$ has a fixed point. Let $z \in B_{+\infty}^0$ be such that $z = \lambda\mathbf{S}(z)$ for some $\lambda \in [0,1)$. Then, for each $t \in [0,n]$ we have

$$\begin{aligned} &\mathbb{E}\|z(t)\|^2 \\ &\quad \leq 2\mathbb{E}\left\|\int_0^t R(t,s)h_1(s,z_s+y_s)\,\mathrm{dW}(s)\right\|^2 + 2\mathbb{E}\left\|\int_0^t R(t,s)h_2(s)\,\mathrm{dW}^H(s)\right\|^2 \\ &\quad \leq 2\mathsf{Tr}(Q)M^2\int_0^t \beta(t)v(\mathbb{E}\|z_s+y_s\|^2)\,\mathrm{d}s + 4\,M^2 n^{2H-1}\int_0^n \|h_2(s)\|_{L_2^0}^2\,\mathrm{d}s \\ &\quad \leq 2\mathsf{Tr}(Q)M^2\int_0^t \beta(t)v\left(2\sup_{p\in[-r,s]}\mathbb{E}\|z(p)\|^2 + 2\sup_{s\in[-r,n]}\mathbb{E}\|y(s)\|^2\right)\mathrm{d}s \\ &\qquad + 4\,M^2 n^{2H-1}\int_0^n \|h_2(s)\|_{L_2^0}^2\,\mathrm{d}s. \end{aligned}$$

Since

$$2\sup_{s\in[-r,n]}\mathbb{E}\|y(s)\|^2 \leq 2(1+M)\mathbb{E}\|\varphi\|^2 = c,$$

we have

$$\mathbb{E}\|z(t)\|^2 \leq 2\mathsf{Tr}(Q)M^2 \int_0^t \beta(t)\upsilon\left(2 \sup_{p\in[0,s]} \mathbb{E}\|z(p)\|^2 + c\right) \mathrm{d}s + 4\, M^2 n^{2H-1} \int_0^n \|h_2(s)\|^2_{L_2^0} \,\mathrm{d}s.$$

Consider the function u defined by $u(t) = 2\sup_{s\in[0,t]} \mathbb{E}\|z(s)\|^2 + c$ for $t \geq 0$. Let $t^* \in [0,t]$ be such that $u(t) = 2\mathbb{E}\|z(t^*)\|^2 + c$. It follows that

$$u(t) \leq c + 8\, M^2 n^{2H-1} \int_0^n \|h_2(s)\|^2_{L_2^0} \,\mathrm{d}s + 4\mathsf{Tr}(Q)M^2 \int_0^t \beta(t)\upsilon(u(s))\,\mathrm{d}s.$$

Recall that $c_n = c + 8\, M^2 n^{2H-1} \int_0^n \|h_2(s)\|^2_{L_2^0} \,\mathrm{d}s$. Then,

$$u(t) \leq c_n + 4\mathsf{Tr}(Q)M^2 \int_0^t \beta(t)\upsilon(u(s))\,\mathrm{d}s.$$

Let us denote the right-hand side of the above inequality by $a(t)$. It follows that $u(t) \leq a(t)$ for all $t \in [0,n]$, $a(0) = c_n$ and $a'(t) = 4\mathsf{Tr}(Q)M^2\beta(t)\upsilon(u(t)) = m(t)\upsilon(u(t))$. Using the non-decreasing character of υ, we get $a'(t) \leq m(t)\upsilon(a(t))$. This implies that for each $t \in [0,n]$ we have

$$\int_{c_n}^{a(t)} \frac{\mathrm{d}s}{\upsilon(s)} \leq \int_0^n m(s)\,\mathrm{d}s < \int_{c_n}^{+\infty} \frac{\mathrm{d}s}{\upsilon(s)}.$$

Thus, by (3.6) for every $t \in [0,n]$ there exists a constant ξ_n such that $a(t) \leq \xi_n$, and hence $u(t) \leq \xi_n$. Since $\|x\|_n \leq u(t)$, we have $\|x\|_n \leq \xi_n$. Consider the set

$$\mathsf{Z} = \left\{x \in B^0_{+\infty} \;:\; \sup_{t\in[0,n]} \mathbb{E}\|x(t)\|^2 \leq \xi_n + 1 \text{ for all } n \in \mathbb{N}\right\},$$

which is a closed subset of $B^0_{+\infty}$. Next, we show that $\mathbf{S}\colon \mathsf{Z} \to B^0_{+\infty}$ is a contraction operator for all $n \in \mathbb{N}$. Using the assumption on h_1, for all $z^1, z^2 \in B^0_{+\infty}$, each $t \in [0,n]$ and $n \in \mathbb{N}$ for $\tau \in [0,t]$ we have

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq\tau\leq t} \|(\mathbf{S}z^1)(\tau) - (\mathbf{S}z^2)(\tau)\|^2 \\
&\quad\leq \int_0^t M\mathbb{E}\|h_1(s, z^1_s + y_s) - h_1(s, z^2_s + y_s)\|^2 \,\mathrm{d}s \\
&\quad\leq \mathsf{Tr}(Q)M^2 \int_0^t L_R(s)\mathbb{E}\|z^1_s - z^2_s\|^2 \,\mathrm{d}s \\
&\quad\leq \mathsf{Tr}(Q)M^2 \int_0^t L_R(s)\mathbb{E} \sup_{0\leq p\leq s} \|z^1(p) - z^2(p)\|^2 \,\mathrm{d}s \\
&\quad\leq \mathsf{Tr}(Q)M^2 \int_0^t L_R(s)e^{\alpha\ell(t)}e^{-\alpha\ell(t)}\mathbb{E} \sup_{0\leq p\leq s} \|z^1(p) - z^2(p)\|^2 \,\mathrm{d}s \\
&\quad\leq \frac{e^{\alpha\ell(t)}}{\alpha}\|z^1 - z^2\|_n.
\end{aligned}$$

It follows that $\|\mathbf{S}z^1 - \mathbf{S}z^2\|_n \leq (1/\alpha)\|z^1 - z^2\|_n$. Hence, the operator $\mathbf{S}$ is a contraction for all $n \in \mathbb{N}$. From the choice of Z there is no $z \in \partial_n \mathsf{Z}^n$ such that $z = \lambda\mathbf{S}(z)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Frigon and Granas the operator $\mathbf{S}$ has a unique fixed point z. Then, $x = z + y$ is the unique mild solution of the problem (1.2). The proof is completed. □

4 Illustratives examples

Example 1. Here, we consider a one-dimensional viscoelastic problem in which the material lies on the interval $0 < z < L$ and is subjected to a displacement given by

$$u(t,z) = f(t,z) - z, \quad u(\theta,z) = u_0(\theta), \quad \theta \in [-r,0],$$

where $f\colon \mathbb{R}^+ \times [0,L] \to \mathbb{R}$, r is a positive constant (a time delay) and $u_0\colon [-r,0] \to \mathbb{R}$ is a continuous function which will be specified later.

If $\rho_0\colon [0,L] \to \mathbb{R}^+$ is the initial density function, applying Newton's second law of motion $F = ma$, we obtain $\partial_z\sigma(t,z) = \rho_0(t,z)f_{tt}$, where σ is the stress. For delayed linear viscoelasticity the stress is given by

$$\sigma(t,z) = \int_0^t G(t,s,z)\partial_{zt}u(s,z)\,\mathrm{d}s + a(t)\partial_{tz}u(t-r,z),$$

where $G\colon \mathbb{R}^2_+ \times [0,L] \to \mathbb{R}^+$ (called the relaxation function) is smooth enough. The term $a\colon \mathbb{R}^+ \to \mathbb{R}$ is a control function. When $r = 0$, a closed form of the above stress is given in [34, Page 98]. Integrating by parts, we get

$$\begin{aligned}\sigma&(t,z)\\ &= G(t,t,z)\partial_z u(t,z) - G(t,0,z)\partial_z u(0,z) - \int_0^t G_s(t,s,z)\partial_z u(s,z)\,\mathrm{d}s + a(t)\partial_{zt}u(t-r,z).\end{aligned}$$

Since $\partial_z u(0,z) \equiv 0$ and $\partial_z\sigma(t,z) = \rho_0(t,z)\partial_{tt}f(t,z)$, we obtain

$$\begin{aligned}&\rho_0(t,z)\partial_{tt}f(t,z)\\ &\quad= \frac{\partial}{\partial z}\left[G(t,t,z)\partial_z u(t,z) - \int_0^t \partial_s G(t,s,z)\partial_z u(s,z)\,\mathrm{d}s + a(t)\partial_{tz}u(t-r,z)\right].\end{aligned} \tag{4.1}$$

In the situation when the material is homogeneous in a certain sense, we assume that $\rho_0(z) \equiv 1$ and G is independent of the space variable z, that is, $G(t,s,z) = G(t,s)$. So, combining all these facts, (4.1) becomes

$$\partial_{tt}u(t,z) = G(t,t)\partial_{zz}u(t,z) - \int_0^t \partial_s G(t,s)\partial_{zz}u(s,z)\,\mathrm{d}s + a(t)\partial_{tzz}u(t-r,z).$$

Assume that all the positions felt some trepidation about the displacement. We separate variables $u(t,z) = g(t)h(z)$ and we get that

$$g''(t)h(z) = G(t,t)g(t)h''(z) - \int_0^t \partial_s G(t,s)g(s)h''(z)\,\mathrm{d}s + a(t)g'(t-r)h''(z).$$

Thus,

$$\frac{h(z)}{h''(z)} = \frac{G(t,t)g(t) - \displaystyle\int_0^t \partial_s G(t,s)g(s)\,\mathrm{d}s + a(t)g'(t-r)}{g''(t)}.$$

Then, there is a constant $K \in \mathbb{R}$ such that

$$g''(t) = KG(t,t)g(t) - \int_0^t K\partial_s G(t,s)g(s)\,\mathrm{d}s + a(t)Kg'(t-r). \tag{4.2}$$

Let $y_1(t) = g(t)$ and $y_2(t) = g'(t)$. We rewrite (4.2) as

$$\begin{aligned}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' &= \begin{pmatrix} 0 & 1 \\ KG(t,t) & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \\ &\quad - \int_0^t \begin{pmatrix} 0 & 0 \\ K\partial_s G(t,s) & 0 \end{pmatrix} \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} \mathrm{d}s + \begin{pmatrix} 0 & 0 \\ 0 & a(t)K \end{pmatrix} \begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}.\end{aligned} \tag{4.3}$$

Now, let $\mathbf{X} := \mathbb{R}^2$ and $\mathcal{F} := \mathcal{C}(\mathbb{R}^+, \mathbf{X})$. Also, set

$$x(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \mathbb{A}(t) := \begin{pmatrix} 0 & 1 \\ KG(t,t) & 0 \end{pmatrix} \quad \text{and} \quad \Upsilon(t,s) := \begin{pmatrix} 0 & 0 \\ K\partial_s G(t,s) & 0 \end{pmatrix}.$$

The nonlinear terms $f_1, f_2 : \mathbb{R}_+^2 \times \mathcal{F} \to \mathbf{X}$ are defined by

$$\begin{aligned} f_1(t,\phi)(s) &\equiv 0 \text{ (the } f_1 \text{ in the above theorems)}, \\ f_2(t,\phi)(s) &= f_2\left(t, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\right)(s) := \begin{pmatrix} 0 & 0 \\ 0 & a(t)K \end{pmatrix} \begin{pmatrix} \phi_1(-r)(s) \\ \phi_2(-r)(s) \end{pmatrix}, \\ \varphi(\theta) &= \begin{pmatrix} u_0(\theta) \\ 0 \end{pmatrix} \end{aligned}$$

for all $(t,s,\theta) \in \mathbb{R}_+^2 \times [-r,0]$, $\phi \in \mathcal{F}$. Then, equation (4.3) can be written as

$$\begin{cases} x'(t) = \mathbb{A}(t)x(t) + \displaystyle\int_0^t \Upsilon(t,s)x(s)\,\mathrm{d}s + f_2(t,x_t), \\ x_0 = \varphi. \end{cases}$$

From the smoothness of G, the resolvent operator $(R(t,s))_{0\le s\le t}$ exists. By construction h is Lipschitz continuous and satisfies the growth condition appearing in the hypothesis of our main result. Consequently, we get the following proposition.

Proposition 4.1 *Assume that*

$$\int_c^{+\infty} \frac{\mathrm{d}s}{s+q(s)} > M \int_0^n \alpha(s)\,\mathrm{d}s \quad \text{for each } n > 0,$$

where $c = M\|u_0\|$. *Then, equation* (4.3) *has at least one solution on* $\mathbb{R}^+$.

Example 2. Here, we apply Theorem 3.2 to study the existence of solutions for a partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,y) - \frac{\partial^2}{\partial y^2}u(t,y) \\ &\quad = \int_0^t b(t-s)\frac{\partial^2}{\partial y^2}u(s,y) + k_1(t)|u(t-r,y)|^\gamma \\ &\qquad + \frac{k_2(t)}{4}\sin(t).u(t-r,y) \quad \text{for } t \ge 0,\ y \in [0,\pi], \end{aligned} \tag{4.4}$$

subjected to condition

$$u(t,0) = u(t,\pi) = 0 \quad \text{for } t \ge 0, \tag{4.5}$$

and finite delay

$$u(\theta, y) = u_0(\theta, y) \quad \text{for } \theta \in [-r, 0],\, y \in [0, \pi]. \tag{4.6}$$

Such equations arise in the study of heat conduction in materials with memory (see [27, 28]). We assume that $(\gamma, r) \in (0,1) \times (0, +\infty)$ are fixed and the delay function $u_0\colon [-r, 0] \times \Omega \to \mathbb{R}$ is continuous. The kernel $b\colon \mathbb{R}^+ \to \mathbb{R}$ is a bounded and $\mathcal{C}^1$ function such that b' is bounded and uniformly continuous. Moreover, $k_1, k_2\colon \mathbb{R}^+ \to \mathbb{R}^+$ are a continuous functions. We set

$$\begin{aligned}
&\mathbf{X} = L^2[0, \pi], \quad \mathcal{F} = C(\mathbb{R}^+, \mathbf{X}),\\
&\mathbb{A}(t) \equiv \mathbb{A} := \frac{\partial^2}{\partial y^2}, \quad \Upsilon(t, s) = b(t - s)\mathbb{A}, \quad \mathsf{Dom}(\mathbb{A}) = \mathsf{Dom}(\Upsilon(t, s)) := H^2[0, \pi] \cap H_0^2[0, \pi],\\
&x(t)(y) = u(t, y), \quad \varphi(\theta)(y) := u_0(\theta, y), \quad f(t, \phi)(y) = k_1(t)|\phi(-r)(y)|^\gamma
\end{aligned}$$

and $h(t, \phi) \equiv \frac{1}{4} k_2(t) \sin(t).\phi$ for all $0 \le s \le t$, $y \in [0, \pi]$ and $\phi \in \mathcal{F}$. Note that $\mathbb{A}$ generates a strongly continuous semigroup $(S(t))_{t \ge 0}$ on $\mathbf{X}$ which is compact for $t > 0$. Combining the semigroup compactness with the smoothness and boundedness of the kernel implies condition $(\mathbf{R})$. Therefore, there exists a unique resolvent operator for equation (1.1) which is a one parameter family of bounded operators. From [17] the resolvent operator $(R(t))_{t \ge 0}$ is compact for $t > 0$. For certain choice of b it follows from [18] that there exists a positive number a such that $\|R(t)\| \le Me^{-at}$ for all $t \ge 0$. Using the Hölder inequality we have

$$\|f(t, \phi)\| \le |k(t)|\pi^{(1-\gamma)/2}\|\phi(-r)\|^\gamma \le \pi^{(1-\gamma)/2}|k(t)|\|\phi\|^\gamma$$

and

$$\|g(t, \phi_1) - g(t, \phi_2)\| \le \frac{k_2(t)}{4}\|\phi_1 - \phi_2\|.$$

Let $q, \nu\colon \mathbb{R}^+ \to \mathbb{R}^+$ be the functions defined by $q(s) = |s|^\gamma$ and $\nu(s) = \pi^{(1-\gamma)/2}|k(s)|$.

Proposition 4.2 *The delay partial differential equation* (4.4)–(4.6) *has at least one mild solution on* $\mathbb{R}^+$ *provided that*

$$\int_c^{+\infty} \frac{\mathrm{d}s}{\pi^{(1-\gamma)/2}(s + |s|^\gamma)} > M \int_0^n |k(s)|\, \mathrm{d}s \quad \textit{for each } n > 0,$$

where $c = M\|\varphi\|$.

Example 3. We consider the stochastic problem

$$\begin{aligned}
\mathrm{d}z(t, \xi) = &\left[\frac{\partial^2 z(t, \xi)}{\partial \xi^2} + \tilde{b}\frac{\partial z(t, \xi)}{\partial \xi} + \tilde{c}z(t, \xi)\right]\\
&+ \int_0^t a(t - s)\left[\frac{\partial^2 z(s, \xi)}{\partial \xi^2} + \tilde{b}\frac{\partial z(s, \xi)}{\partial \xi} + \tilde{c}z(s, \xi)\right] \mathrm{d}s\\
&+ \int_{-\tau_1}^0 e^{b_1 s/2} \sin\left(t|z(t + s, \xi)|\right) \mathrm{d}s\, \mathrm{dW}(t) + e^{-t}\mathrm{dW}^H(t),\\
&\qquad\qquad \text{for } t \in J = [0, +\infty) \text{ and } \xi \in [0, 1],
\end{aligned} \tag{4.7}$$

subjected to condition

$$z(t,0) = z(t,1) = 0 \quad \text{for } t \in [0,+\infty), \tag{4.8}$$

and delay

$$z(\theta,\xi) = z_0(\theta,\xi) \quad \text{for } \theta \in [-\tau,0], \xi \in [0,1], \tag{4.9}$$

where $a\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, $\tilde{b}, \tilde{c} \in \mathbb{R}$ and $0 < \tau \le \tau_1 < \infty$. Let $\mathsf{H} = \mathsf{K} = L^2(0,1)$. Moreover, let $\mathsf{W}(t)$ and W^{H} denote a one-dimensional standard Wiener process and standard fBm process on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t\in[0,1]}, \mathbb{P})$, respectively. We define the operator $\mathbb{A}$ induced on H as follows:

$$\begin{cases} \mathsf{Dom}(\mathbb{A}) = H^2(0,1) \cap H_0^1(0,1), \\ \mathbb{A}x = x'' + \tilde{b}x' + \tilde{c}x, \quad \tilde{b}, \tilde{c} \in \mathbb{R}. \end{cases}$$

From [22, p. 173], we know that $\mathbb{A}$ is the infinitesimal generator of an analytic C_0-semigroup $(T(t))_{t\ge 0}$ on H. Therefore, the linear system corresponding to equation (4.7)–(4.9) has an associated resolvent operator on H. Let $\mathcal{G}\colon \mathsf{Dom}(\mathbb{A}) \subset \mathsf{H} \to \mathsf{H}$ be the operator defined by $\mathcal{G}(t)(x) = a(t)\mathbb{A}x$ for $t \ge 0$ and $x \in \mathsf{Dom}(\mathbb{A})$. We define the operator $h_1\colon \mathbb{R}^+ \times \mathcal{C}([-r,0],\mathsf{H}) \to \mathsf{H}$ by

$$h_1(t,\psi)(\xi) = \int_{-\tau_1}^{0} e^{b_1 t/2} \sin(t|\phi(s)(\xi)|)\,\mathrm{d}s.$$

If we put

$$\begin{cases} \vartheta(t)(\xi) = z(t,\xi) & \text{for } t \in [0,b] \text{ and } \xi \in [0,1], \\ \phi(\theta)(\xi) = z_0(\theta,\xi) & \text{for } \theta \in [-\tau,0] \text{ and } \xi \in [0,1], \end{cases}$$

then equation (4.7)–(4.9) can be written in the abstract form of equation (1.2). Moreover, for all $t \ge 0$, $\phi_1, \phi_2 \in \mathcal{C}([-r,0],\mathsf{H})$ we have

$$\begin{aligned} \mathbb{E}\|h_1(t,\phi)\|_{\mathbb{L}^2}^2 =& \mathbb{E}\left[\int_0^1 \left(\int_{-\tau_1}^0 e^{b_1 s/2} \sin(t|\phi(s)(\xi)|)\,\mathrm{d}s\right)^2 \mathrm{d}\xi\right] \\ \le& \mathbb{E}\left[\int_0^1 \int_{-\tau_1}^0 e^{b_1 s} t^2 |\phi(s)(\xi)|^2\,\mathrm{d}s\,\mathrm{d}\xi\right] \\ \le& \int_{-\tau_1}^0 t^2 e^{b_1 s}\,\mathbb{E}\|\phi(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s := \beta(t)\nu(\mathbb{E}\|\phi\|_{\mathbb{L}^2}^2) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\|h_1(t,\phi_1) - h_1(t,\phi_2)\|_{\mathbb{L}^2}^2 \\ &\quad = \mathbb{E}\left[\int_0^1 \left(\int_{-\tau_1}^0 e^{b_1 s/2}[\sin(t|\phi_1(s)(\xi)|) - \sin(t|\phi_2(s)(\xi)|)]\,\mathrm{d}s\right)^2 \mathrm{d}\xi\right] \\ &\quad \le L_R(t)\mathbb{E}\|\phi_1 - \phi_2\|_{\mathbb{L}^2}^2, \end{aligned}$$

where $L_R(t) \equiv \beta(t) = \frac{(1-e^{-b_1\tau_1})t^2}{b_1}$ and $\nu(t) = t$. Therefore, the assumptions **(B1)**–**(B2)** hold. Thus, all conditions of Theorem 3.7 are satisfied, and equation (4.7)–(4.9) has a mild solution on $\mathbb{R}^+$.

5 Conclusion

In this paper, we mainly discuss the global existence of mild solutions for some non-autonomous deterministic and stochastic integro-differential systems with time-dependent delay in Fréchet spaces. Our approach utilizes two-parameter resolvent operators and nonlinear alternatives of Avramescu and of Granas–Frigon to obtain the results. We have two main results. The first one applies to the case where the resolvent operator is compact and the nonlinear terms are Carathéodory–Lipschitz functions. We use Avramescu's fixed point theorem in this case. The second one deals with the case where the resolvent operator is not necessarily compact and the nonlinear terms are mixed Carathéodory-Lipschitz functions. We use Granas–Frigon's fixed point theorem for this case. Furthermore, we have illustrated the practical applications of our results through some concrete examples.

A possible future direction is to generalize our results to some integro-differential inclusions and fractional evolution equations with state dependent delay in both deterministic and stochastic settings.

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