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STABILITY AND CONVERGENCE OF A NUMERICAL SCHEME FOR ADVECTION-DIFFUSION EQUATIONS INVOLVING A FRACTIONAL LAPLACE OPERATOR

MARTIN NITIEMA*
Université Joseph KI-ZERBO, Ouagadougou, Burkina Faso - Laboratoire LAMIA,
Université des Antilles, Campus Fouillole, 97159 Pointe-à-Pitre Guadeloupe (FWI)

SOMDOUDA SAWADOGO†
Ecole Normale Supérieure, Laboratoire LANIBIO, Ouagadougou, Burkina Faso

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Abstract. We consider an advection-diffusion equation involving a Laplace fractional operator of order $1/2 < s < 1$. We first assume that the solution of this equation is regular. Then we use a combination of left and right fractional Riemann-Liouville derivatives of order $2s$ to approximate the fractional Laplace operator. This allows us to obtain a numerical scheme of Euler explicit type which is proven to be conditionally stable, first order in time and space accurate. Numerical results are given to illustrate the results.

Keywords: fractional Laplace operator, fractional Riemann-Liouville derivatives, stability and convergence.

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*e-mail address: nmartino.nitiema11@gmail.com
†e-mail address: sawasom@yahoo.fr

1 Introduction

In this paper, we are interested in the following fractional evolution equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(x,t) + v\dfrac{\partial \rho}{\partial x}(x,t) + D(-\Delta)^s \rho(x,t) &= f(x,t) \qquad (x,t) \in (0,L)\times(0,T), \\ \rho(0,t) = \rho(L,t) &= 0 \qquad t \in (0,T), \\ \rho(x,0) &= g(x) \qquad x \in (0,L), \end{cases} \tag{1.1}$$

where $T > 0$, $v \in \mathbb{R}$, $D > 0$ and $L > 0$. The functions $g : [0,L] \to \mathbb{R}$ and $f : [0,L] \times [0,T] \to \mathbb{R}$ are given. The operator $(-\Delta)^s$, $0 < s < 1$ denotes the fractional Laplace operator formally given for a suitable function ψ by [3, 8]

$$(-\Delta)^s \psi = \mathcal{F}^{-1}|x|^{2s}\mathcal{F}\psi(x),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and the inverse of Fourier transform operators respectively.

Model (1.1) can be viewed as an advection-diffusion equation involving the Laplace fractional operator of order $0 < s < 1$. Such a model can be used to describe the diffusion and the transport of pollutants in porous media.

Fractional derivatives are an excellent tool to describe memory and heredity effects of various materials and processes. They have been successfully applied by many researchers during the last decade in several scientific fields such as biology, physics, chemistry and even finance.

Many authors have been interested in the methods of numerical resolution of the fractional advection-diffusion problem. In [4], Sadia Arshad *et al.* consider the following fractional advection-diffusion problem involving the Riesz-space fractional operators:

$$\begin{cases} D_t^\alpha \rho(x,t) = K_1 \dfrac{\partial^{\beta_1}}{\partial |x|^{\beta_1}}\rho(x,t) + K_2 \dfrac{\partial^{\beta_2}}{\partial |x|^{\beta_2}}\rho(x,t) + f(x,t) & (x,t) \in (0,L)\times(0,T), \\ \rho(0,t) = \rho(L,t) = 0 & t \in (0,T), \\ \rho(x,0) = g(x) & x \in (0,L), \end{cases} \tag{1.2}$$

where $0 < \alpha, \beta_1 < 1$, $1 < \beta_2 < 2$, $K_1 \geq 0$, $K_2 > 0$ and D_t^α stands for the time fractional Caputo derivative of order α. The operators $\dfrac{\partial^{\beta_1}}{\partial |x|^{\beta_1}}$ and $\dfrac{\partial^{\beta_2}}{\partial |x|^{\beta_2}}$ are respectively the Riesz-space fractional derivatives of order β_1 and β_2. The real functions f and g are suitably given. Using a combination of left and right fractional Riemann-Liouville derivatives to approximate the Riesz-space fractional operators, they constructed a numerical scheme which combines a finite difference method with the trapezoidal method to solve (1.2). They also proved the stability and the convergence to second order in space of their scheme. Considering the same combination of left and right fractional Riemann-Liouville derivatives to approach the Riesz-space fractional operators, Shujun Shen *et al.* [5] constructed an explicit and an implicit finite difference method to solve (1.2). They proved that their scheme is stable and converges to first order in time and second order in space. F. Liu *et al.* [3], consider the problem (1.2) with $f \equiv 0$. They provided three numerical methods to deal with the fractional derivatives in Riesz space, namely the $L1/L2$ approximation method which consists in approximating the diffusion and advection terms, the standard Grünwald/offset method which consists in discretizing the fractional Riesz derivative and finally the matrix transformation method to approximate the fractional Riesz diffusion equation. Jingjun Zhao *et al.* [6] consider a multi-term space and time fractional Riesz advection-diffusion problem which has the space spacial terms as in (1.2) but where the Caputo time fractional derivative D_t^α has been replaced by a combination of Caputo time fractional derivatives of different orders between 0 and 2. They gave an efficient finite element method for the considered advection-diffusion problem using the Diethelm fractional backward finite difference method developed in [7] to discretize the time multiterm. The existence and uniqueness of solution are obtained by means of Lax-Milgram theorem. They proved the stability and convergence of their scheme and also gave some numerical examples to illustrate their results. Note that in all the above papers, the convection term involves the Riesz-space fractional operator of order $\alpha \in (0, 1)$.

In this paper, we are interested in the problem (1.1) which involves a fractional Laplace operator and a classical gradient operator in one dimension in space. Observing that the fractional Laplace operator can be written as a combination of left and right fractional Riemann-Liouville derivatives (see [3, Lemma 1]), we constructed a numerical scheme of Euler explicit type to solve (1.1). We prove that our scheme is stable and converges to the first order in time and in space.

The rest of the paper is organized as follows. In Section 2, we recall some definitions of fractional integral and fractional derivative of Riemann-Liouville. Section 3 is devoted to the construction of the numerical scheme for (1.1). We prove the stability and the convergence of the numerical scheme in Section 4 for $1/2 < s < 1$ and perform numerical tests to verify the orders of convergence of the numerical scheme in Section 5.

2 Preliminaries

Let $L > 0$ be a real number and $f : [0, L] \to \mathbb{R}$ be a given function.

Definition 2.1 [4, 9, 10] *Let $a, L \in \mathbb{R}$. Then the left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ of f, are defined, respectively, by:*

$$({}_aI_x^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad (x > a)$$

and

$$({}_xI_L^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^L (t-x)^{\alpha-1} f(t)\,\mathrm{d}t, \quad (x < L),$$

provided that the integrals exist.

Definition 2.2 [9–11] *Let $a, L \in \mathbb{R}$. Then the left and right Riemann–Liouville fractional derivatives of order $\alpha \in (1, 2)$ of f, are defined, respectively, by:*

$$({}_aD_x^\alpha f)(x) := \frac{\mathrm{d}^2}{\mathrm{d}x^2}({}_aI_x^{2-\alpha} f)(x) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_a^x (x-t)^{1-\alpha} f(t)\,\mathrm{d}t, \qquad (x > a), \qquad (2.1)$$

and

$$({}_xD_L^\alpha f)(x) := \frac{\mathrm{d}^2}{\mathrm{d}x^2}({}_xI_L^{2-\alpha} f)(x) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_x^L (t-x)^{1-\alpha} f(t)\,\mathrm{d}t, \qquad (x < L), \qquad (2.2)$$

provided that the integrals exist.

We recall the following result [3, Lemma 1].

Lemma 2.1 *Let $0 < \alpha < 1$. Then for any real function u defined on $\mathbb{R}$, the following equality holds:*

$$(-\Delta)^\alpha u(x) = \frac{1}{2\cos(\frac{\pi\alpha}{2})}\left[{}_{-\infty}D_x^{2\alpha}u(x) + {}_xD_{+\infty}^{2\alpha}u(x)\right] = \frac{\partial^{2\alpha}}{\partial|x|^{2\alpha}}u(x), \tag{2.3}$$

provided that the right and left Riemann-Liouville fractional derivatives ${}_xD_{+\infty}^{2\alpha}u$ and ${}_{-\infty}D_x^{2\alpha}u$ exist.

3 The finite difference scheme

In this section, we are concerned with the numerical analysis of (1.1). In the interval $[0, L]$, we take the mesh points $x_i = ih$, $i = 0, 1, \cdots, N$ and $t^n = n\tau$, $n = 0, 1, \cdots, M$, where $h = L/N$ and $\tau = T/M$ are respectively the uniform spatial step size and temporal step size. Then, we can write

$$\rho_i^n \approx \rho(ih, n\tau), \quad \text{for } i = 0, \cdots, N \text{ and } n = 0, \cdots, M. \tag{3.1}$$

Since the solution of (1.1) is such that $\rho(0,t) = \rho(L,t)$ for almost every $t \in (0,T)$, using (2.1), (2.2) and (2.3), we have as in [8, 13] that

$$(-\Delta)^s\rho(x,t) = \frac{{}_0D_x^{2s}\rho(x,t) + {}_xD_L^{2s}\rho(x,t)}{2\cos(s\pi)}, \quad (x,t) \in (0,L) \times (0,T), \tag{3.2}$$

provided that the right and left Riemann- Liouville fractional derivatives ${}_xD_L^{2s}\rho$ and ${}_0D_x^{2s}\rho$ exist.

In order to construct our numerical scheme, we need to approximate the fractional Laplace operator $(-\Delta)^s\rho$ and thus the right and left Riemann- Liouville fractional derivatives ${}_xD_L^{2s}\rho$ and ${}_0D_x^{2s}\rho$.

Observing that for any $\zeta \in \mathbb{C}$ such that $|\zeta| \leq 1$, we can write

$$(1-\zeta)^{2s} = \sum_{k=0}^{\infty}(-1)^k \begin{pmatrix} 2s \\ k \end{pmatrix} \zeta^k,$$

where $\begin{pmatrix} 2s \\ k \end{pmatrix} = \dfrac{(2s)!}{k!(2s-k)!}$.

We set

$$g_k^{(2s)} = (-1)^k \begin{pmatrix} 2s \\ k \end{pmatrix} \tag{3.3}$$

and the following relation between $g_k^{(2s)}$ and $g_{k-1}^{(2s)}$ holds true:

$$\begin{cases} g_k^{(2s)} &= (1 - \dfrac{1+2s}{k})g_{k-1}^{(2s)}, \quad \forall k = 1, 2, \cdots \\ g_0^{(2s)} &= 1. \end{cases}$$

Moreover, we have from Lemma 2.3 in [2, 12] that

$$\begin{cases} g_0^{(2s)} = 1,\ g_1^{(2s)} = -2s < 0,\ g_2^{(2s)} = \dfrac{2s(2s-1)}{2}, \\ 1 \geq g_2^{(2s)} \geq g_3^{(2s)} \geq \cdots \geq 0, \\ \displaystyle\sum_{k=0}^{\infty} g_k^{(2s)} = 0,\ \sum_{k=0}^{m} g_k^{(2s)} < 0,\ m \geq 1, \end{cases} \tag{3.4}$$

because $1/2 < s < 1$.

We assume that ρ is regular enough and we then consider as in [1, 2], the following approximation of the Riemann-Liouville fractional derivatives of order $2s,\ 1/2 < s < 1$ of the function ρ:

$$ {}_0D_x^{2s}\rho(x_i, t^n) = h^{-2s}\sum_{k=0}^{i+1} w_k^{(2s)}\rho(x_{i-k+1}, t^n) + O(h^2)$$

and

$$ {}_xD_L^{2s}\rho(x_i, t^n) = h^{-2s}\sum_{k=0}^{N-i+1} w_k^{(2s)}\rho(x_{i+k-1}, t^n) + O(h^2),$$

where the sequence $(w_k^{(2s)})_{k\in\mathbb{N}}$ is defined as follows:

$$w_0^{(2s)} = s \text{ and } w_k^{(2s)} = sg_k^{(2s)} + (1-s)g_{k-1}^{(2s)},\ \ \forall k \geq 1, \tag{3.5}$$

with the sequence $(g_k^{(2s)})_{k\in\mathbb{N}}$ defined as in (3.3). Note that from Lemma 2.8 in [2], the sequence $(w_k^{(2s)})_{k\in\mathbb{N}}$ satisfies the following properties:

$$\begin{cases} w_0^{(2s)} = s > 0,\ w_1^{(2s)} = 1 - s - 2s^2 < 0,\ w_2^{(2s)} = s(2s^2 + s - 2), \\ 1 \geq w_0^{(2s)} \geq w_3^{(2s)} \geq w_4^{(2s)} \geq ... \geq 0, \\ \displaystyle\sum_{k=0}^{\infty} w_k^{(2s)} = 0;\ \sum_{k=0}^{n} w_k^{(2s)} < 0,\ \forall n \geq 2. \end{cases} \tag{3.6}$$

Consequently,

$$(-\Delta)^s\rho(x_i, t^n) = \frac{h^{-2s}}{2\cos(s\pi)}\Big(\sum_{k=0}^{i+1} w_k^{(2s)}\rho(x_{i-k+1}, t^n) + \sum_{k=0}^{N-i+1} w_k^{(2s)}\rho(x_{i+k-1}, t^n)\Big) + O(h^2). \tag{3.7}$$

To approach the convection term, we use the upwind scheme. This means that we approach the first order derivative of ρ by:

$$\begin{aligned} \frac{\partial\rho}{\partial x}(x_i,t^n) &= \frac{\rho(x_{i+1},t^n) - \rho(x_i,t^n)}{h} + O(h) \ \text{ if } \ v < 0, \\ \frac{\partial\rho}{\partial x}(x_i,t^n) &= \frac{\rho(x_i,t^n) - \rho(x_{i-1},t^n)}{h} + O(h) \ \text{ if } \ v > 0. \end{aligned} \tag{3.8}$$

We define the function $H(v)$ as follows:

$$H(v) = \begin{cases} 1 & \text{if} \quad v \geq 0, \\ 0 & \text{if} \quad v < 0. \end{cases} \tag{3.9}$$

Since

$$\frac{\partial \rho}{\partial t}(x_i,t^n) = \frac{\rho(x_i,t^{n+1}) - \rho(x_i,t^n)}{\tau} + O(\tau),$$

using (3.7) and (3.8), we have this approximation of the first equation in (1.1),

$$\begin{aligned} &\tau f(x_i,t^n) \\ &= \rho(x_i,t^{n+1}) - \rho(x_i,t^n)(1-C) - C\left[(1-H(v))\rho(x_{i+1},t^n) + H(v)\rho(x_{i-1},t^n)\right] \\ &- B\big(\sum_{k=0}^{i+1} w_k^{(2s)}\rho(x_{i-k+1},t^n) + \sum_{k=0}^{N-i+1} w_k^{(2s)}\rho(x_{i+k-1},t^n)\big) + \tau\left(O(h^2)+O(\tau)+O(h)\right), \end{aligned} \tag{3.10}$$

where

$$B = \frac{-D\tau}{2\cos(s\pi)h^{2s}} \text{ and } C = \frac{\tau|v|}{h}. \tag{3.11}$$

Remark 3.1 *Note that $B > 0$ because $\cos(s\pi) < 0$ for $1/2 < s < 1$.*

Using (3.10) and (3.1), we have that,

$$\begin{aligned} \rho_i^{n+1} - \rho_i^n(1-C) &= C\left[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n\right] \\ &\quad + B\big(\sum_{k=0}^{i+1} w_k^{(2s)}\rho_{i-k+1}^n + \sum_{k=0}^{N-i+1} w_k^{(2s)}\rho_{i+k-1}^n\big) + \tau f_i^n, \end{aligned} \tag{3.12}$$

and consequently the following numerical scheme for (1.1):

$$\begin{cases} \rho_i^{n+1} = (1-C)\rho_i^n + C[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n] \\ \qquad +B\big(\sum_{k=0}^{i+1} w_k^{(2s)}\rho_{i-k+1}^n + \sum_{k=0}^{N-i+1} w_k^{(2s)}\rho_{i+k-1}^n\big) \\ \qquad +\tau f_i^n,\ i=1,\cdots,(N-1),\ n=0,\cdots,M, \\ \rho_0^n \quad = \rho_L^n = 0,\ n=0,\cdots,M, \\ \rho_i^0 \quad = g_i,\ i=1,\cdots,N. \end{cases} \tag{3.13}$$

4 Stability and convergence analysis

In this section, we assume without loss of generality that $f \neq 0$.

Theorem 4.1 *Let $1/2 < s < 1$. Let C and B be the reals defined by* (3.11) *and $(w_k^{(2s)})_{k\in\mathbb{N}}$ be defined as in* (3.5)*. Assume that the initial condition g in* (1.1) *is bounded. Then the scheme* (3.13) *is stable if the following condition holds:*

$$0 < C - 2Bw_1^{(2s)} < 1. \tag{4.1}$$

Proof. We proceed by induction.

Since the initial condition g is bounded, there exists a real $M > 0$ such that $\left|\rho_i^0\right| \leq M$ for $i = 1,\cdots,N$.

Assume that

$$|\rho_i^n| \leq M, \quad i = 1, \cdots, N \text{ for some } n \geq 1. \tag{4.2}$$

Then using (3.13), we can write

$$\begin{aligned}
\rho_i^{n+1} &= (1-C)\rho_i^n + C\left[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n\right] \\
&+ B\Big(\sum_{k=0}^{i+1} w_k^{(2s)}\rho_{i-k+1}^n + \sum_{k=0}^{N-i+1} w_k^{(2s)}\rho_{i+k-1}^n\Big) \\
&= (1-C)\rho_i^n + C\left[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n\right] \\
&+ B\left(w_0^{(2s)}\rho_{i+1}^n + w_1^{(2s)}\rho_i^n + w_2^{(2s)}\rho_{i-1}^n\right) + B\sum_{k=3}^{i+1} w_k^{(2s)}\rho_{i-k+1}^n \\
&+ B\left(w_0^{(2s)}\rho_{i-1}^n + w_1^{(2s)}\rho_i^n + w_2^{(2s)}\rho_{i+1}^n\right) + B\sum_{k=3}^{N-i+1} w_k^{(2s)}\rho_{i+k-1}^n.
\end{aligned}$$

This means that

$$\begin{aligned}
\rho_i^{n+1} &= [1-(C-2Bw_1^{(2s)})]\rho_i^n + C\left[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n\right] + B\rho_{i-1}^n(w_2^{(2s)} + w_0^{(2s)}) \\
&+ B\rho_{i+1}^n(w_0^{(2s)} + w_2^{(2s)}) + B\sum_{k=3}^{i+1} w_k^{(2s)}\rho_{i-k+1}^n + B\sum_{k=3}^{N-i+1} w_k^{(2s)}\rho_{i+k-1}^n.
\end{aligned} \tag{4.3}$$

Observing that from the definition of H given by (3.9),

$$(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n = \begin{cases} \rho_{i-1}^n & \text{if} \quad v \geq 0, \\ \rho_{i+1}^n & \text{if} \quad v < 0, \end{cases}$$

we have that

$$\left|(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n\right| = \begin{cases} |\rho_{i-1}^n| & \text{if} \quad v \geq 0, \\ |\rho_{i+1}^n| & \text{if} \quad v < 0. \end{cases}$$

Therefore using the fact that $C > 0$, we deduce that,

$$|C[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n]| = \begin{cases} C|\rho_{i-1}^n| & \text{if} \quad v \geq 0, \\ C|\rho_{i+1}^n| & \text{if} \quad v < 0, \end{cases}$$

which in view of (4.2), gives

$$|C[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n]| \leq CM \tag{4.4}$$

because $v \geq 0$ or $v < 0$. Since, from (3.6), we have that,

$$\sum_{k=3}^{i+1} w_k^{(2s)} \geq 0, \quad \sum_{k=3}^{N-i+1} w_k^{(2s)} \geq 0 \text{ and } w_0^{(2s)} + w_2^{(2s)} > 0,$$

using (4.1), (4.2) and (4.4), we obtain that,

$$\begin{aligned}|\rho_i^{n+1}| &\leq [1-(C-2Bw_1^{(2s)})]|\rho_i^n| + |C[(1-H(v))\rho_{i+1}^n + H(v)\rho_{i-1}^n]| + B|\rho_{i-1}^n|(w_2^{(2s)}+w_0^{(2s)})\\ &+ B|\rho_{i+1}^n|(w_0^{(2s)}+w_2^{(2s)}) + B\sum_{k=3}^{i+1} w_k^{(2s)}|\rho_{i-k+1}^n| + B\sum_{k=3}^{N-i+1} w_k^{(2s)}|\rho_{i+k-1}^n|\\ &\leq [1-(C-2Bw_1^{(2s)})]M + CM + 2BM(w_2^{(2s)}+w_0^{(2s)})\\ &+ BM\sum_{k=3}^{i+1} w_k^{(2s)} + BM\sum_{k=3}^{N-i+1} w_k^{(2s)},\end{aligned}$$

which can be rewritten as,

$$\begin{aligned}\rho_i^{n+1} \leq\ & [1-(C-2Bw_1^{(2s)})]M + CM + 2BM(w_2^{(2s)}+w_0^{(2s)})\\ &+ BM\sum_{k=0}^{i+1} w_k^{(2s)} + BM\sum_{k=0}^{N-i+1} w_k^{(2s)} - 2BM(w_0^{(2s)}+w_2^{(2s)}) - 2BMw_1^{(2s)}.\end{aligned}$$

This means that,

$$\rho_i^{n+1} \leq [1-(C-2Bw_1^{(2s)})]M + (C-2Bw_1^{(2s)})M + BM\sum_{k=0}^{i+1} w_k^{(2s)} + BM\sum_{k=0}^{N-i+1} w_k^{(2s)}.$$

In view of (3.6), we have on the one hand that $C - 2Bw_1^{(2s)} > 0$ because $w_1^{(2s)} < 0$ and $C > 0$, and on the other hand that, $\sum_{k=0}^{i+1} w_k^{(2s)} < 0$ and $\sum_{k=0}^{N-i+1} w_k^{(2s)} < 0$. Therefore,

$$\rho_i^{n+1} \leq [1-(C-2Bw_1^{(2s)})]M + (C-2Bw_1^{(2s)})M = M,$$

because $0 < C - 2Bw_1^{(2s)} < 1$ and thus $0 < 1-(C-2Bw_1^{(2s)}) < 1$. □

Remark 4.1 *Note that for $1/2 < s < 1$, we have that $\frac{Dw_1^{(2s)}}{2\cos(s\pi)h^{2s}} > 0$ because $w_1^{(2s)} < 0$ and $\cos(s\pi) < 0$. Therefore, using the definition of B and C given by* (3.11)*, we have from the condition of stability* (4.1) *that the time step is reduced by the explicit character of the fractional Laplace operator and the upwind scheme for the convection:*

$$\tau < \frac{1}{\frac{|v|}{h} + \frac{Dw_1^{(2s)}}{2\cos(s\pi)h^{2s}}}.$$

Theorem 4.2 *Let $1/2 < s < 1$. Let C and B be the reals defined by* (3.11) *and $(w_k^{(2s)})_{k\in\mathbb{N}}$ be defined as in* (3.5)*. Assume that the initial condition g in* (1.1) *is bounded and that* (4.1) *holds. Then, the scheme* (3.13) *is convergent and its error is first order in time and space.*

Proof. Let $\rho(x_i, t^n)$ $(i = 1, 2, \cdots, N-1; n = 1, 2, \cdots, M)$ be the exact solution of equation (1.1) at mesh point (x_i, t^n) and denote the error by

$$e_i^n = \rho(x_i, t^n) - \rho_i^n \quad \text{for } i = 1, 2, \cdots, N-1 \text{ and } n = 0, \cdots, M-1. \tag{4.5}$$

Then in view of (3.13) and (4.5) we have that,

$$e_i^0 = 0 \quad \text{for } i = 1, 2, \cdots, N-1. \tag{4.6}$$

Using (3.10) and (3.12), we obtain that,

$$\begin{aligned}
&\Delta t(f(x_i, t^n) - f_i^n)\\
&= (\rho(x_i, t^{n+1}) - \rho_i^{n+1}) - (1-C)(\rho(x_i, t^n) - \rho_i^n)\\
&- C(1-H(v))(\rho(x_{i+1}, t^n) - \rho_{i+1}^n) - CH(v)(\rho(x_{i-1}, t^n) - \rho_{i-1}^n)\\
&- B\big(\sum_{k=0}^{i+1} w_k^{(2s)}(\rho(x_{i-k+1}, t^n) - \rho_{i-k+1}^n) + \sum_{k=0}^{N-i+1} w_k^{(2s)}(\rho(x_{i+k-1}, t^n) - \rho_{i+k-1}^n)\big)\\
&= e_i^{n+1} - (1-C)e_i^n - C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n]\\
&- B\big(\sum_{k=0}^{i+1} w_k^{(2s)} e_{i-k+1}^n + \sum_{k=0}^{N-i+1} w_k^{(2s)} e_{i+k-1}^n\big) + \tau\,(O(h^2) + O(\tau) + O(h)).
\end{aligned}$$

Since $f(x_i, t^n) - f_i^n = 0$, we have that,

$$\begin{aligned}
e_i^{n+1} &= (1-C)e_i^n + C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n]\\
&+ B\big(\sum_{k=0}^{i+1} w_k^{(2s)} e_{i-k+1}^n + \sum_{k=0}^{N-i+1} w_k^{(2s)} e_{i+k-1}^n\big) + \tau\,(O(h^2) + O(\tau) + O(h)).
\end{aligned} \tag{4.7}$$

To complete the proof of Theorem 4.2, we need to prove that there exists $K > 0$ such that,

$$\max_{1 \le i \le N-1} |e_i^n| \le K(O(\tau) + O(h));\, n = 1, 2, \cdots, M-1.$$

To this end, we proceed by induction.

So, for $n = 0$, using (4.6), we have that,

$$|e_i^1| = \tau\,|(O(h^2) + O(\tau) + O(h))| = \tau\,R,\ i = 1, \cdots, N-1, \tag{4.8}$$

with $R = |(O(h^2) + O(\tau) + O(h))|$.

For $n = 1$,

$$\begin{aligned}
e_i^2 &= (1-C)e_i^1 + C[(1-H(v))e_{i+1}^1 + H(v)e_{i-1}^1]\\
&+ B\big(\sum_{k=0}^{i+1} w_k^{(2s)} e_{i-k+1}^1 + \sum_{k=0}^{N-i+1} w_k^{(2s)} e_{i+k-1}^1\big) + \tau\,(O(h^2) + O(\tau) + O(h))\\
&= (1 - C + 2Bw_1^{(2s)})e_i^1 + C[(1-H(v))e_{i+1}^1 + H(v)e_{i-1}^1]\\
&+ Be_{i-1}^1(w_0^{(2s)} + w_2^{(2s)}) + B\sum_{k=3}^{i+1} w_k^{(2s)} e_{i-k+1}^1 + B\sum_{k=3}^{N-i+1} w_k^{(2s)} e_{i+k-1}^1\\
&+ Be_{i+1}^1(w_0^{(2s)} + w_2^{(2s)}) + \tau\,(O(h^2) + O(\tau) + O(h)).
\end{aligned}$$

Using the definition of H given by (3.9) and the fact that $C > 0$, we have that

$$|C[(1-H(v))e_{i+1}^1 + H(v)e_{i-1}^1]| = \begin{cases} C|e_{i-1}^1| & \text{if} \quad v \ge 0,\\ C|e_{i+1}^1| & \text{if} \quad v < 0,\end{cases}$$

which in view of (4.8) and the fact that $v \geq 0$ or $v < 0$, gives

$$|C[(1-H(v))e_{i+1}^1 + H(v)e_{i-1}^1]| \leq C\tau\, R, \quad i = 1, \cdots, N-1. \tag{4.9}$$

Using (4.1) and (3.6), we can write

$$\begin{aligned}|e_i^2| &\leq (1 - C + 2Bw_1^{(2s)})|e_i^1| + |C[(1-H(v))e_{i+1}^1 + H(v)e_{i-1}^1]| \\ &+ B|e_{i-1}^1|(w_0^{(2s)} + w_2^{(2s)}) + B\sum_{k=3}^{i+1} w_k^{(2s)}|e_{i-k+1}^1| + B\sum_{k=3}^{N-i+1} w_k^{(2s)}|e_{i+k-1}^1| \\ &+ B|e_{i+1}^1|(w_0^{(2s)} + w_2^{(2s)}) + \tau\,|(O(h^2) + O(\tau) + O(h))|,\end{aligned}$$

which in view of (4.8) and (4.9), gives

$$\begin{aligned}|e_i^2| &\leq [1 - (C - 2Bw_1^{(2s)})]\Delta\, tR + C\Delta\, tR + 2B\Delta\, tR(w_0^{(2s)} + w_2^{(2s)}) \\ &+ B\Delta\, tR\sum_{k=0}^{i+1} w_k^{(2s)} + B\sum_{k=0}^{N-i+1} w_k^{(2s)} - 2B\Delta\, tR(w_0^{(2s)} + w_2^{(2s)}) - 2B\Delta\, tRw_1^{(2s)} + \tau\, R.\end{aligned}$$

Using (3.6), we obtain that,

$$|e_i^2| \leq [1 - (C - 2Bw_1^{(2s)})]\Delta\, tR + \Delta\, tR(C - Bw_1^{(2s)}) + \tau\, R,$$

which in view of (4.1) implies that, $|e_i^2| \leq 2\tau\, R$.

Now assume that for some $n \geq 1$,

$$|e_i^n| \leq n\tau\, R, \quad \text{for } i = 1, 2, \cdots, M-1. \tag{4.10}$$

Then in view of (4.7), we have that,

$$\begin{aligned}e_i^{n+1} &= (1-C)e_i^n + C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n] \\ &+ B\big(\sum_{k=0}^{i+1} w_k^{(2s)} e_{i-k+1}^n + \sum_{k=0}^{N-i+1} w_k^{(2s)} e_{i+k-1}^n\big) + \tau\,(O(h^2) + O(\tau) + O(h)) \\ &= [1 - (C - Bw_1^{(2s)})]e_i^n + C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n] + Be_{i+1}^n(w_0^{(2s)} + w_2^{(2s)}) \\ &+ B\sum_{k=3}^{i+1} w_k^{(2s)} e_{i-k+1}^n + B\sum_{k=3}^{N-i+1} w_k^{(2s)} e_{i+k-1}^n \\ &+ Be_{i-1}^n(w_0^{(2s)} + w_2^{(2s)}) + \tau\,(O(h^2) + O(\tau) + O(h)).\end{aligned}$$

Using the definition of H given by (3.9) and the fact that $C > 0$, we get

$$|C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n]| = \begin{cases} C|e_{i-1}^n| & \text{if} \quad v \geq 0, \\ C|e_{i+1}^n| & \text{if} \quad v < 0, \end{cases}$$

which in view of (4.10) and the fact that $v \geq 0$ or $v < 0$, gives

$$|C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n]| \leq Cn\tau\, R, \quad i = 1, \cdots, N-1. \tag{4.11}$$

Hence, using (4.1) and (3.6), we obtain that,

$$\begin{aligned}|e_i^{n+1}| &\le [1-(C-Bw_1^{(2s)})]|e_i^n| + |C[(1-H(v))e_{i+1}^n + H(v)e_{i-1}^n]| + B|e_{i+1}^n|(w_0^{(2s)}+w_2^{(2s)})\\ &+ B\sum_{k=3}^{i+1} w_k^{(2s)}|e_{i-k+1}^n| + B\sum_{k=3}^{N-i+1} w_k^{(2s)}|e_{i+k-1}^n| + B|e_{i-1}^n|(w_0^{(2s)}+w_2^{(2s)})\\ &+ \tau\,(O(h^2)+O(\tau)+O(h)),\end{aligned}$$

which in view of (4.10) and (4.11) implies that,

$$\begin{aligned}|e_i^{n+1}| &\le [1-(C-Bw_1^{(2s)})]n\tau\, R + Cn\tau\, R + 2Bn\tau\, R(w_0^{(2s)}+w_2^{(2s)})\\ &+ Bn\tau\, R\sum_{k=3}^{i+1} w_k^{(2s)} + Bn\tau\, R\sum_{k=3}^{N-i+1} w_k^{(2s)} + \tau\, R.\end{aligned}$$

Thus,

$$\begin{aligned}|e_i^{n+1}| &\le [1-(C-Bw_1^{(2s)})]n\tau\, R + (C-Bw_1^{(2s)})n\tau\, R + 2Bn\tau\, R(w_0^{(2s)}+w_2^{(2s)})\\ &+ Bn\tau\, R\sum_{k=0}^{i+1} w_k^{(2s)} + Bn\tau\, R\sum_{k=0}^{N-i+1} w_k^{(2s)} - 2Bn\tau\, R(w_0^{(2s)}+w_2^{(2s)})\\ &+ \tau\,(O(h^2)+O(\tau)+O(h)),\\ |e_i^{n+1}| &\le (n+1)\Delta\, t\, R.\end{aligned}$$

We thus have proved that for any $n = 1, 2, \cdots, M-1$,

$$|e_i^n| \le n\Delta\, t\,(O(h^2)+O(\tau)+O(h)), i = 1, \cdots, N-1.$$

Consequently, $\max\limits_{1\le i\le M} |e_i^n| \le T\,(O(\tau)+O(h))$ because $n\Delta\, t = T$. □

5 Numerical results

In this section, we illustrate the effectiveness of our numerical method with the following advection-diffusion equation involving the fractional Laplace operator in one dimension:

$$\begin{cases} \dfrac{\partial\rho(x,t)}{\partial t} + v\dfrac{\partial\rho(x,t)}{\partial x} + D(-\Delta)^s\rho(x,t) = f(x,t) & \text{in } (0,L)\times(0,T),\\ \rho(0,t) = \rho(L,t) = 0 & \text{in } (\mathbb{R}\setminus(0,L))\times(0,T),\\ \rho(x,0) = x^2(L-x)^2 & \text{in } (0,L), \end{cases} \tag{5.1}$$

where $1/2 < s < 1$, the fractional Laplace operator is approached as in (3.2):

$$(-\Delta)^s\rho(x,t) = \frac{{}_0D_x^{2s}\rho(x,t) + {}_xD_L^{2s}\rho(x,t)}{2\cos(s\pi)},$$

the function f is given by:

$$\begin{aligned}f(x,t) = \Big[&\frac{D}{2\cos(s\pi)}\Big[\frac{24}{\Gamma(5-2s)}(x^{4-2s}+(L-x)^{4-2s}) - \frac{12L}{\Gamma(4-2s)}(x^{3-2s}+(L-x)^{3-2s})\\ &+ \frac{2L^2}{\Gamma(3-2s)}(x^{2-2s}+(L-x)^{2-2s})\Big] - x^2(L-x)^2\cos t + 2xv(L-x)(L-2x)\Big]\mathrm{e}^{-\sin t}\end{aligned}$$

and the exact solution is

$$\rho(x,t) = x^2(L-x)^2 \mathrm{e}^{-\sin t}.$$

Since the fractional Laplace operator of order $1/2 < s < 1$ converges to classical Laplace operator when $s \to 1$, in order to compare our numerical results with those of classical advection-diffusion equation, we will consider the limit case: $s = 1$.

	$s=0.6$		$s=0.7$		$s=0.9$		$s=1$	
$1/N$	$Error_{L_\infty}$	rate	$Error_{L_\infty}$	rate	$Error_{L_\infty}$	rate	$Error_{L_\infty}$	rate
1/20	3.955859e-04	-	3.672168e-04	-	2.902796e-04	-	2.473525e-04	-
1/40	2.02935e-04	0.96	1.898970e-04	0.95	1.530945e-04	0.92	1.319796e-04	0.90
1/80	1.026621e-04	0.98	9.643893e-05	0.97	7.847235e-05	0.96	6.803680e-05	0.95
1/160	5.157342e-05	0.99	4.854499e-05	0.99	3.968540e-05	0.98	3.450898e-05	0.97
1/320	2.578609e-05	1.00	2.430069e-05	0.99	1.992239e-05	0.99	1.735728e-05	0.99

Table 1: $v = \pm 0.01;\ D = 0.01;\ \Delta x = 1/N;\ \ \Delta t = 10^{-7}$

Table 1 gives the error and the convergence of the Euler explicit scheme at time $T = 1$ when the velocity $v = \pm 0.01$, the diffusion factor $D = 0.01$ and $s \in]1/2, 1]$. It can be observed that the scheme converges at the first order in the space direction. This can be explained by the fact that we approached the advection term with an upwind scheme of first order which is higher than that of the Laplace operator.

	$s=0.6$		$s=0.7$		$s=0.9$		$s=1$	
$1/N$	$Error_{L_\infty}$	rate	$Error_{L_\infty}$	rate	$Error_{L_\infty}$	rate	$Error_{L_\infty}$	rate
1/20	1.916460e-04	-	1.632000e-04	-	1.703046e-04	-	1.894108e-04	-
1/40	5.871214e-05	1.70	4.825218e-05	1.75	4.191585e-05	2.02	4.739295e-05	1.99
1/80	1.565643e-05	1.90	1.286137e-05	1.90	1.030367e-05	2.02	1.183844e-05	2.00
1/160	4.207778e-06	1.90	3.565211e-06	1.90	2.543565e-06	2.01	2.946694e-06	2.00
1/320	1.113265e-06	1.91	9.486200e-07	1.91	6.402313e-07	1.99	7.235601e-07	2.02

Table 2: $v = 0;\ D = 0.01;\ \Delta x = 1/N;\ \ \Delta t = 10^{-7}$

The estimate of the error as well as the rate of the convergence are given in Table 2 for the same scheme correspond to the case where the time $T = 1$, the velocity $v = 0$, the diffusion factor $D = 0.01$ and $s \in]1/2, 1]$. We can observe that the rate of convergence is of second order. This result is in agreement with the fact that this data (5.1) describe a space fractional diffusion equation.

In what follows, we compare the numerical solutions obtained with different values of the velocity v and the diffusion constant D.

Figure 1 deals with a diffusion equation involving only the fractional Laplace operator since $v = 0$. We can observe that as the fractional power of the Laplace operator decreases, the scheme is diffusive.

For Figures 2-4, we are concerned with the cases $|v| = D$, $|v| < D$ and $|v| > D$ respectively. We can observe that the numerical solutions of our scheme evolve as those of the scheme for classical advection-diffusion equation when $|v| < D$ and $|v| > D$. In the three cases, when the fractional power of the Laplace operator decreases, the scheme is diffusive.

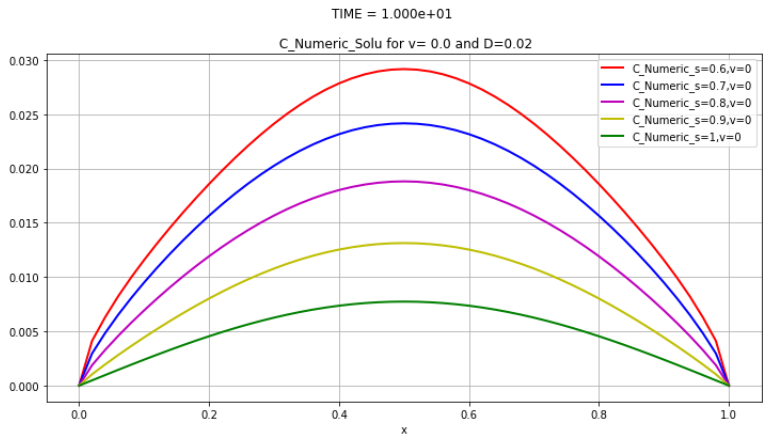

Figure 1: $T = 10$, $v = 0$, $D = 0.02$ and $s \in]1/2, 1]$

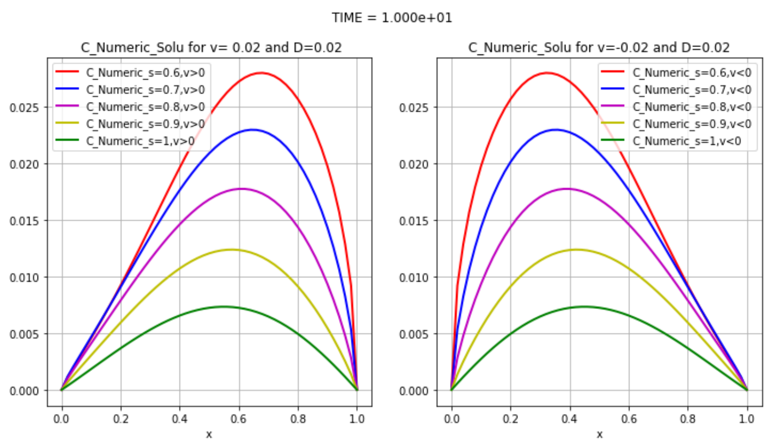

Figure 2: $T = 10$, $v = \pm 0.02$, $D = 0.02$ and $s \in]1/2, 1]$

6 Conclusion

We have constructed a numerical scheme of explicit Euler type for an advection-diffusion equation involving a fractional Laplace operator of order $1/2 < s < 1$, in a one-dimension space. We proved that the scheme is stable and convergent. The results of the tests proved that when the velocity is equal to zero the order of the numerical approximation is that of the fractional Laplace operator. Moreover, when $s = 1$, our results are the same as those obtained in the classical advection-diffusion equation. We also observe that for a decreasing value of the order of the fractional derivative s, the solution increases in the same given time. Our next objective is to construct a scheme with a convergence order greater than one.

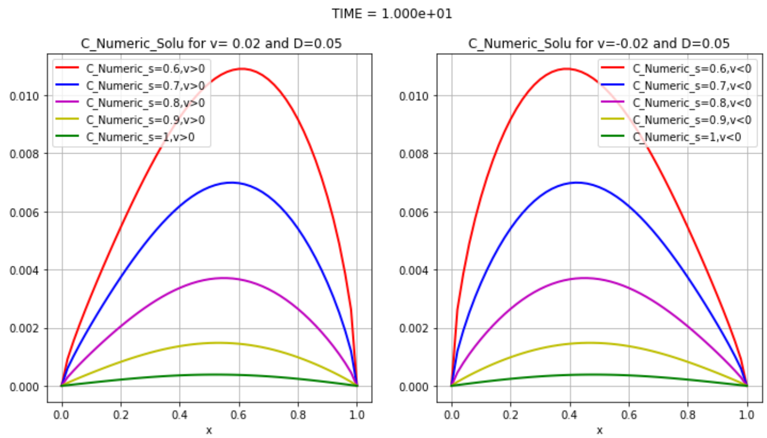

Figure 3: $T = 10$, $v = \pm 0.02$, $D = 0.05$ and $s \in]1/2, 1]$

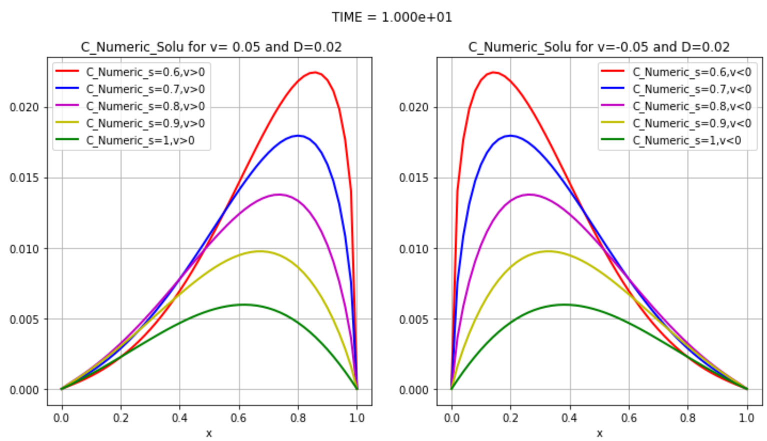

Figure 4: $T = 10$, $v = \pm 0.05$, $D = 0.02$ and $s \in]1/2, 1]$

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