Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2022, Number 7, pp. 127–142 (April 2023) http://www.jneea.com

REGULARITY CRITERION FOR THE 3D MAGNETO–MICROPOLAR FLUID FLOWS IN TERMS OF PRESSURE

SIMIAO YAN* AND XIAOLI CHEN†
School of Mathematics and Statistics, Jiangxi Normal University,
Nanchang, Jiangxi 330022, P. R. China

Received on February 6, 2022, revised version on December 3, 2022,

Accepted on January 27, 2023

Communicated by Stanislas Ouaro

Abstract. In this note we obtain a new regularity criterion for the three-dimensional magneto–micropolar fluid flows in terms of pressure. More precisely, we prove that if $\pi \in L^{\frac{2}{2-r}}(0,T;L^{\frac{3}{r}}(\mathbb{R}^3))$ with $0 < r \leq 1$, then the local strong solution (u,b,ω) to the magneto–micropolar fluid flows can be extended beyond time $t = T$. Meanwhile, we also show that provided that $\pi \in L^p(0,T;\dot{F}^0_{q,\frac{10q}{5q+6}}(\mathbb{R}^3))$ with $\frac{2}{p}+\frac{3}{q}<\frac{7}{4}$, $\frac{12}{5}<q\leq\infty$ or $\nabla\pi \in L^p(0,T;\dot{F}^0_{q,\frac{8q}{12-3q}}(\mathbb{R}^3))$ with $\frac{2}{p}+\frac{3}{q}=\frac{11}{4}$, $\frac{12}{11}<q<4$, the weak solution (u,b,ω) to the magneto–micropolar fluid flows can also be extended smoothly beyond $t = T$.

Keywords: Blow up criterion, magneto–micropolar equations, pressure, regularity criterion, Triebel–Lizorkin spaces, weak solution.

2010 Mathematics Subject Classification: 35Q35, 76D03.

*e-mail address: smyanjxnu@163.com
†e-mail address: littleli_chen@163.com

1 Introduction

This paper focuses on the regularity criterion for the following three dimensional magneto–micropolar fluid flows

$$\begin{cases} \partial_t u + u\cdot\nabla u - (\mu+\chi)\Delta u - b\cdot\nabla b + \nabla\pi - \chi\nabla\times\omega = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \partial_t\omega - \gamma\Delta\omega - \kappa\nabla\operatorname{div}\omega + 2\chi\omega + u\cdot\nabla\omega - \chi\nabla\times u = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \partial_t b - \nu\Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \operatorname{div} u = \operatorname{div} b = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0,\ \omega|_{t=0} = \omega_0, & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$, $b = (b_1, b_2, b_3)$ and π denote the unknown velocity field, the micro-rotational velocity, the magnetic field and the unknown scalar pressure at the point $(x, t) \in \mathbb{R}^3 \times (0, T)$, respectively. Moreover, u_0, ω_0, b_0 are the prescribed initial data, and $\operatorname{div} u = \operatorname{div} b = 0$ in the sense of distributions. The constants $\mu, \chi, \kappa, \gamma, \nu$ are positive numbers associated to properties of the material: μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities, and $\frac{1}{\nu}$ is the magnetic Reynolds number (for more details see [11]). The magneto–micropolar fluid flows illustrate a model of incompressible Navier–Stokes equations, micro-rotational inertia and micro-rotational effects (see [7] for more details). In 1977, Galdi and Rionero [13] stated (without a proof) the theorem on the existence and uniqueness of strong solutions. Ahmadi and Shahinpoor [1] studied the stability of solutions for the system in 1974. By using spectral Galerkin method, in 1997, Rojas-Medar [16] established local existence and uniqueness of strong solutions. In 1998, Ortega-Torres and Rojas-Medar [15] proved global existence of strong solutions with small initial data. As regards weak solutions, Rojas-Medar and Boldrini [17] established their local existence in two and three dimensions by using Galerkin method and also proved their uniqueness in the 2D case.

Without loss of generality, we set $\mu = \chi = \frac{1}{2}$ and $\kappa = \gamma = \nu = 1$ in the rest of the paper. Then, equation (1.1) becomes

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u - b\cdot\nabla b + \nabla\pi - \frac{1}{2}\nabla\times\omega = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \partial_t\omega - \Delta\omega - \nabla\operatorname{div}\omega + \omega + u\cdot\nabla\omega - \frac{1}{2}\nabla\times u = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \partial_t b - \Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \operatorname{div} u = \operatorname{div} b = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0,\ \omega|_{t=0} = \omega_0, & x\in\mathbb{R}^3. \end{cases} \tag{1.2}$$

When the micro-rotational velocity $\omega = 0$ and the magnetic field $b = 0$, the equation (1.1) becomes the incompressible Navier–Stokes equations (see [4, 18]). The well-known regularity criterion for this system due to Serrin states that if u is the Leray–Hope weak solution of the 3D Navier–Stokes equations satisfying one of the following conditions:

$$u \in L^q(0,T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} = 1, \quad 3 < p \le \infty,$$

or

$$\nabla u \in L^q(0,T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} < p \le \infty,$$

then it can be extended beyond time $t = T$. There is a large number of literature on the regularity criterion for the Navier–Stokes equations by imposing the growth conditions on the pressure field. For example, if the pressure π satisfies

$$\pi \in L^s(0,T;L^r(\mathbb{R}^3)), \quad \frac{2}{s}+\frac{3}{r}=2, \quad \frac{3}{2}<p\le\infty, \tag{1.3}$$

or the gradient of the pressure $\nabla\pi$ satisfies

$$\nabla\pi \in L^s(0,T;L^r(\mathbb{R}^3)), \quad \frac{2}{s}+\frac{3}{r}=3, \quad 1<p\le\infty, \tag{1.4}$$

then the weak solution u of the Navier–Stokes equations can be extended beyond time $t = T$ (see Chae and Lee [5], Berselli and Galdi [3] and Zhou [22, 23, 24]).

When the magnetic field $b = 0$, the equation (1.1) is the incompressible micropolar fluid flows. Dong, Yuan and Chen [10] proved that if the pressure π satisfies either (1.3) or (1.4), then the weak solution (u, ω) can be extended beyond time $t = T$. Very recently, Zhou [21] showed that if the gradient of the pressure satisfies the condition

$$\nabla\pi \in L^r(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{r}+\frac{3}{s}\le 3, \quad \text{for} \quad s\in[1,\infty],$$

then the weak solution u is regular. In 2006, Chen and Zhang [6] proved that if the pressure π satisfies the condition

$$\int_0^T \|\pi\|_{B^0_{\infty,\infty}}\,\mathrm{d}t<\infty,$$

then u is regular in $(0, T]$. When the micro-rotational velocity $\omega = 0$ and $\chi = 0$, the equation (1.1) becomes the standard magneto–hydrodynamic (MHD) equations, which have been studied extensively in [8, 14, 25, 26]. Duan [9] showed that if the pressure π satisfies (1.3) or (1.4), then the local strong solution (u, b) to the MHD equations can be extended smoothly beyond $t = T$. Motivated by [3, 5, 6, 9, 10, 22, 23, 24, 27] and [28], we will investigate the regularity criteria for the solutions to the magneto–micropolar flows in terms of the pressure satisfying (1.3) or the pressure and its gradient in some Triebel–Lizorkin spaces. Our results can be stated as follows.

Theorem 1.1 *Let $T > 0$. Assume that (u, b, ω) is the local strong solution to the magneto–micropolar fluid flows* (1.2) *defined on $[0, T)$. Suppose that the initial data $(u_0, b_0, \omega_0) \in H^2(\mathbb{R}^3)$ satisfy $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. If the pressure π satisfies*

$$\pi \in L^{\frac{2}{2-r}}(0,T;L^{\frac{3}{r}}(\mathbb{R}^3))\ \textit{for}\ 0<r\le 1, \tag{1.5}$$

then the local strong solution (u, b, ω) can be extended smoothly beyond $t = T$.

Remark 1.2 *Theorem* 1.1 *can be seen as a generalization of* [3, 5, 9, 10] *and* [22, 23, 24].

Theorem 1.3 *Let the initial data $(u_0, b_0, \omega_0) \in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$ satisfy $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Assume that (u, b, ω) is the weak solution to the magneto–micropolar fluid flows* (1.2) *defined on $[0, T)$. If the pressure π satisfies*

$$\pi \in L^p(0,T;\dot{F}^0_{q,\frac{10q}{5q+6}}(\mathbb{R}^3)) \quad \textit{for} \quad \frac{2}{p}+\frac{3}{q}=1+\frac{9}{5q}, \quad \frac{12}{5}<q<4, \tag{1.6}$$

or the gradient of the pressure satisfies

$$\nabla\pi \in L^p(0,T;\dot{F}^0_{q,\frac{8q}{12-3q}}(\mathbb{R}^3)) \quad for \quad \frac{2}{p}+\frac{3}{q}=\frac{11}{4}, \quad \frac{12}{11}<q<4, \tag{1.7}$$

then the weak solution (u,b,ω) *can be extended smoothly beyond* $t=T$.

Remark 1.4 *In our best knowledge, we first establish the blow up criterion for the weak solution of the magneto–micropolar equations with the help of the pressure in a Triebel–Lizorkin space. When* $\omega=0,\chi=0$*, the magneto–micropolar equation is just the MHD equations. From this viewpoint, Theorem* 1.3 *implies the blow up criterion for the weak solution of the incompressible MHD equations under the assumption that the pressure or its gradient belongs to some Triebel–Lizorkin spaces. We can also deduce a similar regularity criteria for the weak solution of micropolar fluid flow on the above Triebel–Lizorkin spaces.*

The rest of this note is organized as follows. Section 2 contains some crucial lemmas. The proof of Theorem 1.1 can be found in Section 3, and the proof of Theorem 1.3 can be found in Section 4.

2 Preliminaries

In this section, we will introduce the definition of a weak solution to the magneto–micropolar equation (1.1) and some useful lemmas.

Definition 2.1 (see [12]) *Let* $(u_0,b_0)\in L^2_\sigma(\mathbb{R}^3)$, $\omega\in L^2(\mathbb{R}^3)$ *and* $T>0$. *A measurable function* (u,b,ω) *is said to be a weak solution to* (1.1) *on* $(0,T)$ *if*

(i) $(u,b)\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$,

(ii) $\omega\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$,

(iii) *for every* $\phi,\varphi\in H^1(0,T;H^1_\sigma(\mathbb{R}^3))$ *and* $\psi\in H^1(0,T;H^1(\mathbb{R}^3))$ *with* $\phi(T)=\varphi(T)=\psi(T)=0$,

$$\int_0^T \langle -u,\partial_t\phi\rangle + \langle u\cdot\nabla u,\phi\rangle + (\mu+\chi)\langle\nabla u,\nabla\phi\rangle\,\mathrm{d}t - \int_0^T \langle b\cdot\nabla b,\phi\rangle + \chi\langle\nabla\times\omega,\phi\rangle\,\mathrm{d}t = -\langle u_0,\phi_0\rangle,$$

$$\int_0^T \langle -\omega,\partial_t\varphi\rangle + \gamma\langle\omega,\nabla\varphi\rangle + \kappa\langle\nabla\cdot u,\nabla\cdot\varphi\rangle\,\mathrm{d}t + \int_0^T \langle u\cdot\nabla\omega,\varphi\rangle + 2\chi\langle\omega,\varphi\rangle - 2\chi\langle\nabla\times u,\varphi\rangle\,\mathrm{d}t = -\langle\omega_0,\phi_0\rangle$$

and

$$\int_0^T \langle -b,\partial_t\psi\rangle + \langle u\cdot\nabla b,\psi\rangle + \nu\langle\nabla b,\nabla\psi\rangle - \langle b\cdot\nabla u,\psi\rangle\,\mathrm{d}t = -\langle b_0,\psi_0\rangle,$$

where $L_2^\sigma=\{u\in L^2:\nabla\cdot u=0\}$.

As in [17], for any $0 \le t \le T$ it is easy to establish the following inequality

$$\begin{aligned} \|(u(t), \omega(t), b(t))\|_{L^2}^2 + 2(\mu+\chi)\int_0^t \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + 2\gamma \int_0^t \|\nabla\omega\|_{L^2}^2 \,\mathrm{d}\tau \\ + 2\kappa \int_0^t \|\nabla\cdot\omega\|_{L^2}^2 \,\mathrm{d}\tau + 2\chi \int_0^t \|\omega\|_{L^2}^2 \,\mathrm{d}\tau \le \|(u_0, \omega_0, b_0)\|_{L^2}^2; \end{aligned} \tag{2.1}$$

it suffices to apply the L^2-inner product to the resulting equation with (u, ω, b) and then integrate over $[0, t]$ in the time variable.

The following lemma plays a crucial role in proving the regularity criterion for the magneto–micropolar fluid flows (1.1).

Lemma 2.2 (see [20]) *Let $2 \le p \le \infty$ and $s > n(\frac{1}{2} - \frac{1}{p})$. Then, there exists a constant $C = C(n, p, s)$ such that for any n-dimensional function $f \in H^s(\mathbb{R}^n)$,*

$$\|f\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^2(R^n)}^{1-\frac{n}{s}(\frac{1}{2}-\frac{1}{p})} \| \wedge^s f\|_{L^2(\mathbb{R}^n)}^{\frac{n}{s}(\frac{1}{2}-\frac{1}{p})}.$$

When $p \ne \infty$, the above inequality also holds for $s = n(\frac{1}{2} - \frac{1}{p})$.

In order to define the Besov and Triebel–Lizorkin spaces, we first introduce the Littlewood–Paley decomposition theory. Let $S(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. For a given $f \in S(\mathbb{R}^n)$, its Fourier transform $F(f) = \hat{f}$ and its inverse Fourier transform $F^{-1}(f) = \check{f}$ are given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x,$$

and

$$\check{f}(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,\mathrm{d}\xi,$$

respectively. Let us choose two non-negative radial functions $\chi, \varphi \in S(\mathbb{R}^n)$ satisfying $\operatorname{supp}\chi \subset B = \{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\}$ and $\operatorname{supp}\varphi \subset C = \{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\sum_{j\in Z} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \in \mathbb{R}^n \backslash \{0\}$$

and

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \in \mathbb{R}^n.$$

Let $\dot{\Delta}_j$ be a frequency projection to the annulus $\{|\xi| \sim 2^j\}$. And assume $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \le 2^j\}$. Let $s \in \mathbb{R}$, $p, r \in [1, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ consists of those distributions $f \in S'_h$ such that

$$\left(\sum_{j\in Z} 2^{jsr} \|\dot{\Delta}_j f\|_{L^p}^r \right)^{\frac{1}{r}} < \infty$$

with the norm

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j\in Z} 2^{jsr}\|\dot{\Delta}_j f\|^r_{L^p}\right)^{\frac{1}{r}}, & 1 \le r < \infty, \\ \sup_{j\in Z}\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}, & r = \infty. \end{cases}$$

The homogeneous Triebel–Lizorkin space $\dot{F}^s_{p,r}(\mathbb{R}^n)$ consists of those distributions $f \in S'_h$ such that

$$\left\|\left(\sum_{j\in Z} 2^{jsr}|\dot{\Delta}_j f|^r\right)^{\frac{1}{r}}\right\|_{L^p} < \infty$$

with the norm

$$\|f\|_{\dot{F}^s_{p,r}(\mathbb{R}^n)} = \begin{cases} \left\|\left(\sum_{j\in Z} 2^{jsr}|\dot{\Delta}_j f|^r\right)^{\frac{1}{r}}\right\|_{L^p}, & 1 \le r < \infty, \\ \left\|\sup_{j\in Z} |2^{js}\dot{\Delta}_j f|\right\|_{L^p}, & r = \infty. \end{cases}$$

We also need the following Bernstein's inequality.

Lemma 2.3 *Let $\mathcal{C}$ be an annulus and B a ball. Then, there exists a constant C such that for any non-negative integer k, any couple $(p,q) \in [1,\infty]^2$ with $q \ge p \ge 1$, and any function u of L^p, we have*

(i) *if* $\operatorname{supp}\hat{u} \subset \lambda B$, *then* $\sup_{|\alpha|=k}\|D^\alpha u\|_{L^q} \le C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}$,

(ii) *if* $\operatorname{supp}\hat{u} \subset \lambda\mathcal{C}$, *then* $C^{-k-1}\lambda^k\|u\|_{L^p} \le \|D^k u\|_{L^p} \le C^{k+1}\lambda^k\|u\|_{L^p}$.

The proof of this lemma can be found in [2].

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

Proof. Let $z^+ = u + b$ and $z^- = u - b$. We can convert the 3D magneto–micropolar fluid flows (1.2) into the following form

$$\begin{cases} \partial_t z^+ + z^- \cdot \nabla z^+ - \Delta z^+ + \nabla\pi - \frac{1}{2}\nabla\times\omega = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \partial_t z^- + z^+ \cdot \nabla z^- - \Delta z^- + \nabla\pi - \frac{1}{2}\nabla\times\omega = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \partial_t\omega - \Delta\omega - \nabla\nabla\cdot\omega + \omega + \frac{1}{2}(z^+ + z^-)\cdot\nabla\omega - \frac{1}{4}\nabla\times(z^+ + z^-) = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ \operatorname{div} z^+ = 0,\ \operatorname{div} z^- = 0, & (x,t)\in\mathbb{R}^3\times(0,T), \\ (z^+, z^-, \omega)|_{t=0} = (z_0^+(x), z_0^-(x), \omega_0(x)), & x\in\mathbb{R}^3, \end{cases} \tag{3.1}$$

where $z_0^+(x) = u_0(x) + b_0(x)$, $z_0^-(x) = u_0(x) - b_0(x)$.

L^4-energy estimate. Taking the inner products of $(3.1)_1$ with $|z^+|^2 z^+$, of $(3.1)_2$ with $|z^-|^2 z^-$ and of $(3.1)_3$ with $|\omega|^2\omega$, then integrating by parts and summing together, we conclude that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\\
&\quad+4\big(\||z^+||\nabla z^+|\|_{L^2}^2+\||z^-||\nabla z^-|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\big)\\
&\quad+2\big(\|\nabla|z^+|^2\|_{L^2}^2+\|\nabla|z^-|^2\|_{L^2}^2\big)+2\||\omega|\nabla\cdot\omega\|_{L^2}^2+4\|\omega\|_{L^4}^4\\
&\le 2\bigg(\int_{\mathbb{R}^3}(\nabla\times\omega)\cdot\big(|z^+|^2z^++|z^-|^2z^-\big)\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\times(z^++z^-)\cdot|\omega|^2\omega\,\mathrm{d}x\bigg)\\
&\quad-4\int_{\mathbb{R}^3}\nabla\pi\cdot\big(|z^+|^2z^++|z^-|^2z^-\big)\,\mathrm{d}x\\
&=:I_1+I_2,
\end{aligned}\tag{3.2}$$

where we have used the divergence free condition $(3.1)_4$ and the inequality

$$\begin{aligned}
\big(-\nabla\nabla\cdot\omega,|\omega|^2\omega\big)&=\int_{\mathbb{R}^3}\nabla\cdot\omega(\omega\nabla|\omega|^2+|\omega|^2\nabla\cdot\omega)\,\mathrm{d}x\\
&\ge-\tfrac{1}{2}\|\nabla|\omega|^2\|_{L^2}^2+\tfrac{1}{2}\||\omega|\nabla\cdot\omega\|_{L^2}^2.
\end{aligned}$$

Now, we estimate the first term I_1. Integrating by parts and using the Hölder and Young inequalities, we obtain

$$\begin{aligned}
I_1\le&2\bigg(\int_{\mathbb{R}^3}(\nabla\times\omega)\cdot(|z^+|^2z^++|z^-|^2z^-)\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\times(z^++z^-)\cdot|\omega|^2\omega\,\mathrm{d}x\bigg)\\
\le&C\bigg(\int_{\mathbb{R}^3}\omega\cdot\big(|z^+|^2\nabla\times z^++z^+\times(\nabla|z^+|^2)+|z^-|^2\nabla\times z^-+z^-\times(\nabla|z^-|^2)\big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(z^++z^-)\cdot(|\omega|^2\nabla\times\omega+\omega\nabla|\omega|^2)\,\mathrm{d}x\bigg)\\
\le&C\|\omega\|_{L^4}\big(\|z^+\|_{L^4}\||z^+||\nabla z^+|\|_{L^2}+\|z^-\|_{L^4}\||z^-||\nabla z^-|\|_{L^2}\big)\\
&\quad+C\big(\|z^+\|_{L^4}+\|z^-\|_{L^4}\big)\big(\|\omega\|_{L^4}\||\omega||\nabla\omega|\|_{L^2}\big)\\
\le&C\|\omega\|_{L^4}^2\|z^+\|_{L^4}^2+\tfrac{1}{2}\||z^+||\nabla z^+|\|_{L^2}^2+C\|\omega\|_{L^4}^2\|z^-\|_{L^4}^2+\tfrac{1}{2}\||z^-||\nabla z^-|\|_{L^2}^2\\
&\quad+C\|\omega\|_{L^4}^2\big(\|z^+\|_{L^4}^2+\|z^-\|_{L^4}^2\big)+\tfrac{1}{2}\||\omega||\nabla\omega|\|_{L^2}^2\\
\le&C\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\\
&\quad+\tfrac{1}{2}\big(\||z^+||\nabla z^+|\|_{L^2}^2+\||z^-||\nabla z^-|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\big).
\end{aligned}\tag{3.3}$$

From the integration by parts and the Hölder inequality we can infer that

$$\begin{aligned}
I_2\le&-\int_{\mathbb{R}^3}\nabla\pi\cdot(|z^+|^2z^++|z^-|^2z^-)\,\mathrm{d}x\\
=&\int_{\mathbb{R}^3}\pi(z^+\nabla|z^+|^2+z^-\nabla|z^-|^2)\,\mathrm{d}x\\
\le&\int_{\mathbb{R}^3}|\pi|^{\frac12}|\pi|^{\frac12}\big(|z^+|+|z^-|\big)\big(|\nabla|z^+|^2|+|\nabla|z^-|^2|\big)\,\mathrm{d}x\\
\le&C\|\pi^{\frac12}\|_{L^{\frac{6}{r}}}\|\pi^{\frac12}\|_{L^4}\||z^+|+|z^-|\|_{L^{\frac{12}{3-2r}}}\||z^+||\nabla z^+|+|z^-||\nabla z^-|\|_{L^2}\\
\le&C\|\pi\|_{L^{\frac{3}{r}}}^{\frac12}\|\pi\|_{L^2}^{\frac12}\||z^+|^2+|z^-|^2\|_{L^{\frac{6}{3-2r}}}^{\frac12}\||z^+||\nabla z^+|+|z^-||\nabla z^-|\|_{L^2},
\end{aligned}\tag{3.4}$$

where we used the fact that $\operatorname{div} z^+ = 0$, $\operatorname{div} z^- = 0$. Applying the divergence operator $\nabla\cdot$ to the first equation of (3.1), we get $\pi = (-\Delta)^{-1}\operatorname{div}\operatorname{div}(z^+\cdot z^-)$. Thanks to the Calderón–Zygmund inequality, we have

$$\|\pi\|_{L^p} \le C\||z^+||z^-|\|_{L^p} \le C\||z^+|^2+|z^-|^2\|_{L^p} \tag{3.5}$$

for $1<p<\infty$. With the help of Lemma 2.2, we get

$$\||z^+|^2+|z^-|^2\|^{\frac12}_{L^{\frac{6}{3-2r}}} \le C\||z^+|^2+|z^-|^2\|^{\frac{1-r}{2}}_{L^2}\|\nabla|z^+|^2+\nabla|z^-|^2\|^{\frac r2}_{L^2}.$$

This together with (3.5) with $p=2$, (3.4), and the Young inequality gives rise to

$$\begin{aligned} I_2 \le&\|\pi\|^{\frac12}_{L^{\frac3r}}\||z^+|^2+|z^-|^2\|^{\frac12}_{L^2}\||z^+|^2+|z^-|^2\|^{\frac{1-r}{2}}_{L^2}\|\nabla|z^+|^2\\ &+\nabla|z^-|^2\|^{\frac r2}_{L^2}\|\nabla|z^+|^2+\nabla|z^-|^2\|_{L^2}\\ \le&C\|\pi\|^{\frac12}_{L^{\frac3r}}\||z^+|^2+|z^-|^2\|^{\frac{2-r}{2}}_{L^2}\|\nabla|z^+|^2+\nabla|z^-|^2\|^{\frac{r+2}{2}}_{L^2}\\ \le&C\|\pi\|^{\frac{2}{2-r}}_{L^{\frac3r}}\||z^+|^2+|z^-|^2\|^2_{L^2}+\tfrac14\|\nabla|z^+|^2+\nabla|z^-|^2\|^2_{L^2}\\ \le&C\|\pi\|^{\frac{2}{2-r}}_{L^{\frac3r}}\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}\big)+\tfrac12\big(\|\nabla|z^+|^2\|^2_{L^2}+\|\nabla|z^-|^2\|^2_{L^2}\big). \end{aligned} \tag{3.6}$$

Substituting (3.3) and (3.6) into (3.2) yields

$$\begin{aligned} &\frac{\mathrm d}{\mathrm dt}\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}+\|\omega\|^4_{L^4}\big)\\ &\quad+3\big(\||z^+||\nabla z^+|\|^2_{L^2}+\||z^-||\nabla z^-|\|^2_{L^2}+\||\omega||\nabla\omega|\|^2_{L^2}\big)\\ &\quad+\tfrac32\big(\|\nabla|z^+|^2\|^2_{L^2}+\|\nabla|z^-|^2\|^2_{L^2}\big)+\||\omega|\nabla\cdot\omega\|^2_{L^2}+2\|\omega\|^2_{L^2}\\ &\le C\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}\big)+C\|\pi\|^{\frac{2}{2-r}}_{L^{\frac3r}}\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}\big)\\ &\le C\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}+\|\omega\|^4_{L^4}\big)\Big(\|\pi\|^{\frac{2}{2-r}}_{L^{\frac3r}}+1\Big). \end{aligned} \tag{3.7}$$

The Gronwall inequality says

$$\begin{aligned} &\sup_{0\le t\le T}\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}+\|\omega\|^4_{L^4}\big)\\ &\quad\le C\big(\|z_0^+\|^4_{L^4}+\|z_0^-\|^4_{L^4}+\|\omega_0\|^4_{L^4}\big)\exp\Big\{C\int_0^T\Big(1+\|\pi(t)\|^{\frac{2}{2-r}}_{L^{\frac3r}}\Big)\,\mathrm dt\Big\}\\ &\quad\le C\big(\|u_0\|^4_{H^2}+\|b_0\|^4_{H^2}+\|\omega_0\|^4_{H^2}\big)\exp\Big\{C\Big(T+\int_0^T\|\pi(t)\|^{\frac{2}{2-r}}_{L^{\frac3r}}\,\mathrm dt\Big)\Big\}, \end{aligned}$$

which implies

$$\sup_{0\le t\le T}\big(\|u\|^4_{L^4}+\|b\|^4_{L^4}+\|\omega\|^4_{L^4}\big)\le C\sup_{0\le t\le T}\big(\|z^+\|^4_{L^4}+\|z^-\|^4_{L^4}+\|\omega\|^4_{L^4}\big)<\infty.$$

This ends the proof of Theorem 1.1. □

4 Proof of Theorem 1.3

In this section, we will give the proof of Theorem 1.3. The estimate of the nonlinear term is different from the procedure in the proof Theorem 1.1.

Step 1. We deal with the first case (1.6). According to the L^4-energy estimate (3.2) we have

$$I_2 = -4\int_{\mathbb{R}^3} \nabla\pi \cdot |z^+|^2 z^+ \,\mathrm{d}x - 4\int_{\mathbb{R}^3} \nabla\pi \cdot |z^-|^2 z^- \,\mathrm{d}x =: I_{21} + I_{22}.$$

Thanks to the Littlewood–Paley decomposition, we decompose π into the following form:

$$\pi = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j \pi = \sum_{j<-N} \dot{\Delta}_j \pi + \sum_{j=-N}^{N} \dot{\Delta}_j \pi + \sum_{j>N} \dot{\Delta}_j \pi, \tag{4.1}$$

where N is a positive integer to be determined later. Then, inserting (4.1) into I_{21} yields

$$\begin{aligned} I_{21} =& -4\int_{\mathbb{R}^3} \sum_{j<-N} \dot{\Delta}_j \nabla\pi \cdot z^+ |z^+|^2 \,\mathrm{d}x + 4\int_{\mathbb{R}^3} \sum_{j=-N}^{N} \dot{\Delta}_j \pi z^+ \cdot \nabla |z^+|^2 \,\mathrm{d}x \\ & + 4\int_{\mathbb{R}^3} \sum_{j>N} \dot{\Delta}_j \pi z^+ \cdot \nabla |z^+|^2 \,\mathrm{d}x \\ =:& I_{211} + I_{212} + I_{213}. \end{aligned} \tag{4.2}$$

Now, we estimate I_{211}. With the help of the Hölder inequality, Lemma 2.3, the interpolation inequality, the Calderón–Zygmund inequality (3.5) for $p = 2$ and the Young inequality, we obtain

$$\begin{aligned} I_{211} \le& \int_{\mathbb{R}^3} \Big| \sum_{j<-N} \dot{\Delta}_j \nabla\pi \Big| |z^+|^2 |z^+| \,\mathrm{d}x \\ \le& \sum_{j<-N} \|\dot{\Delta}_j \nabla\pi\|_{L^6} \|z^+\|_{L^2} \|z^+\|_{L^6}^2 \\ \le& \sum_{j<-N} 2^j \|\dot{\Delta}_j \nabla\pi\|_{L^2} \|z^+\|_{L^2} \|z^+\|_{L^4} \|z^+\|_{L^{12}} \\ \le& \sum_{j<-N} 2^j \|\nabla\pi\|_{L^2} \|z^+\|_{L^2} \|z^+\|_{L^4} \|\nabla |z^+|^2\|_{L^2}^{\frac{1}{2}} \\ \le& C2^{-N} \|z^- \cdot \nabla z^+\|_{L^2} \|z^+\|_{L^4} \|\nabla |z^+|^2\|_{L^2}^{\frac{1}{2}} \\ \le& \tfrac{1}{16} \|z^- \cdot \nabla z^+\|_{L^2}^2 + C2^{-2N} \|z^+\|_{L^4}^2 \|\nabla |z^+|^2\|_{L^2} \\ \le& \tfrac{1}{16} \|z^- \cdot \nabla z^+\|_{L^2}^2 + \tfrac{1}{16} \|\nabla |z^+|^2\|_{L^2}^2 + C2^{-4N} \big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4 \big). \end{aligned} \tag{4.3}$$

Using again the Young inequality for convolution, we get

$$
\begin{aligned}
I_{212} \leq & C \int_{\mathbb{R}^3} \left(\sum_{j=-N}^{N} |\dot{\Delta}_j \pi|^{q_1} \right)^{\frac{1}{q_1}} N^{1-\frac{1}{q_1}} |\nabla |z^+|^2| |z^+| \, \mathrm{d}x \\
\leq & C N^{1-\frac{1}{q_1}} \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)} \|\nabla |z^+|^2\|_{L^2} \|z^+\|_{L^{\frac{2q}{q-2}}} \\
\leq & C N^{1-\frac{1}{q_1}} \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)} \|\nabla |z^+|^2\|_{L^2} \|z^+\|_{L^2}^{1-\frac{12}{5q}} \|z^+\|_{L^{12}}^{\frac{12}{5q}} \\
\leq & C N^{1-\frac{1}{q_1}} \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)} \|z^+\|_{L^2}^{1-\frac{12}{5q}} \|\nabla |z^+|^2\|_{L^2}^{\frac{5q+6}{5q}} \\
\leq & \tfrac{1}{16} \|\nabla |z^+|^2\|_{L^2}^2 + C N \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{10q}{5q-6}} \|z^+\|_{L^2}^{\frac{10q-24}{5q-6}} \\
\leq & \tfrac{1}{16} \|\nabla |z^+|^2\|_{L^2}^2 + C N \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}},
\end{aligned}
\tag{4.4}
$$

where $q_1 = \frac{10q}{5q+6}$ with $q > \frac{12}{5}$. With the help of (2.1), we can verify that

$$
\begin{aligned}
I_{213} \leq & C \sum_{j>N} \|\dot{\Delta}_j \pi\|_{L^5} \|z^+\|_{L^{\frac{10}{3}}} \|\nabla |z^+|^2\|_{L^2} \\
\leq & C \sum_{j>N} 2^{-j} \|\dot{\Delta}_j \nabla \pi\|_{L^5} \|z^+\|_{L^2}^{\frac{1}{5}} \|z^+\|_{L^4}^{\frac{4}{5}} \|\nabla |z^+|^2\|_{L^2} \\
\leq & C \sum_{j>N} 2^{-\frac{j}{10}} \|\dot{\Delta}_j \nabla \pi\|_{L^2} \|z^+\|_{L^2}^{\frac{1}{5}} \|z^+\|_{L^4}^{\frac{4}{5}} \|\nabla |z^+|^2\|_{L^2} \\
\leq & C 2^{-\frac{N}{10}} \|\nabla \pi\|_{L^2} \|z^+\|_{L^4}^{\frac{4}{5}} \|\nabla |z^+|^2\|_{L^2} \\
\leq & C 2^{-\frac{N}{10}} \|z^- \cdot \nabla z^+\|_{L^2} \|z^+\|_{L^4}^{\frac{4}{5}} \|\nabla |z^+|^2\|_{L^2} \\
\leq & \tfrac{1}{16} \|z^- \cdot \nabla z^+\|_{L^2}^2 + C 2^{-\frac{N}{5}} \|z^+\|_{L^4}^{\frac{8}{5}} \|\nabla |z^+|^2\|_{L^2}^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
I_{213} \leq & \tfrac{1}{16} \|z^- \cdot \nabla z^+\|_{L^2}^2 + C 2^{-\frac{N}{5}} \left(\|z^+\|_{L^4}^{\frac{8}{5}} + \|z^-\|_{L^4}^{\frac{8}{5}} + \|\omega\|_{L^4}^{\frac{8}{5}} \right) \\
& \times \left(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2 \right).
\end{aligned}
\tag{4.5}
$$

This inequality together with (4.3) and (4.4) gives rise to

$$
\begin{aligned}
I_{21} \leq & \tfrac{1}{8} \|z^- \cdot \nabla z^+\|_{L^2}^2 + \tfrac{1}{8} \|\nabla |z^+|^2\|_{L^2}^2 + C 2^{-4N} \left(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4 \right) \\
& + C N \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}} + C 2^{-\frac{N}{5}} \left(\|z^+\|_{L^4}^{\frac{8}{5}} + \|z^-\|_{L^4}^{\frac{8}{5}} + \|\omega\|_{L^4}^{\frac{8}{5}} \right) \\
& \qquad \times \left(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2 \right) \\
\leq & \tfrac{1}{8} \|z^- \cdot \nabla z^+\|_{L^2}^2 + \tfrac{1}{8} \|\nabla |z^+|^2\|_{L^2}^2 + C_0 2^{-4N} \left(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4 \right) \\
& + C_1 N \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}} + \left[C_2 2^{-\frac{N}{2}} \left(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4 \right) \right]^{\frac{2}{5}} \\
& \qquad \times \left(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2 \right).
\end{aligned}
\tag{4.6}
$$

Similarly,

$$
\begin{aligned}
I_{22} \leq & \tfrac{1}{8}\|z^- \cdot \nabla z^+\|_{L^2}^2 + \tfrac{1}{8}\|\nabla |z^-|^2\|_{L^2}^2 + C_0 2^{-4N}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big) \\
& + C_1 N \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}} + \Big[C_2 2^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big)\Big]^{\frac{2}{5}} \\
& \qquad \times \big(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2\big).
\end{aligned} \tag{4.7}
$$

Substituting (3.3), (4.6) and (4.7) into (3.2) yields

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big) + \tfrac{7}{2}\big(\||z^+||\nabla z^+|\|_{L^2}^2 + \||z^-||\nabla z^-|\|_{L^2}^2 + \||\omega||\nabla \omega|\|_{L^2}^2\big) \\
& \quad + \tfrac{15}{8}\big(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2\big) + 2\|\omega\|_{L^4}^4 \\
\leq & C\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4\big) + \tfrac{1}{4}\|z^- \cdot \nabla z^+\|_{L^2}^2 + C_0 2^{-4N}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big) \\
& \quad + C_1 N \|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}} + \Big[C_2 2^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big)\Big]^{\frac{2}{5}} \\
& \qquad \times \big(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2\big).
\end{aligned} \tag{4.8}
$$

We can choose N in (4.8) so that

$$
\Big[C_2 2^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big)\Big]^{\frac{2}{5}} \leq \frac{1}{8}. \tag{4.9}
$$

This implies that

$$
N \geq 2\bigg[\frac{\log^+\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4 + e\big)}{\log 2} + 1\bigg];
$$

here, $\log^+ t = \log t$ for $t \geq 1$ and $\log^+ t = 0$ for $0 < t < 1$. On the other hand, it is easy to deduce that

$$
C_0 2^{-4N}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big) \leq C_0 2^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big) \leq C.
$$

Hence, (4.8) becomes

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4\big) + \tfrac{7}{4}\big(\|\nabla |z^+|^2\|_{L^2}^2 + \|\nabla |z^-|^2\|_{L^2}^2\big) \\
& \quad + \tfrac{7}{2}\big(\||z^+||\nabla z^+|\|_{L^2}^2 + \||z^-||\nabla z^-|\|_{L^2}^2 + \||\omega||\nabla \omega|\|_{L^2}^2\big) \\
\leq & \tfrac{1}{4}\|z^- \cdot \nabla z^+\|_{L^2}^2 + C\Big(\|\pi\|_{\dot{F}^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}} + 1\Big)\big(\|z^+\|_{L^4}^4 + \|z^-\|_{L^4}^4 + \|\omega\|_{L^4}^4 + e\big).
\end{aligned} \tag{4.10}
$$

This inequality together with the following inequalities (here we refer the readers to the references [29, 30] for details)

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^s} &\leq \tfrac{1}{2}\big(\|z^+(\cdot,t)\|_{L^s} + \|z^-(\cdot,t)\|_{L^s}\big), \\
\|b(\cdot,t)\|_{L^s} &\leq \tfrac{1}{2}\big(\|z^+(\cdot,t)\|_{L^s} + \|z^-(\cdot,t)\|_{L^s}\big),
\end{aligned} \tag{4.11}
$$

leads to

$$
\begin{aligned}
&\left(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4\right)+\frac{7}{4}\int_0^t\left(\|\nabla|u|^2\|_{L^2}^2+\|\nabla|b|^2\|_{L^2}^2\right)\mathrm{d}\tau\\
&\quad+\frac{7}{2}\int_0^t\left(\||u||\nabla u|\|_{L^2}^2+\||u||\nabla b|\|_{L^2}^2+\||b||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\right)\mathrm{d}\tau\\
&\le\left(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4\right)\\
&\quad+\int_0^t\left(\||u||\nabla u|\|_{L^2}^2+\||u||\nabla b|\|_{L^2}^2+\||b||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2\right)\mathrm{d}\tau\\
&\quad+C\int_0^t\left(\|\pi\|_{\dot F^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}}+1\right)\left(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\right)\mathrm{d}\tau.
\end{aligned}
\tag{4.12}
$$

Then, one has

$$
\begin{aligned}
&\left(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4\right)\\
&\quad+\frac{5}{2}\int_0^t\left(\||u||\nabla u|\|_{L^2}^2+\||u||\nabla b|\|_{L^2}^2+\||b||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\right)\mathrm{d}\tau\\
&\le\left(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4\right)\\
&\quad+C\int_0^t\left(\|\pi\|_{\dot F^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}}+1\right)\left(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\right)\mathrm{d}\tau.
\end{aligned}
\tag{4.13}
$$

The Gronwall inequality guarantees that

$$
\begin{aligned}
&\|u(t)\|_{L^4}^4+\|b(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\\
&\qquad\le\left(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4+e\right)\exp\left(CT+C\int_0^T\|\pi\|_{\dot F^0_{q,q_1}(\mathbb{R}^3)}^{\frac{q_1}{q_1-1}}\mathrm{d}\tau\right)
\end{aligned}
$$

holds for all $0\le t\le T$.

Step 2. To deal with the second case (1.7), we only need to estimate I_{212}. We have

$$
\begin{aligned}
I_{212} =& -\int_{\mathbb{R}^3}\sum_{j=-N}^{N}\dot{\Delta}_j\nabla\pi\cdot z^+|z^+|^2\,\mathrm{d}x\\
\le& C\int_{\mathbb{R}^3}\left|\sum_{j=-N}^{N}\dot{\Delta}_j\nabla\pi\right||z^+|^3\,\mathrm{d}x\\
\le& C\int_{\mathbb{R}^3}\left(\sum_{j=-N}^{N}|\dot{\Delta}_j\nabla\pi|^{q_2}\right)^{\frac{1}{q_2}}N^{1-\frac{1}{q_2}}|z^+|^3dx\\
\le& CN^{1-\frac{1}{q_2}}\|\nabla\pi\|_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\|z^+\|^3_{L^{\frac{3q}{q-1}}}\\
\le& CN^{1-\frac{1}{q_2}}\|\nabla\pi\|_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\|z^+\|_{L^4}^{\frac{3(3q-4)}{2q}}\|z^+\|_{L^{12}}^{\frac{3(4-q)}{2q}}\\
\le& CN^{1-\frac{1}{q_2}}\|\nabla\pi\|_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\|z^+\|_{L^4}^{\frac{3(3q-4)}{2q}}\|\nabla|z^+|^2\|_{L^2}^{\frac{3(4-q)}{4q}}\\
\le& \tfrac{1}{16}\|\nabla|z^+|^2\|_{L^2}^2+CN\|\nabla\pi\|^{\frac{q_2}{q_2-1}}_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\|z^+\|_{L^4}^{\frac{36q-48}{11q-12}}\\
\le& \tfrac{1}{16}\|\nabla|z^+|^2\|_{L^2}^2+CN\|\nabla\pi\|^{\frac{q_2}{q_2-1}}_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\big(\|z^+\|_{L^4}^4+1\big)\\
\le& \tfrac{1}{16}\|\nabla|z^+|^2\|_{L^2}^2+CN\|\nabla\pi\|^{\frac{q_2}{q_2-1}}_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big),
\end{aligned}
\tag{4.14}
$$

where $q_2=\frac{8q}{12-3q}$ with $\frac{12}{11}<q<4$. Then, by summing up (4.3), (4.5) and (4.14), we obtain

$$
\begin{aligned}
I_{21}\le& \tfrac{1}{8}\|z^-\cdot\nabla z^+\|_{L^2}^2+\tfrac{1}{8}\|\nabla|z^+|^2\|_{L^2}^2+C2^{-4N}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\\
&+C_1N\|\nabla\pi\|^{\frac{q_2}{q_2-1}}_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\\
&+\Big[C_22^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\Big]^{\frac{2}{5}}\big(\|\nabla|z^+|^2\|_{L^2}^2+\|\nabla|z^-|^2\|_{L^2}^2\big).
\end{aligned}
\tag{4.15}
$$

Similarly,

$$
\begin{aligned}
I_{22}\le& \tfrac{1}{8}\|z^-\cdot\nabla z^+\|_{L^2}^2+\tfrac{1}{8}\|\nabla|z^-|^2\|_{L^2}^2+C2^{-4N}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\\
&+C_1N\|\nabla\pi\|^{\frac{q_2}{q_2-1}}_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\\
&+\Big[C_22^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\Big]^{\frac{2}{5}}\big(\|\nabla|z^+|^2\|_{L^2}^2+\|\nabla|z^-|^2\|_{L^2}^2\big).
\end{aligned}
\tag{4.16}
$$

Then, summing up (4.15) and (4.16), we get

$$
\begin{aligned}
I_{2}\le& \tfrac{1}{4}\|z^-\cdot\nabla z^+\|_{L^2}^2+\tfrac{1}{8}\|\nabla|z^+|^2\|_{L^2}^2+\tfrac{1}{8}\|\nabla|z^-|^2\|_{L^2}^2\\
&+C_1N\|\nabla\pi\|^{\frac{q_2}{q_2-1}}_{\dot{F}^0_{q,q_2}(\mathbb{R}^3)}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\\
&+\Big[C_22^{-\frac{N}{2}}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)\Big]^{\frac{2}{5}}\big(\|\nabla|z^+|^2\|_{L^2}^2+\|\nabla|z^-|^2\|_{L^2}^2\big),
\end{aligned}
\tag{4.17}
$$

where N is just the same as in (4.9). Hence, substituting (3.3) and (4.17) into (3.2) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)+\tfrac{7}{4}\big(\|\nabla|z^+|^2\|_{L^2}^2+\|\nabla|z^-|^2\|_{L^2}^2+\|\nabla|\omega|^2\|_{L^2}^2\big)\\
&\quad+\tfrac{7}{2}\big(\||z^+||\nabla z^+|\|_{L^2}^2+\||z^-||\nabla z^-|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\big)\\
&\le C\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+e\big)+\tfrac{1}{4}\|z^-\cdot\nabla z^+\|_{L^2}^2+C\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}\\
&\qquad\times\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\log\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\\
&\le\tfrac{1}{4}\|z^-\cdot\nabla z^+\|_{L^2}^2+C\bigg(\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}+1\bigg)\Big[\log\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)+1\Big]\\
&\qquad\times\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4\big)+\tfrac{7}{2}\big(\||z^+||\nabla z^+|\|_{L^2}^2+\||z^-||\nabla z^-|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\big)\\
&\le\tfrac{1}{4}\|z^-\cdot\nabla z^+\|_{L^2}^2+C\bigg(\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}+1\bigg)\Big[\log\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)+1\Big]\\
&\qquad\qquad\times\big(\|z^+\|_{L^4}^4+\|z^-\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big).
\end{aligned}
$$

Similar to what we did to (4.13), we can write

$$
\begin{aligned}
&\big(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\\
&\quad+\frac{5}{2}\int_0^t\big(\||u||\nabla u|\|_{L^2}^2+\||u||\nabla b|\|_{L^2}^2+\||b||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2+\||\omega||\nabla\omega|\|_{L^2}^2\big)\,\mathrm{d}\tau\\
&\le\big(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4+e\big)\\
&\quad+C\int_0^t\bigg(\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}+1\bigg)\Big[\log\big(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)+1\Big]\\
&\qquad\qquad\times\big(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4+e\big)\,\mathrm{d}\tau.
\end{aligned}
$$

Then Gronwall inequality reads

$$
\begin{aligned}
&\big(\|u(t)\|_{L^4}^4+\|b(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4+e\big)\\
&\quad\le\big(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4+e\big)\\
&\qquad\times\exp\bigg\{C\int_0^t\bigg(\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}+1\bigg)\Big(\log(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4+e)+1\Big)\,\mathrm{d}\tau\bigg\}.
\end{aligned}
$$

Let $X(t)=\log(\|u\|_{L^4}^4+\|b\|_{L^4}^4+\|\omega\|_{L^4}^4+e)$. The Gronwall inequality guarantees that

$$
X(t)\le X(0)+C\int_0^t\bigg(\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}+1\bigg)(X(t)+1)\,\mathrm{d}\tau,
$$

which yields

$$
X(t)\le X(0)\exp\bigg\{CT+C\int_0^t\|\nabla\pi\|_{\dot F^0_{q,q_2}(\mathbb{R}^3)}^{\frac{q_2}{q_2-1}}\,\mathrm{d}\tau\bigg\}.
$$

This completes the proof of Theorem 1.3.

Competing interests. The authors declare that they have no competing interests.

Authors' contributions. The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 11961032 and No. 11971209), the Natural Science Foundation of Jiangxi Province, China (No. 20191BAB201003).

References

[1] G. Ahmadi, M. Shahinpoor, *Universal stability of magneto–micropolar fluid motions*, International Journal of Engineering Science **12** (1974), 657–663.

[2] H. Bahouri, J. Y. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften 343, Springer, Heidelberg Dordrecht London New York, 2011.

[3] L. Berselli, G. Galdi, *Regularity criteria involving the pressure for the weak solutions of the Navier–Stokes equations*, Proceedings of the American Mathematical Society **130** (2002), 3585–3595.

[4] H. Beiráo da Veiga, *A new regularity class for the Navier–Stokes equations in* $\mathbb{R}^n$, Chinese Annals of Mathematics, Series B **16** 1995, 407–412.

[5] D. Chae, J. Lee, *Regularity criterion in terms of pressure for the Navier–Stokes equations*, Nonlinear Analysis: Theory, Methods & Applications **46** (2001), 727–735.

[6] Q. Chen, Z. Zhang, *Regularity criterion via the pressure on weak solutions to the 3D Navier–Stokes equations*, Proceedings of the American Mathematical Society **135** (2007), 1829–1837.

[7] S. Doi, M. Edwards, *The theory of polymer dynamics*, Oxford Science Publications, Oxford, 1986.

[8] G. Duvaut, J. L. Lions, *Inéquations en thermoélasticité et magnétohydrodynamique*, Archive for Rational Mechanics and Analysis **46** (1972), 241–279.

[9] H. Duan, *On regularity criterion in terms of pressure for the 3D viscous MHD equations*, Applicable Analysis **91** (2012), 947–952.

[10] B. Dong, J. Yan, Z. Chen, *Pressure regularity criteria of the three-dimensional micropolar fluid flows*, Mathematical Methods in the Applied Sciences **34** (2011), 595-606.

[11] G. Lukaszewicz, *Micropolar fluids*: *Theory and applications*, Birkhäuser, Berlin, 1998.

[12] S. Gala, *A note on the uniqueness of mild solutions to the Navier–Stokes equations*, Archive Der Mathematik (Basel) **84** (2007), 448–454.

[13] G. P. Galdi, S. Rionero, *A note on the existence and uniqueness of solutions of the micropolar fluid equations*, International Journal of Engineering Science **15** (1977), 105–108.

[14] C. He, Z. Xin, *On the regularity of weak solutions to the magnetohydrodynamic equations*, Journal Differential Equations **213** (2005), 235–254.

[15] E. E. Ortega-Torres, M. A. Rojas-Medar, *Magneto–micropolar fluid motion*: *global existence of strong solutions*, Abstract and Applied Analysis **4** (1999), 109–125.

[16] M. A. Rojas-Medar, *Magneto–micropolar fluid motion*: *existence and uniqueness of strong solution*, Mathematische Nachrichten **188** (1997), 301–319.

[17] M. A. Rojas-Medar, J. L. Boldrini, *Magneto–microploar fluid motion: existence of weak solutions*, Revista Matemática Complutense **11** (1998), 443–460.

[18] J.Serrin, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Archive for Rational Mechanics and Analysis **19** (1962), 187–195.

[19] M. Sermange, R. Temam, *Some mathematical questions related to the MHD equations*, Communications on Pure Applied Mathematics **36** (1983), 635–664.

[20] W. Yang, Q. Jiu, J. Wu, *The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation*, Journal Differential Equations **266** (2019), 630–652.

[21] Y. Zhou, *On the regularity criteria in terms of pressure of the Navier–Stokes equations in* $\mathbb{R}^3$, Proceedings of the American Mathematical Society **134** (2006), 149–156.

[22] Y. Zhou, *Regularity criteria in terms of pressure for the 3D Navier–Stokes equations in a generic domain*, Mathematische Annalen **328** (2004), 173–192.

[23] Y. Zhou, *On a regularity criterion in terms of the gradient of pressure for the Navier–Stokes equations in* $\mathbb{R}^n$, Zeitschrift für angewandte Mathematik und Physik **57** (2006), 384–392.

[24] Y. Zhou, *On regularity criteria in terms of pressure for the Navier–Stokes equations in* $\mathbb{R}^3$, Proceedings of the American Mathematical Society **134** (2006), 149–156.

[25] Y. Zhou, *Remarks on regularities for the 3D MHD equations*, Discrete Continuous Dynamic Systems **12** (2005), 881–886.

[26] Y. Zhou, *Regularity criteria for the 3D MHD equations in terms of the pressure*, International Journal of Non-Linear Mechanics **41** (2006), 1174–1180.

[27] R. Agarwal, S. Gala, M. A. Ragusa, *A regularity criterion in weak spaces to Boussinesq equations*, Mathematics **8** (2020), no. 6, article no. 920.

[28] L. Son, L. Ngoc, L. Long, *Existence, blow-up and exponential decay estimates for the nonlinear Kirchhoff-carrier wave equation in an annular with nonhomogeneous Dirichlet conditions*, Filomat **33** (2019), no. 17, 5561–5588.

[29] S. Gala, M. A. Ragusa, *A note on regularity criteria in terms of pressure for the 3D viscous MHD equations*, Mathematical Notes **102** (2017), 475–479.

[30] F. Wu, *A refined regularity criteria of weak solutions to the magneto–micropolar fluid equations*, Journal of Evolution Equations **21** (2021), 725–734.