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BESICOVITCH MULTI-DIMENSIONAL ALMOST AUTOMORPHIC TYPE FUNCTIONS AND APPLICATIONS

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Abstract. In this paper, we analyze various classes of multi-dimensional Besicovitch almost automorphic type functions, working with general Lebesgue spaces with variable exponents. We provide several illustrative examples and applications to abstract Volterra integro-differential equations.

Keywords: Abstract Volterra integro-differential equations, Besicovitch almost automorphic functions in $\mathbb{R}^n$, Besicovitch almost periodic functions in $\mathbb{R}^n$, Lebesgue spaces with variable exponents, Weyl almost automorphic functions in $\mathbb{R}^n$.

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1 Introduction and preliminaries

The notion of almost automorphy was discovered by the American mathematician S. Bochner in 1955 while he was studying problems related to differential geometry ([4]). The study of almost automorphy on (semi-)topological groups starts presumably with the papers of W. A. Veech [28, 29], which were published during the period 1965–1967. For more details about almost automorphic functions on semi-topological groups we refer the reader to [7, Section 4].

Suppose that $F\colon \mathbb{R}^n \to X$ is a continuous function, where $(X, \|\cdot\|)$ is a complex Banach space. Then, we say that the function $F(\cdot)$ is almost automorphic if and only if for every sequence $(\mathbf{b}_k)$ in

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$\mathbb{R}^n$ there exist a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ and a mapping $G\colon \mathbb{R}^n \to X$ such that

$$\lim_{k\to\infty} F\big(\mathbf{t}+\mathbf{a}_k\big) = G(\mathbf{t}) \text{ and } \lim_{k\to\infty} G\big(\mathbf{t}-\mathbf{a}_k\big) = F(\mathbf{t}), \tag{1.1}$$

pointwisely for $\mathbf{t} \in \mathbb{R}^n$. The range of an almost automorphic function $F(\cdot)$ is relatively compact in X, and the limit function $G(\cdot)$ is bounded on $\mathbb{R}^n$ but not necessarily continuous on $\mathbb{R}^n$. If the convergence of limits appearing in (1.1) is uniform on compact subsets of $\mathbb{R}^n$ (resp., the whole space $\mathbb{R}^n$), then we say that the function $F(\cdot)$ is compactly almost automorphic (resp., almost periodic). It is well-known that an almost automorphic function $F(\cdot)$ is compactly almost automorphic if and only if $F(\cdot)$ is uniformly continuous. For more details about almost periodic functions, almost automorphic functions, various generalizations and applications we refer the reader to the research monographs and articles [2, 3, 9, 10, 13, 14, 15, 18, 24, 30]; see especially [7, 19] and the lists of references quoted therein.

Various classes of multi-dimensional almost automorphic functions have been analyzed by A. Chávez *et al.* in the above-mentioned paper [7]. This research study has recently been continued in [21] and [22], where we have analyzed the Stepanov classes and the Weyl classes of multi-dimensional almost automorphic functions, respectively. (Let us recall that in the one-dimensional setting the notion of Stepanov almost automorphy was introduced by V. Casarino [8] in 2000 and later reconsidered by G. M. N'Guérékata and A. Pankov [17] in 2008, while the notion of Weyl almost automorphy was introduced by S. Abbas [1] in 2012.) The main aim of this study is to introduce and analyze the multi-dimensional Besicovitch almost automorphic functions as well as to present certain applications in the analysis of the existence and uniqueness of the Besicovitch almost automorphic type solutions for various classes of abstract Volterra integro-differential equations and partial differential equations. Some classes of Besicovitch almost automorphic functions introduced here seem to be new even in the one-dimensional setting. On the other hand, this is probably the first research article which seeks for spatially Besicovitch almost automorphic solutions of (abstract) PDEs.

The organization and main ideas of this paper can be briefly described as follows. The main part of the paper is Section 2, in which we analyze various notions of multi-dimensional Besicovitch almost automorphy in Lebesgue spaces with variable exponent. In Definition 2.1, we introduce the notion of Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphy, weak Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphy, Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphy of type 1 and weak Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphy of type 1. Proposition 2.2 states that any Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic function $F(\cdot;\cdot)$ is Besicovitch-$(\mathrm{R}, \mathcal{B}, \phi, \mathrm{F})$-$B^{p(\cdot)}$-normal, where $\mathrm{F}(l) \equiv \mathbb{F}(l, 0)$. Proposition 2.5 continues our analysis from [22, Proposition 2.10]. After that, in Definition 2.6, we introduce the notions of Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy and the Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy, and explain how these notions can be introduced for all other classes of functions from Definition 2.1. In Proposition 2.8, we consider the pointwise products of Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions with the scalar Besicovitch almost automorphic functions of a similar type. A composition principle for Besicovitch-$(\mathbb{F}, \phi, p, \mathrm{R}, \mathcal{B})$-multi-almost automorphic functions of type 1 is deduced in Theorem 2.10.

Some applications of our results to abstract Volterra integro-differential equations and partial differential equations are provided in Section 3. It is worth noticing that in Definition 3.1 we introduce the class of Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, w)$-multi-almost automorphic type functions, in which we aim to control the growth order of the limit function $F^*(\cdot)$ by the weight function $w(\cdot)$. This idea seems to be completely new and not explored elsewhere even in the one-dimensional setting. The notion introduced in Definition 3.1 plays a fundamental role in Proposition 3.2, where we investigate the invariance of Besicovitch almost automorphy under the actions of infinite convolution products, and Theorem 3.3, where we investigate the convolution invariance of multi-dimensional Besicovitch almost automorphy. Because of a certain similarity with our previous investigations of the existence and uniqueness of Besicovitch almost periodic solutions of abstract nonautonomous differential equations of first order and the classical wave equation, we have skipped here some irrelevant details concerning the existence and uniqueness of Besicovitch almost automorphic solutions for these classes of PDEs ([20]). The final conclusions and remarks about the introduced classes of functions are given in Section 4. In addition to the above, we also provide many useful comments, illustrative examples and propose some open problems. It is also worth noting that we give some new definitions, observations and examples regarding multi-dimensional Weyl almost automorphic type functions.

We use the standard notation throughout the paper. We assume henceforth that $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are complex Banach spaces. By $L(X, Y)$ we denote the Banach algebra of all bounded linear operators from X into Y, with $L(X, X)$ being denoted $L(X)$. If A is a closed linear operator on X, then its domain and range are denoted by $D(A)$ and $R(A)$, respectively. We assume henceforth that $\mathcal{B}$ is any collection of non-empty subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$; R denotes a general non-empty collection of sequences in $\mathbb{R}^n$. For further information concerning the Lebesgue spaces with variable exponents $L^{p(x)}$ we refer the reader to the research monograph [11] by L. Diening, P. Harjulehto, P. Hästüso and M. Ruzicka, as well as to [12, 19] and [26]. We will only note here that $\mathcal{P}(\mathbb{R}^n)$ denotes the space of all Lebesgue measurable functions $p\colon \mathbb{R}^n \to [1, \infty]$.

We need to recall the following definitions.

Definition 1.1 ([7, Definition 2.1]) *Suppose that $F\colon \mathbb{R}^n \times X \to Y$ is a continuous function. Then, we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and every sequence $(\mathbf{b}_k) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and a function $F^*\colon \mathbb{R}^n \times X \to Y$ such that*

$$\lim_{l\to+\infty} F\big(\mathbf{t} + \mathbf{b}_{k_l}; x\big) = F^*(\mathbf{t}; x)$$

and

$$\lim_{l\to+\infty} F^*\big(\mathbf{t} - \mathbf{b}_{k_l}; x\big) = F(\mathbf{t}; x),$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. If the convergence in the above-limit equations is uniform with respect to the sets B of collection $\mathcal{B}$, for a fixed number $\mathbf{t} \in \mathbb{R}^n$, then we say that the function $F(\cdot;\cdot)$ is uniformly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.

Definition 1.2 ([20, Definition 2.11]) *Suppose that $F\colon \mathbb{R}^n \times X \to Y$, $\phi\colon [0, \infty) \to [0, \infty)$ and $\mathrm{F}\colon (0, \infty) \to (0, \infty)$ are given. Then, we say that the function $F(\cdot;\cdot)$ is Besicovitch-$(\mathrm{R}, \mathcal{B}, \phi, \mathrm{F})$-$B^{p(\cdot)}$-normal if and only if for every set $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k)_{k\in\mathbb{N}}$ in R there exists a subsequence $(\mathbf{b}_{k_m})_{m\in\mathbb{N}}$ of $(\mathbf{b}_k)_{k\in\mathbb{N}}$ such that for any $\epsilon > 0$ an integer $m_0 \in \mathbb{N}$ can be found so that for all $m, m' \geq m_0$ we have*

$$\limsup_{t\to+\infty} \mathrm{F}(t) \sup_{x\in B} \Big[\phi\big(\|F(\mathbf{t} + \mathbf{b}_{k_m}; x) - F(\mathbf{t} + \mathbf{b}_{k_{m'}}; x)\|_Y\big)\Big]_{L^{p(\mathbf{t})}(A_t)} < \epsilon,$$

where $A_t = \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t}| \leq t\}$ for $t > 0$; here, $|\mathbf{t}|$ denotes the Euclidean norm of the point $\mathbf{t} \in \mathbb{R}^n$.

2 Multi-dimensional Besicovitch almost automorphy in Lebesgue spaces with variable exponent

The main aim of this section is to introduce and analyze various classes of multi-dimensional Besicovitch almost automorphic functions in Lebesgue spaces with variable exponent. Unless stated otherwise, we will always assume henceforth that $\Omega := [-1,1]^n \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $\mathbb{F} \colon (0,\infty) \times \mathbb{R}^n \to (0,\infty)$.

We start by introducing the following notion.

Definition 2.1 *Suppose that $F \colon \mathbb{R}^n \times X \to Y$ is a given function. Let for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \cdot\cdot\cdot, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* \colon \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $\phi(F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x) - F^*(\mathbf{t}+\mathbf{u};x)) \in L^{p(\mathbf{u})}(l\Omega : Y)$, $\phi(F^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x) - F(\mathbf{t}+\mathbf{u};x)) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as:*

(i)
$$\lim_{m\to+\infty}\limsup_{l\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)-F^*(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0 \tag{2.1}$$

and

$$\lim_{m\to+\infty}\limsup_{l\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x)-F(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0, \tag{2.2}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(ii)
$$\lim_{m\to+\infty}\liminf_{l\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)-F^*(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0$$

and

$$\lim_{m\to+\infty}\liminf_{l\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x)-F(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(iii)
$$\lim_{l\to+\infty}\limsup_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)-F^*(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0$$

and

$$\lim_{l\to+\infty}\limsup_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x)-F(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(iv)
$$\lim_{l\to+\infty}\liminf_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)-F^*(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0$$

and

$$\lim_{l\to+\infty}\liminf_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\sup_{x\in B}\Big[\phi\big(\|F^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x)-F(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^{p(\mathbf{u})}(l\Omega)}=0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

If (i) (*resp.,* (ii), (iii) *or* (iv)) *holds, then we say that the function* $F(\cdot;\cdot)$ *is Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic* (*resp., weakly Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic, Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic of type* 1 *or weakly Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic of type* 1). *By* $AAB^{\mathbb{F},\phi,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n\times X:Y)$ (*resp.,* $w\text{-}AAB^{\mathbb{F},\phi,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n\times X:Y)$, $AAB^{\mathbb{F},\phi,p(\mathbf{u}),1}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n\times X:Y)$ *and* $w\text{-}AAB^{\mathbb{F},\phi,p(\mathbf{u}),1}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n\times X:Y)$) *we denote the collection of all Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic* (*resp., weakly Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic, Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic of type* 1 *and weakly Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic of type* 1) *functions* $F\colon \mathbb{R}^n\times X\to Y$.

Trivially, if the requirements in (i) (resp., (iii)) of Definition 2.1 hold, then the requirements in (ii) (resp., (iv)) of Definition 2.1 hold as well. The notion introduced in [19, Definition 8.3.17 and Definition 8.3.32] is a special case of the notion introduced in Definition 2.1. The interested reader may simply clarify some sufficient conditions ensuring that the spaces introduced in Definition 2.1 are translation invariant or have a linear vector structure with the usual operations (see also the items [20, (i)–(iv)] clarified at the beginning of the second section). An analogue of [20, Proposition 2.13] holds in our new framework.

The case $\phi(x)\equiv x$, $p(\mathbf{u})\equiv p\in[1,\infty)$ and $\mathbb{F}(l,\mathbf{t})\equiv l^{-n/p}$ is the most important; we then say that the function $F\colon \mathbb{R}^n\times X\to Y$ is (weakly) Besicovitch p-$(\mathrm{R},\mathcal{B})$-multi-almost automorphic (of type 1). If, in addition to the above, the collection $\mathcal{B}$ consists of bounded subsets of X, then the notion of Besicovitch p-$(\mathrm{R},\mathcal{B})$-multi-almost automorphy is equivalent with the notion of Besicovitch-$(\mathrm{R},\mathcal{B})$-B^p-normality, since, in this case, an extension of the well-known result of J. Marcinkiewicz [25] holds (see [20, Theorem 2.5]) and the equations (2.1)–(2.2) hold for arbitrary $\mathbf{t}\in\mathbb{R}^n$ if and only the equations (2.1)–(2.2) hold with $\mathbf{t}=0$ (the value of the limit superior in these equations does not depend on $\mathbf{t}\in\mathbb{R}^n$; see the proof of [20, Proposition 3.3]). Furthermore, these two notions are equivalent in the case when $\phi(x)\equiv x^\alpha$ for some $\alpha\in(0,1]$. If we replace all the operations $\limsup$ and $\liminf$ in Definition 2.1 with the classical limits, then we obtain the corresponding notion of Weyl p-almost automorphy (of type 1).

If the function $F\colon \mathbb{R}^n\times X\to Y$ is Besicovitch p-$(\mathrm{R},\mathcal{B})$-multi-almost automorphic, for example, and R denotes the collection of all sequences in $\mathbb{R}^n$, then we omit the term “R” from the notation. Furthermore, if $X=\{0\}$, then we omit the term “$\mathcal{B}$” from the notation. This particularly means that the function $F\colon \mathbb{R}^n\to Y$ is Besicovitch-p-almost automorphic if and only if $F(\cdot)$ is Besicovitch-p-R-multi-almost automorphic with R being the collection of all sequences in $\mathbb{R}^n$. In [22, Theorem 2.9], we constructed an example of a Weyl (Besicovitch)-p-almost automorphic function which is not Besicovitch-p-bounded and therefore not Besicovitch-p-almost periodic ($p\geq 1$). Further on, if $p=1$, then we say that the function $F(\cdot)$ is Besicovitch almost periodic, automorphic, *etc*.

The proof of following result is not difficult and can be omitted.

Proposition 2.2 *Suppose that the function* $F(\cdot;\cdot)$ *is Besicovitch-*$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B})$*-multi-almost automorphic, and* $\mathrm{F}(l)\equiv\mathbb{F}(l,0)$. *Then, the function* $F(\cdot;\cdot)$ *is Besicovitch-*$(\mathrm{R},\mathcal{B},\phi,\mathrm{F})$*-*$B^{p(\cdot)}$*-normal.*

We continue by providing the following illustrative examples. (The first one is in support of our recent investigation of Weyl almost automorphy [22].)

Example 2.3 (based on the example of D. Brindle [5, Example 2.2]) *Let l_∞ denote the Banach space of all bounded numerical sequences, equipped with the supremum norm. Consider the function $f\colon \mathbb{R} \to l_\infty$ given by $f(t) := (e^{-|t|/k})_{k\in\mathbb{N}}$ for $t \in \mathbb{R}$. We know that this function is uniformly continuous, bounded, slowly oscillating and has no mean value so that $f(\cdot)$ is not Besicovitch almost periodic (for the notion and more details see* [19, Example 9.0.20]). *On the other hand, we can simply prove that the function $f(\cdot)$ is not Stepanov almost automorphic. In fact, if we assume the contrary, then the function $f(\cdot)$ needs to be almost automorphic (see,* e.g., [19, Lemma 2.3.4]), *since it is uniformly continuous. This is not the case, because we can use the sequence $(b_k \equiv k)$ in the corresponding definition of almost automorphy with $t = 0$ in order to conclude that for each $\epsilon \in (0, e^{-1})$ there exists an integer $k_0 \in \mathbb{N}$ such that*

$$\sup_{k\in\mathbb{N}}\left|e^{-l/k} - e^{-m/k}\right| < \epsilon, \quad l, m \in \mathbb{N}.$$

If we plug $k = k_0 = l$ here, then we obtain

$$\left|e^{-1} - e^{-m/k_0}\right| \le \sup_{k\in\mathbb{N}}\left|e^{-l/k} - e^{-m/k}\right| < \epsilon,$$

which gives a contradiction, since the first term tends to e^{-1} as $m \to +\infty$. Moreover, we can simply prove that the function $f(\cdot)$ is Weyl-p-almost automorphic for any finite real exponent $p \ge 1$. Actually,, for every sequence (b_k) in $\mathbb{R}$ we can take the same subsequence $(b_{k_m}) = (b_k)$ and the limit function $f^* \equiv f$ in the corresponding definition, since the function $f(\cdot)$ is slowly oscillating and bounded. It can be also proved that the function $f(\cdot)$ is Weyl-p-almost automorphic of type 1 (jointly Weyl-p-almost automorphic; see [22, Definition 2.5] *for the notion) for any finite real exponent $p \ge 1$, since for any sequence (b_k) tending to plus or minus infinity we can take the limit function $f^* \equiv 0$ in the corresponding definition. (The situation is much simpler for bounded sequences (b_k), when we can take an appropriate translation of the function $f(\cdot)$ as the limit function $f^*(\cdot)$.) This example is important because for the vector-valued function $f(\cdot)$ the limit $\lim_{t\to+\infty} t^{-1}\int_0^t f(s)\,ds$ does not exist in l_∞, but $\lim_{t\to+\infty} t^{-1}\int_0^t \|f(s)\|\,ds = 0$.*

Example 2.4

(i) *Suppose that $\omega \in \mathbb{R}^n \setminus \{0\}$, $\phi(0) = 0$ and a continuous function $F\colon \mathbb{R}^n \to Y$ is ω-periodic,* i.e., *$F(\mathbf{t} + \omega) = F(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$. Let R denote the collection of all sequences in the set $\omega \cdot \mathbb{Z}$. Then, the function $F(\cdot)$ is Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R})$-multi-almost automorphic (resp., weakly Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R})$-multi-almost automorphic, Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R})$-multi-almost automorphic of type 1 and weakly Besicovitch-$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R})$-multi-almost automorphic of type 1).*

(ii) *Let $p(\cdot) \equiv p \in [1, \infty)$. Then, it is very simple to construct an example of an ω-periodic continuous function $F\colon \mathbb{R}^n \to Y$ which is not weakly Besicovitch-$(\mathbb{F}, \phi, p, \mathrm{R})$-multi-almost automorphic of type 1. Suppose, for simplicity, that $\phi(x) \equiv x$, $n = 2$, $Y := \mathbb{C}$, and $\mathbb{F}(\cdot;\cdot)$ is arbitrary. Suppose, further, that $F_0\colon \{(x, y) \in \mathbb{R}^2 : 0 \le x + y \le 2\} \to [0, \infty)$ is any continuous function such that $F_0(x, y) = F_0(x + 1, y + 1)$ for every $(x, y) \in \mathbb{R}^2$ with $x + y = 0$.*

Assume also that a sequence (a_k) in $\mathbb{N}$ and a sequence (r_k) in $(0, \infty)$ satisfy $\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} r_k = +\infty$, $a_k + 3r_k < a_{k+1} - 3r_{k+1}$ for all $k \in \mathbb{N}$, and the value of the function $F_0(\cdot;\cdot)$

on the projection of the rectangle $a_k + [-r_k, r_k]^2$ *to the strip* $\{(x, y) \in \mathbb{R}^2 : 0 \leq x + y \leq 2\}$ *is greater or equal than* k.

After that, we extend the function $F_0(\cdot;\cdot)$ *to a continuous* $(1, 1)$*-periodic function defined on the whole space* $\mathbb{R}^2$ *in the usual way. Then, it can be simply shown that for each* $l > 0$ *we have*

$$\lim_{k \to +\infty} \int_{[-l,l]^2} \bigl|F(x + a_k, y)\bigr|^p \, \mathrm{d}x \, \mathrm{d}y = +\infty.$$

This implies the required conclusion with R *being the collection of all sequences in* $\{(k, 0) : k \in \mathbb{N}\}$.

The following result, extending [22, Proposition 2.10], can be also formulated in the multi-dimensional setting.

Proposition 2.5 *Suppose that* $p \geq 1$, $\sigma > 0$, $\mathbb{F}(l) \equiv l^{-\sigma}$ *and* $f \in L^p_{loc}(\mathbb{R} : X)$. *Moreover, assume that there exist a strictly increasing sequence* (l_k) *of positive real numbers tending to plus infinity, a sequence* (b_k) *of real numbers and a positive real number* $\epsilon_0 > 0$ *such that for every* $k \in \mathbb{N}$ *and for every subsequence of* (b_{k_m}) *of* (b_k) *we have*

$$\lim_{m \to +\infty} l_k^{-\sigma} \int_{b_{k_m} - l_k}^{b_{k_m} + l_k} \bigl\|f(x)\bigr\|^p \, \mathrm{d}x = +\infty. \tag{2.3}$$

Then, the function $f(\cdot)$ *is not Besicovitch-*$(\mathbb{F}, x, p)$*-almost automorphic of type* 1.

Proof. Suppose that the function $f(\cdot)$ is Besicovitch-$(\mathbb{F}, x, p)$-almost automorphic of type 1. Let $\epsilon > 0$ be arbitrary. Then, there exist a subsequence (b_{k_m}) of (b_k), a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ and a finite real number $l_0 > 0$ such that for every $l \geq l_0$ an integer $m_l \in \mathbb{N}$ can be found so that for any integer $m \geq m_l$ we have

$$l^{-\sigma} \int_{-l}^{l} \bigl\|f\bigl(x + b_{k_m}\bigr) - f^*(x)\bigr\|^p \, \mathrm{d}x < \epsilon.$$

Let $k \in \mathbb{N}$ be such that $l_k \geq l_0$. Then, due to (2.3), we have

$$\begin{aligned}
\epsilon &> l_k^{-\sigma} \int_{-l_k}^{l_k} \bigl\|f\bigl(x + b_{k_m}\bigr) - f^*(x)\bigr\|^p \, \mathrm{d}x \\
&\geq l_k^{-\sigma} 2^{1-p} \left[\int_{-l_k}^{l_k} \bigl\|f\bigl(x + b_{k_m}\bigr)\bigr\|^p \, \mathrm{d}x - \int_{-l_k}^{l_k} \bigl\|f^*(x)\bigr\|^p \, \mathrm{d}x \right] \\
&= l_k^{-\sigma} 2^{1-p} \left[\int_{-l_k + b_{k_m}}^{l_k + b_{k_m}} \bigl\|f\bigl(x\bigr)\bigr\|^p \, \mathrm{d}x - \int_{-l_k}^{l_k} \bigl\|f^*(x)\bigr\|^p \, \mathrm{d}x \right] \\
&\to +\infty, \quad m \to +\infty.
\end{aligned}$$

This ends the proof. □

We can simply reformulate [22, Example 3.6] in our new framework, as well as the conclusions established in [22, Proposition 3.7 and Proposition 3.8]. Further on, the convergence of limits in Definition 2.1 is pointwise for any $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. For our further work, it will be important to note that we can impose further requirements about the convergence of limits in Definition 2.1 and consider, in such a way, several new classes of multi-dimensional Besicovitch almost automorphic type functions. For example, consider the class $AAB^{\mathbb{F},\phi,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$ and assume that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$ we have that $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then, we can introduce the following notion (cf. also [7, Definition 2.2] and the example following it).

Definition 2.6 *We say that a function* $F \colon \mathbb{R}^n \times X \to Y$ *is*

(a) *Besicovitch-*$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic if and only if* (2.1)–(2.2) *hold pointwisely for all* $x \in B$ *and* $\mathbf{t} \in \mathbb{R}^n$*, as well as that for each* $x \in B$ *the convergence in* $\mathbf{t}$ *is uniform for any element of the collection* $W_{B,(\mathbf{b}_k)}(x)$*;*

(b) *Besicovitch-*$(\mathbb{F}, \phi, p(\mathbf{u}), \mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic if and only if* (2.1)–(2.2) *hold pointwisely for all* $x \in B$ *and* $\mathbf{t} \in \mathbb{R}^n$*, as well as that the convergence in* (2.1)–(2.2) *is uniform in* $(\mathbf{t}; x)$ *for any set of the collection* $\mathrm{P}_{B,(\mathbf{b}_k)}$*.*

We similarly define the $W_{\mathcal{B},\mathrm{R}}$- and $\mathrm{P}_{\mathcal{B},\mathrm{R}}$-classes of multi-dimensional Besicovitch almost periodic functions from the parts (ii)–(v) of Definition 2.1. We can also introduce the corresponding classes of Weyl almost automorphic functions considered in [22]; we only need to replace the operations $\limsup$ and $\liminf$ in Definition 2.1 with the usual limits. In connection with Definition 2.6 and the above observations, we will present the following example (cf. also [7, Example 5] and [21, Example 2.4]).

Example 2.7 *Suppose that* $\varphi \colon \mathbb{R} \to \mathbb{C}$ *is an almost automorphic function, and* $(T(t))_{t\in\mathbb{R}} \subseteq L(X, Y)$ *is an operator family which is strongly locally integrable and not strongly continuous at zero. Suppose, further, that there exist a finite real number* $M \geq 1$ *and a real number* $\gamma \in (0, 1)$ *such that*

$$\|T(t)\|_{L(X,Y)} \leq \frac{M}{|t|^\gamma}, \quad t \in \mathbb{R} \setminus \{0\}.$$

Let also R *be the collection of all sequences in* $\Delta_2 \equiv \{(t, t) : t \in \mathbb{R}\}$ *and let* $\mathcal{B}$ *be the collection of all bounded subsets of* X*. Define*

$$F(t, s; x) := e^{\int_s^t \varphi(\tau)\, \mathrm{d}\tau} T(t - s)x, \quad (t, s) \in \mathbb{R}^2,\ x \in X,$$

and assume that for each bounded subset B *of* X *and for each sequence* $(\mathbf{b}_k = (b_k, b_k))$ *in* R *the collection* $P_{B,(\mathbf{b}_k)}$ *consists of all sets of the form* $\{(t, s) \in \mathbb{R}^2 : |t - s| \leq L\} \times B$*, where* $L > 0$*. Define*

$$F^*(t, s; x) := e^{\int_s^t \varphi^*(r)\, \mathrm{d}r} T(t - s)x, \quad (t, s) \in \mathbb{R}^2,\ x \in X.$$

If the function $\varphi(\cdot)$ *is almost periodic, then it is not difficult to show, with the help of the computation established in* [7, Example 5]*, that the function* $F(\cdot, \cdot; \cdot)$ *is Stepanov* $(\Omega, 1)$*-*$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic*; *see* [21] *for the notion. But, this is no longer possible if the function* $\varphi(\cdot)$ *is almost automorphic but not almost periodic. If this is the case, then we can simply prove that the function* $F(\cdot, \cdot; \cdot)$ *is Weyl-*$(\mathbb{F}, x, 1, \mathrm{R}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic of type* 1*, since for each fixed real number* $l > 0$ *we have*

$$\lim_{m\to+\infty} \int_{[-l,l]^2} \sup_{x\in B} \big\| F\big(t + u_1 + b_{k_m}, s + u_2 + b_{k_m}; x\big) - F^*\big(t + u_1, s + u_2; x\big)\big\|_Y \, \mathrm{d}u_1\, \mathrm{d}u_2 = 0,$$

which follows from an application of the dominated convergence theorem and a simple calculation.

Now, we will state and prove the following result for the class of Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions. The same result holds for the Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions, and the interested reader may try to formulate this result for the functions introduced in parts (iii)–(iv) of Definition 2.1.

Proposition 2.8 *Suppose that* $p,\ q,\ r\in[1,\infty)$, $1/r=1/p+1/q$, $\mathbb{F}_1(l,\mathbf{t})\equiv l^{-n/p}$, $\mathbb{F}_2(l,\mathbf{t})\equiv l^{-n/q}$, $\mathbb{F}(l,\mathbf{t})\equiv l^{-n/r}$, *and* $\phi(x)\equiv x^{\alpha}$ *for some real number* $\alpha>0.$ *If for each sequence in* R *any its subsequence also belongs to* R, *the function* $F_1\colon\mathbb{R}^n\to\mathbb{C}$ *is Besicovitch-*$(\mathbb{F},\phi,p,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic* (*resp., weakly Besicovitch-*$(\mathbb{F}_1,\phi,p,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic*), $F_2\colon\mathbb{R}^n\to Y$ *is Besicovitch-*$(\mathbb{F},\phi,q,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic* (*resp., weakly Besicovitch-*$(\mathbb{F}_1,\phi,q,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic*), *and for each set* $B\in\mathcal{B}$ *there exist finite real numbers* $l_0>0$ *and* $m_B>0$ *such that*

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\,x\in B}\|[\phi(|F_1(\mathbf{t}+\cdot)|)]\|_{L^p(l\Omega)}\le m_B l^{n/p},\ l\ge l_0,$$

and

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\,x\in B}\|[\phi(\|F_2(\mathbf{t}+\cdot)\|_Y)]\|_{L^q(l\Omega)}\le m_B l^{n/q},\ l\ge l_0,$$

(*resp., there exists a strictly increasing sequence* (l_k) *of positive real numbers tending to plus infinity such that*

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\,x\in B;\,k\in\mathbb{N}}\|[\phi(|F_1(\mathbf{t}+\cdot)|)]\|_{L^p(l_k\Omega)}\le m_B l_k^{n/p}$$

and

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\,x\in B;\,k\in\mathbb{N}}\|[\phi(\|F_2(\mathbf{t}+\cdot)\|_Y)]\|_{L^q(l_k\Omega)}\le m_B l_k^{n/q}),$$

then the function $F\colon\mathbb{R}^n\times X\to Y$ *given by* $F(\mathbf{t};x):=F_1(\mathbf{t};x)F_2(\mathbf{t};x)$, $\mathbf{t}\in\mathbb{R}^n$, $x\in X$, *is Besicovitch-*$(\mathbb{F},\phi,r,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic* (*resp., weakly Besicovitch-*$(\mathbb{F},\phi,r,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic*).

Proof. We will consider the Besicovitch-$(\mathbb{F},\phi,p,\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions only. Let $(\mathbf{b}_k)\in\mathrm{R}$ and $B\in\mathcal{B}$ be given. Since for every sequence in R any its subsequence also belongs to R, we can extract a subsequence $(\mathbf{b}_{k_m})$ of $(\mathbf{b}_k)$ such that

$$\lim_{m\to+\infty}\limsup_{l\to+\infty}\mathbb{F}_1(l)\sup_{x\in B}\Big[\phi\big(|F_1(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)-F_1^*(\mathbf{t}+\mathbf{u};x)|\big)\Big]_{L^p(l\Omega)}=0,\tag{2.4}$$

$$\lim_{m\to+\infty}\limsup_{l\to+\infty}\mathbb{F}_1(l)\sup_{x\in B}\Big[\phi\big(|F_1^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x)-F_1(\mathbf{t}+\mathbf{u};x)|\big)\Big]_{L^p(l\Omega)}=0\tag{2.5}$$

as well as

$$\lim_{m\to+\infty}\limsup_{l\to+\infty}\mathbb{F}_2(l)\sup_{x\in B}\Big[\phi\big(\|F_2(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)-F_2^*(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^q(l\Omega)}=0\tag{2.6}$$

and

$$\lim_{m\to+\infty}\limsup_{l\to+\infty}\mathbb{F}_2(l)\sup_{x\in B}\Big[\phi\big(\|F_2^*(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_m};x)-F_2(\mathbf{t}+\mathbf{u};x)\|_Y\big)\Big]_{L^q(l\Omega)}=0.\tag{2.7}$$

Our assumption simply implies that for each set $B \in \mathcal{B}$ there exist finite real numbers $l_0' > 0$ and $m_B' > 0$ such that

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\, x\in B} \|[\phi(|F_1^*(\mathbf{t}+\cdot)|)]\|_{L^p(l\Omega)} \leq m_B' l^{n/p},\ l \geq l_0',$$

and

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\, x\in B} \|[\phi(\|F_2^*(\mathbf{t}+\cdot)\|_Y)]\|_{L^q(l\Omega)} \leq m_B' l^{n/q}, l \geq l_0'.$$

Keeping in mind these estimates and the equality $1/r = 1/p + 1/q$, the required first limit equality follows from (2.4)–(2.6), the existence of a finite real number $c_\alpha > 0$ such that

$$\begin{aligned}
&\phi\Big(\big\|F_1(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)F_2(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x) - F_1^*(\mathbf{t}+\mathbf{u};x)F_2^*(\mathbf{t}+\mathbf{u};x)\big\|_Y\Big) \\
&\quad \leq c_\alpha\Bigg[\phi\Big(\big|F_1(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x) - F_1^*(\mathbf{t}+\mathbf{u};x)\big|\Big)\cdot\phi\Big(\big\|F_2(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x)\big\|_Y\Big) \\
&\qquad + \phi\Big(\big|F_1^*(\mathbf{t}+\mathbf{u};x)\big|\Big)\cdot\phi\Big(\big\|F_2(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x) - F_2^*(\mathbf{t}+\mathbf{u};x)\big\|_Y\Big)\Bigg], \quad \mathbf{t}\in\mathbb{R}^n,
\end{aligned}$$

and the Hölder inequality. The second limit equality can be proved analogously, by using (2.5) and (2.7). □

Example 2.9 *It is worth noting that Proposition* 2.8 *can be applied to construct multi-dimensional almost automorphic functions of the form* $F(\mathbf{t}) := F_1(t_1)\cdot F_2(t_2)\cdot ...\cdot F_n(t_n)$, $\mathbf{t} = (t_1, ..., t_n) \in \mathbb{R}^n$, *where all functions* $F_j(\cdot)$ *are Besicovitch-*p*-almost automorphic in a certain sense* (*see also* [19, Example 8.1.6] *and* [20, Example 2.8]).

We close this section by stating and proving a composition principle for Besicovitch-$(\mathbb{F}, \phi, p, \mathrm{R}, \mathcal{B})$-multi-almost automorphic functions of type 1, which continues our analysis from [7, Theorem 2.20] and [20, Theorem 2.10]. We consider here the Besicovitch-p-almost automorphy of the multi-dimensional Nemytskii operator $W\colon \mathbb{R}^n \times X \to Z$ given by

$$W(\mathbf{t};x) := G(\mathbf{t};F(\mathbf{t};x)), \quad \mathbf{t}\in\mathbb{R}^n,\ x\in X, \tag{2.8}$$

where $F\colon \mathbb{R}^n \times X \to Y$ and $G\colon \mathbb{R}^n \times Y \to Z$.

Theorem 2.10 *Suppose that* $1 \leq p, q < +\infty$, $\alpha > 0$, $p = \alpha q$, $\mathrm{F}(t) \equiv t^{-n/p}$, $\phi(x) \equiv x^\zeta$ *for some real number* $\zeta > 0$. *Moreover, asssume that* $F(\cdot;\cdot)$ *is Besicovitch-*$(\mathbb{F}, \phi, p, \mathrm{R}, \mathcal{B})$*-multi-almost automorphic of type* 1 *and such that for every* $B \in \mathcal{B}$ *and* $(\mathbf{b}_k) \in \mathrm{R}$ *the subsequence* $(\mathbf{b}_{k_m})$ *of* $(\mathbf{b}_k)$ *and the function* $F^*\colon \mathbb{R}^n \times X \to Y$ *from* Definition 2.1 *satisfy* $F^*(\mathbf{t};x) \in \overline{\bigcup_{\mathbf{s}\in\mathbb{R}^n} F(\mathbf{s};x)}$, $\mathbf{t}\in\mathbb{R}^n$, $x \in X$. *Define* $B' := \overline{\bigcup_{\mathbf{t}\in\mathbb{R}^n} F(\mathbf{t};B)}$ *for each set* $B \in \mathcal{B}$, *and* $\mathcal{B}' := \{B' : B \in \mathcal{B}\}$. *Assume that for every sequence from* R *any its subsequence also belongs to* R.

(i) *Suppose that* $G\colon \mathbb{R}^n \times Y \to Z$ *is uniformly* $(\mathrm{R}, \mathcal{B}')$*-almost automorphic and there exists a finite real constant* $a > 0$ *such that*

$$\big\|G(\mathbf{t};y) - G(\mathbf{t};y')\big\|_Z \leq a\big\|y - y'\big\|_Y^\alpha, \quad \mathbf{t}\in\mathbb{R}^n,\ y,\ y' \in Y. \tag{2.9}$$

Then, the function $W(\cdot;\cdot)$, *given by* (2.8), *is Besicovitch-*$(\mathbb{F}^{p/q}, \phi, q, \mathrm{R}, \mathcal{B})$*-multi-almost automorphic of type* 1.

(ii) *By* $AAB^{\mathbb{F},\phi,q,1,a,\alpha}_{(\mathrm{R},\mathcal{B}')}(\mathbb{R}^n \times Y : Z)$ *we denote the class of all functions* $G_1 \in AAB^{\mathbb{F},\phi,q,1}_{(\mathrm{R},\mathcal{B}')}(\mathbb{R}^n \times Y : Z)$ *such that for each set* $B' \in \mathcal{B}'$ *there exists a sequence of uniformly* $(\mathrm{R}, \mathcal{B}')$*-multi-almost automorphic functions* $(G_1^k(\cdot;\cdot))$ *such that* (2.9) *holds with the function* $G(\cdot;\cdot)$ *replaced therein by the function* $G_1^k(\cdot;\cdot)$ *for all* $k \in \mathbb{N}$*, as well as that for each* $\epsilon > 0$ *there exist a sufficiently large real number* $l_0 > 0$ *and an integer* $k_0 \in \mathbb{N}$ *such that for every* $l \geq l_0$ *and* $k \geq k_0$ *we have*

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\, y\in B'} \mathbb{F}(\mathbf{t}, l)^{p/q} \left(\int_{[-l,l]^n} \bigl\| G_k(\mathbf{t}+\mathbf{u}; y) - G(\mathbf{t}+\mathbf{u}; y) \bigr\|_Z^{\zeta q} \, d\mathbf{u} \right)^{1/q} < \epsilon.$$

If $G \in AAB^{\mathbb{F},\phi,q,1,a,\alpha}_{(\mathrm{R},\mathcal{B}')}(\mathbb{R}^n \times Y : Z)$*, then the function* $W(\cdot;\cdot)$ *is Besicovitch-*$(\mathbb{F}^{p/q}, \phi, q, \mathrm{R}, \mathcal{B})$*-multi-almost automorphic of type* 1.

Proof. Let the set $B \in \mathcal{B}$ and the sequence $(\mathbf{b}_k) \in \mathrm{R}$ be given. By definition, there exist a subsequence $(\mathbf{b}_{k_m})$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that the requirements of Definition 2.1 (iii) hold and $F^*(\mathbf{t}; x) \in \overline{\bigcup_{\mathbf{s}\in\mathbb{R}^n} F(\mathbf{s}; x)}$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$. Since we have assumed that for every sequence from R any its subsequence also belongs to R, we may assume that the limit function $G^* : \mathbb{R}^n \times Y \to Z$ satisfies the corresponding limit equations pointwisely for $\mathbf{t} \in \mathbb{R}^n$, uniformly on the set B', and with the functions $F(\cdot;\cdot)$ and $F^*(\cdot;\cdot)$ replaced therein with the functions $G(\cdot;\cdot)$ and $G^*(\cdot;\cdot)$, respectively. Using (2.9) and the first limit equation for $G(\cdot;\cdot)$ and $G^*(\cdot;\cdot)$, we get that

$$\bigl\| G^*(\mathbf{t}; y) - G^*(\mathbf{t}; y') \bigr\|_Z \leq a\|x - y\|_Y^{\alpha}, \quad \mathbf{t} \in \mathbb{R}^n,\ y,\ y' \in B'. \tag{2.10}$$

In order to see that the function $W(\cdot;\cdot)$ is Besicovitch-$(\mathbb{F}^{p/q}, \phi, q, \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1, we first observe that for every $\mathbf{t} \in \mathbb{R}^n$, $x \in B$ and $m \in \mathbb{N}$ we have (here, we designate $(\tau_m := \mathbf{b}_{k_m})$)

$$\begin{aligned}
&\bigl\| G\bigl(\mathbf{t}+\tau_m; F\bigl(\mathbf{t}+\tau_m; x\bigr)\bigr) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \bigr\|_Z \\
&\quad \leq \bigl\| G\bigl(\mathbf{t}+\tau_m; F\bigl(\mathbf{t}+\tau_m; x\bigr)\bigr) - G(\mathbf{t}+\tau_m; F^*(\mathbf{t}; x)) \bigr\|_Z \\
&\qquad + \bigl\| G(\mathbf{t}+\tau_m; F^*(\mathbf{t}; x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \bigr\|_Z \\
&\quad \leq a\bigl\| F\bigl(\mathbf{t}+\tau_m; x\bigr) - F^*(\mathbf{t}; x) \bigr\|_Y^{\alpha} + \bigl\| G(\mathbf{t}+\tau_m; F^*(\mathbf{t}; x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \bigr\|_Z.
\end{aligned}$$

Since $x \in B$ and $F^*(\mathbf{t}; x) \in B'$ for all $\mathbf{t} \in \mathbb{R}^n$, we simply deduce the required conclusion from an elementary argument involving the fact that for every fixed real number $l > 0$, we have

$$\lim_{m\to+\infty} \int_{[-l,l]^n} \sup_{y\in B'} \bigl\| G\bigl(\mathbf{t}+\mathbf{u}+\tau_{\mathbf{m}}; y\bigr) - G(\mathbf{t}+\mathbf{u}; y) \bigr\|_Z^{\alpha q} \, d\mathbf{u} = 0, \quad \mathbf{t} \in \mathbb{R}^n,$$

which, in turn, follows from a simple application of the dominated convergence theorem. Keeping in mind (2.10) and the estimate

$$\begin{aligned}
&\bigl\| G^*\bigl(\mathbf{t}-\tau_l; F^*\bigl(\mathbf{t}-\tau_l; x\bigr)\bigr) - G(\mathbf{t}; F(\mathbf{t}; x)) \bigr\|_Z \\
&\quad \leq \bigl\| G^*\bigl(\mathbf{t}-\tau_l; F^*\bigl(\mathbf{t}-\tau_l; x\bigr)\bigr) - G^*(\mathbf{t}-\tau_l; F(\mathbf{t}; x)) \bigr\|_Z \\
&\qquad + \bigl\| G^*(\mathbf{t}-\tau_l; F(\mathbf{t}; x)) - G(\mathbf{t}; F(\mathbf{t}; x)) \bigr\|_Z, \quad l \in \mathbb{N},
\end{aligned}$$

the proof of the second limit equation is quite analogous. This proves the first part of the theorem. The second part of the theorem follows from the first part and a simple approximation argument. □

Before we switch to the next section, let us only note that an analogue of Theorem 2.10 can be formulated, under certain extra conditions, for the functions spaces introduced in Definition 2.6; see [7, Theorem 2.20] for more details.

3 Applications to abstract Volterra integro-differential eqautions

The main aim of this section is to furnish some applications of our results to abstract Volterra integro-differential equations and partial differential equations.

1. In this issue, we will first continue our analysis of the invariance of Besicovitch almost periodicity under the actions of an infinite convolution product

$$t \mapsto F(t) := \int_{-\infty}^{t} R(t-s)f(s)\ \mathrm{d}s, \quad t \in \mathbb{R}; \tag{3.1}$$

as mentioned in the first application of [20, Section 4], this result can be also given in the multi-dimenisonal setting and applied to a wide class of abstract (degenerate) Volterra integro-differential equations without initial conditions. For example, we can apply this result in the analysis of the existence and uniqueness of Besicovitch-p-almost automorphic type solutions of the fractional Poisson heat equation in $L^p(\mathbb{R}^n)$, and a class of abstract fractional differential equations with the higher-order elliptic operators in the Hölder spaces ([18]).

We assume that for the operator family $(R(t))_{t>0} \subseteq L(X,Y)$ there exist finite real constants $M>0$, $\beta \in (0,1]$ and $\gamma > 1$ such that

$$\|R(t)\|_{L(X,Y)} \leq M\frac{t^{\beta-1}}{1+t^{\gamma}}, \quad t>0. \tag{3.2}$$

Before stating our result, we need to introduce the following notion, which can be constituted in a much more general situation for the classes introduced in Definition 2.1.

Definition 3.1 *Suppose that the function $F\colon \mathbb{R}^n \to X$ is Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R})$-multi-almost automorphic (resp., Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},W_{\mathrm{R}})$-multi-almost automorphic and Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},P_{\mathrm{R}})$-multi-almost automorphic). Let $w\colon \mathbb{R} \to (0,\infty)$. Then, we say that $F(\cdot)$ is Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},w)$-multi-almost automorphic (resp., Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},W_{\mathrm{R}},w)$-multi-almost automorphic and Besicovitch-$(\mathbb{F},\phi,p(\mathbf{u}),\mathrm{R},P_{\mathrm{R}},w)$-multi-almost automorphic) if and only if for each sequence $(\mathbf{b}_k) \in \mathrm{R}$ the growth of the corresponding limit function $F^*\colon \mathbb{R} \to X$ from Definition 2.1 can be controlled by w, that is, there exists a finite real number $M>0$ such that $\|F^*(\mathbf{t})\| \leq Mw(|\mathbf{t}|)$, $\mathbf{t} \in \mathbb{R}^n$.*

The idea of controlling the growth order of the limit function $F^*(\cdot)$ by the weight function $w(\cdot)$ seems to be new within the theory of almost automorphic functions, and it is generally applicable in the analysis of many other classes of (generalized) almost automorphic functions known in the existing literature.

Now, we are ready to formulate the following analogue of [20, Proposition 4.1, Proposition 4.2].

Proposition 3.2 *Suppose that the operator family $(R(t))_{t>0} \subseteq L(X,Y)$ satisfies (3.2). Moreover, assume that $a>0$, $\alpha>0$, $1 \leq p < +\infty$, $\alpha p \geq 1$, $ap \geq 1$, $\alpha p(\beta-1)/(\alpha p-1) > -1$ if $\alpha p > 1$, and $\beta = 1$ if $\alpha p = 1$. If $b \in [0,\gamma-\beta)$, $w(t) := (1+|t|)^b$, $t \in \mathbb{R}$, the function $f\colon \mathbb{R} \to X$ is Besicovitch-$(t^{-a},x^{\alpha},p,\mathrm{R},w)$-multi-almost automorphic (resp., Besicovitch-$(t^{-a},x^{\alpha},p,\mathrm{R},W_{\mathrm{R}},w)$-multi-almost automorphic and Besicovitch-$(t^{-a},x^{\alpha},p,\mathrm{R},P_{\mathrm{R}},w)$-multi-almost automorphic) and there exists a finite real constant $M'>0$ such that $\|f(t)\|_Y \leq M'w(t)$, $t \in \mathbb{R}$, then the* *function* $F(\cdot)$*, given by* (3.1)*, is continuous, Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, w)$*-multi-almost automorphic* (*resp., Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, W_{\mathrm{R}}, w)$*-multi-almost automorphic and Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, P_{\mathrm{R}}, w)$*-multi-almost automorphic*) *and there exists a finite real constant* $M'' > 0$ *such that* $\|F(t)\|_Y \leq M''w(t)$*,* $t \in \mathbb{R}$*.*

Proof. We will consider the class of Besicovitch-$(t^{-a}, x^{\alpha}, p, \mathrm{R}, w)$-multi-almost automorphic functions only. Since we assumed that there exists a finite real constant $M' > 0$ such that $\|f(t)\|_Y \leq M'w(t)$, $t \in \mathbb{R}$, the function $F(\cdot)$ is well-defined and there exists a finite real constant $M'' > 0$ such that $\|F(t)\|_Y \leq M''w(t)$, $t \in \mathbb{R}$. The continuity of $F(\cdot)$ can be shown using the argument contained in the proof of [20, Proposition 4.2]. Let a sequence $(\mathbf{b}_k) \in \mathrm{R}$ be given. Then, there exist a subsequence $(\mathbf{b}_{k_m})$ of $(\mathbf{b}_k)$, a function $f^* \colon \mathbb{R} \to X$ and a finite real constant $M > 0$ such that $\|f^*(t)\| \leq Mw(t)$, $t \in \mathbb{R}$, and the equations (2.1)–(2.2) hold with the prescribed parameters, and the meaning clear. Define $F^* \colon \mathbb{R} \to Y$ by $F^*(t) := \int_{-\infty}^{t} R(t-s)f^*(s)\, \mathrm{d}s$, $t \in \mathbb{R}$. Then, it is clear that $F^*(\cdot)$ is well-defined as well as that there exists a finite real constant $M''' > 0$ such that $\|F^*(t)\| \leq M'''w(t)$, $t \in \mathbb{R}$. In order to see that the estimate (2.2) holds for the functions $F(\cdot)$ and $F^*(\cdot)$, take any real number $\zeta \in ((1/(\alpha p)) + b, (1/(\alpha p)) + \gamma - \beta)$. Then, we can argue as in the computation carried out in the proof of [20, Proposition 4.1]. For any $t \in \mathbb{R}$ we have

$$
\begin{aligned}
&\frac{1}{2l^{ap}} \int_{-l}^{l} \big\|F(s+b_{k_m}+t) - F^*(s+t)\big\|^{\alpha p}\, \mathrm{d}s \\
&\leq \frac{1}{2l^{ap}} \int_{-l}^{l} \left| \int_{-\infty}^{0} \|R(-z)\| \cdot \big\|F(s+b_{k_m}+t+z) - F^*(s+t+z)\big\|\, \mathrm{d}z \right|^{\alpha p} \mathrm{d}s \\
&\leq \frac{M}{2l^{ap}} \int_{-l}^{l} \left| \int_{-\infty}^{0} \frac{|z|^{\beta-1}(1+|z|)^{\zeta}}{(1+|z|^{\gamma})} \cdot (1+|z|)^{-\zeta} \big\|F(s+b_{k_m}+t+z) - F^*(s+t+z)\big\|\, \mathrm{d}z \right|^{\alpha p} \mathrm{d}s \\
&\leq \frac{M_1}{2l^{ap}} \int_{-l}^{l} \int_{-\infty}^{0} \frac{1}{(1+|z|^{\alpha\zeta})^{p}} \big\|F(s+b_{k_m}+t+z) - F^*(s+t+z)\big\|^{\alpha p}\, \mathrm{d}z\, \mathrm{d}s \\
&= \frac{M_1}{2l^{ap}} \int_{-l}^{l} \int_{z-s}^{l} \frac{1}{(1+|z-s|^{\alpha\zeta})^{p}} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}s\, \mathrm{d}z \\
&\quad + \frac{M_1}{2l^{ap}} \int_{-\infty}^{-l} \int_{-l}^{l} \frac{1}{(1+|z-s|^{\alpha\zeta})^{p}} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}s\, \mathrm{d}z \\
&\leq \frac{M_1}{l^{ap}} \int_{-l}^{l} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}z \cdot \int_{-\infty}^{+\infty} \frac{\mathrm{d}s}{(1+|s|^{\zeta})^{\alpha p}} \\
&\quad + \frac{M_1}{2l^{ap}} \int_{-\infty}^{-3l} \int_{-l}^{l} \frac{1}{(1+|z-s|^{\alpha\zeta})^{p}} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}s\, \mathrm{d}z \\
&\quad + \frac{M_1}{2l^{ap}} \int_{-3l}^{3l} \int_{-l}^{l} \frac{1}{(1+|z-s|^{\alpha\zeta})^{p}} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}s\, \mathrm{d}z \\
&\leq \frac{M_1}{l^{ap}} \int_{-l}^{l} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}z \cdot \int_{-\infty}^{+\infty} \frac{\mathrm{d}s}{(1+|s|^{\zeta})^{\alpha p}} \\
&\quad + \frac{M_1}{l^{ap}} \int_{-3l}^{3l} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}z \cdot \int_{-\infty}^{+\infty} \frac{\mathrm{d}s}{(1+|s|^{\zeta})^{\alpha p}} \\
&\quad + \frac{cM_1 l}{2l^{ap}} \int_{-\infty}^{-3l} \frac{1}{(1+|z/2|^{\alpha\zeta})^{p}} \big\|F(b_{k_m}+t+z) - F^*(t+z)\big\|^{\alpha p}\, \mathrm{d}z;
\end{aligned}
$$

here, we have used the Hölder inequality, the Fubini theorem and an elementary change of variables in the double integral. The estimate (2.2) for the functions $F(\cdot)$ and $F^*(\cdot)$ can be proved analogously. This completes the proof. □

It is worth noting that Proposition 3.2 can be reformulated for all other classes of one-dimensional Besicovitch almost automorphic type functions introduced in Definition 2.1, but not for multi-dimensional Weyl almost automorphic functions considered in [22], since it is not clear how one can prove the existence of the right limits (classical ones) in the equations from Definition 2.1. Using the method proposed in the proofs of [18, Theorem 3.7.1] and Proposition 3.2, we can study the existence and uniqueness of Besicovitch-p-almost automorphic solutions for a class of abstract nonautonomous differential equations of first order; see also [20, Theorem 4.5].

2. Concerning the convolution invariance of multi-dimensional Besicovitch almost automorphy, the notion introduced in Definition 3.2 again plays a crucial role. We need to control the growth order of limit functions in order to obtain any relevant result. The conclusions established in this application can be also formulated for all other classes of functions introduced in Definition 2.1.

We will consider first the actions of the Gaussian semigroup

$$F \mapsto (G(t)F)(x) \equiv \big(4\pi t\big)^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2/4t} F(x-y)\ \mathrm{d}y, \quad t > 0,\ x \in \mathbb{R}^n. \tag{3.3}$$

Assume that there exist two finite real numbers $b \geq 0$ and $c > 0$ such that $|F(x)| \leq c(1 + |x|)^b \equiv cw(x)$, $x \in \mathbb{R}^n$. Moreover, assume that $a > 0$, $\alpha > 0$, $1 \leq p < +\infty$, $\alpha p \geq 1$ and $1/(\alpha p) + 1/q = 1$. Also, let $F(\cdot)$ be Besicovitch-$(t^{-a}, x^{\alpha}, p, \mathrm{R}, w)$-multi-almost automorphic, where R is a general collection of sequences in $\mathbb{R}^n$. Let us fix a real number t_0 in (3.3). Then, the mapping $x \mapsto (G(t_0)F)(x)$, $x \in \mathbb{R}^n$, is well-defined and has the same growth as the inhomogeneity $F(\cdot)$. Writing the term $e^{-|y|^2/4t_0}$ as $e^{-|y|^2/8t_0} \cdot e^{-|y|^2/8t_0}$ and applying the Hölder inequality, we may conclude that the function $F(\cdot)$ is Besicovitch-$(t^{-a}, x^{\alpha}, p, \mathrm{R}, w)$-multi-almost automorphic; see, *e.g.*, the argument given in the fourth application of [20, Section 4].

We can similarly prove the following analogue of [20, Theorem 4.6] (see also [22, Theorem 3.9 and Theorem 3.13]).

Theorem 3.3 *Suppose that* $b \geq 0$, $\alpha > 0$, $a > 0$, $1 \leq p < +\infty$, $\alpha p \geq 1$ *and* $1/(\alpha p) + 1/q = 1$. *Moreover, assume that* $f \colon \mathbb{R}^n \to Y$ *is Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, w)$*-multi-almost automorphic* (*resp., Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, W_{\mathrm{R}}, w)$*-multi-almost automorphic and Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, P_{\mathrm{R}}, w)$*-multi-almost automorphic*)*, where* $w(t) \equiv (1 + |t|)^b$, $t \in \mathbb{R}$. *If there exist two functions* $h_1 \colon \mathbb{R}^n \to \mathbb{C}$ *and* $h_2 \colon \mathbb{R}^n \to \mathbb{C}$ *such that* $h = h_1 h_2$, $h_1 \in L^q(\mathbb{R}^n)$ *and* $|h_1(\cdot)|^{\alpha}[1 + |\cdot|]^{\zeta} \in L^p(\mathbb{R}^n)$ *with* $\zeta = \max(b\alpha, a)$*, then the function* $F(\cdot)$*, given by*

$$F(x) \equiv \int_{\mathbb{R}^n} h(x-y) f(y)\ \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

is Besicovitch-$(t^{-a}, x^{\alpha}, p, \mathrm{R}, w)$*-multi-almost automorphic* (*resp., Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, W_{\mathrm{R}}, w)$*-multi-almost automorphic and Besicovitch-*$(t^{-a}, x^{\alpha}, p, \mathrm{R}, P_{\mathrm{R}}, w)$*-multi-almost automorphic*).

The notion introduced in Definition 3.2 is important if we want to reconsider the fifth and sixth application of [20, Section 4]. We will only note that the analysis of the existence and uniqueness of Besicovitch-p-almost automorphic type solutions of the wave equation whose solutions are given by the famous d'Alembert formula (the Kirchhoff formula, the Poisson formula) can be carried out in almost the same way as in the almost periodic case, and the same conclusions can be achieved. The analysis of the existence and uniqueness of Besicovitch-p-almost automorphic type solutions connected with the use of evolution systems considered in the above-mentioned sixth application of [20, Section 4] and the final application of [19, Section 6.3, pp. 426–428] can be carried out as in the almost periodic case as well.

The notion introduced in Definition 3.2 is also important if we want to reconsider the application from [7, Example 1], given directly before Subsection 1.1 of that paper. More precisely, suppose that A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X whose elements are certain complex-valued functions defined on $\mathbb{R}^n$. Under some assumptions, the function

$$u(t,x)=\bigl(T(t)u_0\bigr)(x)+\int_0^t [T(t-s)f(s)](x)\ \mathrm{d}s,\quad t\geq 0,\ x\in\mathbb{R}^n,$$

is a unique classical solution of the abstract Cauchy problem

$$u_t(t,x)=Au(t,x)+F(t,x),\ t\geq 0,\ x\in\mathbb{R}^n;\ u(0,x)=u_0(x),$$

where $F(t,x):=[f(t)](x)$, $t\geq 0$, $x\in\mathbb{R}^n$. In many concrete situations (for example, this holds for the Gaussian semigroup on $\mathbb{R}^n$), we have the existence of a kernel $(t,y)\mapsto E(t,y)$, $t>0$, $y\in\mathbb{R}^n$, which is integrable on any set $[0,T]\times\mathbb{R}^n$ $(T>0)$ and satisfies the condition

$$[T(t)f(s)](x)=\int_{\mathbb{R}^n} F(s,x-y)E(t,y)\ \mathrm{d}y,\quad t>0,\ s\geq 0,\ x\in\mathbb{R}^n.$$

If a real number $t_0>0$ is fixed and the above requirement holds, then we observed in [6, Example 0.1] that the almost periodic behaviour of the function $x\mapsto u_{t_0}(x)\equiv\int_0^{t_0}[T(t_0-s)f(s)](x)\,\mathrm{d}s$, $x\in\mathbb{R}^n$, depends on the almost periodic behaviour of the function $F(t,x)$ in the space variable x. The argument given there is applicable not only for almost periodicity but also for almost automorphy and various generalizations of these concepts provided that the exponent $p(\cdot)$ has a constant value 1. For example, if the function $F(t,x)$ is Besicovitch-$(\mathbb{F},x,1,\mathrm{R},1)$-multi-almost automorphic with respect to the variable $x\in\mathbb{R}^n$, uniformly in the variable t on compact subsets of $[0,\infty)$, the solution $u_{t_0}(\cdot)$ will be Besicovitch-$(\mathbb{F},x,1,\mathrm{R},1)$-multi-almost automorphic as well; see, *e.g.*, the computations carried out in [19, pp. 402–403] for more details. In connection with the abstract Cauchy problems considered above, we can also recommend the research article [16], where G. M. N'Guérékata investigated the existence and uniqueness of almost automorphic mild solutions for certain classes of abstract semilinear differential equations.

3. Without going into full details, we will only note that Theorem 2.10 can be applied in the analysis of the existence and uniqueness of bounded, continuous, Besicovitch-$(\mathbb{F},\phi,p,\mathrm{R},\mathcal{B})$-multi-almost automorphic of type 1 solutions for a various classes of abstract (fractional) semilinear Cauchy problems; here, $\mathcal{B}=\mathcal{B}'$ can be chosen to be the collection consisting of all bounded subsets of the Banach space X and $p\geq 1$ is any finite real exponent $(\alpha=1)$. See also the third application of [20, Section 4] for more details.

4 Conclusions and final remarks

In this paper, we have introduced and investigated various classes of multi-dimensional Besicovitch almost automorphic type functions. We have presented many illustrative examples and certain applications to abstract Volterra integro-differential equations, working with general Lebesgue spaces with variable exponents.

Concerning certain drawbacks and possibilities for further investigations of multi-dimensional Besicovitch almost automorphic type functions, we want to mention first that, besides the classes of functions introduced above, we can also consider some other classes. For example, in parts (i) and (ii) of Definition 2.1 we can replace $\lim_{m\to+\infty}$ with $\liminf_{m\to+\infty}$, while in parts (iii) and (iv) of Definition 2.1 we can replace $\lim_{l\to+\infty}$ with $\liminf_{l\to+\infty}$; the notion introduced in [19, Definition 8.3.18 and Definition 8.3.28] can be extended only if we use the function $\phi(\cdot)$ in the analysis. We have not considered these topics here. Also, we have not studied the integration and differentiation of multi-dimensional Besicovitch (Weyl) almost automorphic type function and multi-dimensional Besicovitch (Weyl) almost automorphic type functions with values in nonlocally convex spaces (cf. [23] for some results obtained in the almost periodic setting).

Concerning some open problems, we would like to recall first that we asked in [22, Question 5.1] whether for a given real exponent $p \geq 1$ we can find a Weyl p-almost automorphic function $f : \mathbb{R} \to Y$ of type 1 which is not Weyl p-almost automorphic. The same question can be raised for Besicovitch almost automorphic type functions. Moreover, in [22, Question 5.3] we asked whether an almost automorphic function $f : \mathbb{R} \to Y$ is automatically Weyl-p-almost automorphic. It is also not clear whether an almost automorphic function is Besicovitch-p-almost automorphic for some finite real exponent $p \geq 1$. Recall only that any Stepanov-p-almost automorphic function $f : \mathbb{R} \to Y$ is Weyl-p-almost automorphic of type 1, which shows the full importance of this class of Weyl almost automorphic type functions.

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