Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2022, Number 6, pp. 105–126 (April 2023) http://www.jneea.com

# ON $k$-GENERALIZED $\psi$-HILFER IMPULSIVE BOUNDARY VALUE PROBLEM IN BANACH SPACES

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Received April 26, 2022

Accepted June 9, 2022

Communicated by Gaston M. N'Guérékata

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**Abstract.** This paper deals with the existence and uniqueness results for a class of boundary value problem for implicit nonlinear fractional differential equations with instantaneous impulses and $k$-Generalized $\psi$-Hilfer fractional derivative involving both retarded and advanced arguments. The results are based on Mönch fixed point theorem associated with the technique of measure of non-compactness. An illustrative example is provided to indicate the applicability of our results.

**Keywords:** $\psi$-Hilfer fractional derivative, $k$-generalized $\psi$-Hilfer fractional derivative, impulsions, retarded arguments, Banach space, advanced arguments, existence, uniqueness.

**2010 Mathematics Subject Classification:** 34A08, 26A33, 34A12.

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## 1 Introduction

Fractional calculus has long been an important research topic in functional space theory due to its relevance in the modeling and scientific understanding of natural phenomena. Indeed, several applications in viscoelasticity and electrochemistry have been investigated. Non-integer derivatives of fractional order have been successfully used to generalize the fundamental laws of nature. For

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more details, we recommend [1–3, 7, 11, 13, 19–23, 25]. The authors of [6, 12, 14] explored the existence, stability and uniqueness of solutions for various problems with fractional differential equation and inclusions concerning retarded or advanced arguments. Recently in [9], Diaz presented the definitions of the special functions $k$-gamma and $k$-beta. Several findings and generalizations for various fractional integrals and derivatives based on the properties of these special functions can be found in [8, 16, 17]. In [24], Sousa *et al.* introduced another so-called $\psi$-Hilfer fractional derivative with respect to another function and gave some important properties concerning this type of fractional operators.

Recently in [20], we established existence, uniqueness and Ulam stability results to the boundary value problem with nonlinear implicit generalized Hilfer type fractional differential equation with impulses:

$$\begin{cases} \left({}^{\rho}D^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}D^{\alpha,\beta}_{t_k^+}u\right)(t)\right);\ t \in J_k,\ k = 0,\dots,m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + \varpi_k(u(t_k^-)); k = 1,\dots,m, \\ c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) = c_3, \end{cases}$$

where ${}^{\rho}D^{\alpha,\beta}_{t_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma(\gamma = \alpha+\beta-\alpha\beta)$ respectively, $c_1$ and $c_2$ are reals with $c_1 + c_2 \neq 0$, $c_3 \in E$, $J_k := (t_k, t_{k+1}]; k = 0,\dots,m$, $a = t_0 < t_1 < \dots < t_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon\to 0^+} u(t_k+\epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon\to 0^-} u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : (a,b] \times E \times E \to E$ is a given function and $\varpi_k : E \to E$; $k = 1,\dots,m$ are given continuous functions, where $(E, \|\cdot\|)$ is a Banach space. The proved results rely on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness.

In the aim of generalizing the prior results, in this paper, we establish an existence result to the following $k$-generalized $\psi$-Hilfer problem with nonlinear implicit fractional differential equation with impulses involving both retarded and advanced arguments :

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}x\right)(t) = f\left(t, x^t(\cdot), \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}x\right)(t)\right),\ \ t \in J_i,\ i = 0,\dots,m, \tag{1.1}$$

$$\left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_i^+}x\right)(t_i^+) = \left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_{i-1}^+}x\right)(t_i^-) + L_i(x(t_i^-)); i = 1,\dots,m, \tag{1.2}$$

$$\alpha_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+) + \alpha_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_m^+}x\right)(b) = \alpha_3, \tag{1.3}$$

$$x(t) = \varpi(t),\ \ t \in [a-\lambda, a],\ \lambda > 0, \tag{1.4}$$

$$x(t) = \tilde{\varpi}(t),\ \ t \in \left[b, b+\tilde{\lambda}\right],\ \tilde{\lambda} > 0, \tag{1.5}$$

where ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}$, $\mathcal{J}^{k(1-\xi),k;\psi}_{a+}$ are the $k$-generalized $\psi$-Hilfer fractional derivative of order $\vartheta \in (0,k)$ and type $r \in [0,1]$ defined in Section 2, and $k$-generalized $\psi$-fractional integral of order $k(1-\xi)$ defined in [18] respectively, where $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k > 0$, $\varpi \in C([a-\lambda, a], E)$, $\tilde{\varpi} \in C\left(\left[b, b+\tilde{\lambda}\right], E\right)$, $f : [a,b] \times PC_{\xi;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right) \times E \longrightarrow E$ is a given appropriate function specified later, $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$, $\alpha_3 \in E$, $J_i := (t_i, t_{i+1}]; i = 0,\dots,m$, $a = t_0 < t_1 < \dots < t_m < t_{m+1} = b < \infty$, $x(t_i^+) = \lim\limits_{\epsilon\to 0^+} x(t_i+\epsilon)$ and $x(t_i^-) = \lim\limits_{\epsilon\to 0^-} x(t_i+\epsilon)$

represent the right and left hand limits of $x(t)$ at $t = t_i$ and $L_i : E \to E$; $i = 1, \dots, m$ are given continuous functions, where $(E, \|\cdot\|)$ is a Banach space. For each function $x$ defined on $\left[a-\lambda, b+\tilde{\lambda}\right]$ and for any $t \in (a, b]$, we denote by $x^t$ the element defined by

$$x^t(\tau) = x(t+\tau), \ \ \tau \in \left[-\lambda, \tilde{\lambda}\right].$$

This paper has the following structure: In Section 2, some notations are introduced and we recall some preliminaries about $k$-generalized $\psi$-Hilfer fractional integral, the functions $k$-Gamma, $k$-Beta and some auxiliary results. Further, we give the definition of the $k$-generalized $\psi$-Hilfer type fractional derivative and some essential theorems and lemmas. In Section 3, we present an existence result for the problem (1.1)-(1.5) that is founded on Mönch fixed point theorem associated with the technique of measure of noncompactness. Finally, in the last section, we give an example to illustrate the applicability of our main result.

## 2 Preliminaries

First, we present the weighted spaces, notations, definitions, and preliminary facts which are used in this paper. Let $0 < a < b < \infty$, $J = [a, b]$, $\vartheta \in (0, k)$, $r \in [0, 1]$, $k > 0$ and $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$. Let $(E, \|\cdot\|)$ be a Banach space. By $C(J, E)$ we denote the Banach space of all continuous functions from $J$ into $E$ with the norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

Let $\mathcal{C} = C([a-\lambda, a], E)$ and $\tilde{\mathcal{C}} = C\left(\left[b, b+\tilde{\lambda}\right], E\right)$ be the spaces endowed, respectively, with the norms

$$\|x\|_{\mathcal{C}} = \sup\{\|x(t)\| : t \in [a-\lambda, a]\},$$

and

$$\|x\|_{\tilde{\mathcal{C}}} = \sup\left\{\|x(t)\| : t \in \left[b, b+\tilde{\lambda}\right]\right\}.$$

Consider the weighted Banach space

$$C_{\xi;\psi}(J_i) = \left\{x : J_i \to E : t \to \Psi_\xi^\psi(t, t_i)x(t) \in C([t_i, t_{i+1}], E)\right\},$$

where $\Psi_\xi^\psi(t, t_i) = (\psi(t) - \psi(t_i))^{1-\xi}$ and $i = 0, \dots, m$. And, we consider

$$PC_{\xi;\psi}(J) = \Bigg\{x : (a, b] \to E : x \in C_{\xi;\psi}(J_i); i = 0, \dots, m, \text{and there exist}$$
$$x(t_i^-) \text{ and } \left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}x\right)(t_i^+); i = 1, \dots, m, \text{with } x(t_i^-) = x(t_i)\Bigg\},$$

with the norm

$$\|x\|_{PC_{\xi;\psi}} = \max_{i=0,\dots,m}\left\{\sup_{t\in[t_i,t_{i+1}]}\left\|\Psi_\xi^\psi(t, t_i)x(t)\right\|\right\}.$$

Consider the weighted Banach space

$$PC_{\xi;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)=\Bigg\{x:\left[-\lambda,\tilde{\lambda}\right]\to E:\tau\to\Psi_{\xi}^{\psi}(t,t_i)x(\tau)\in C([\tau_i,\tau_{i+1}],E);i=0,\ldots,m,$$
$$\text{and there exist } x(\tau_i^-) \text{ and } \left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}x\right)(\tau_i^+);i=1,\ldots,m,$$
$$\text{with } x(\tau_i^-)=x(\tau_i) \text{ and } \tau_i=t_i-t, \text{ for each } t\in J_i\Bigg\},$$

with the norm

$$\|x^t\|_{[-\lambda,\tilde{\lambda}]}=\max\left\{\max_{i=0,\ldots,m}\left\{\sup_{\tau\in[\tau_i,\tau_{i+1}]}\left\|\Psi_{\xi}^{\psi}(t,t_i)x^t(\tau)\right\|\right\},\sup_{\tau\in[-\lambda,0]}\|x^a(\tau)\|,\sup_{\tau\in[0,\tilde{\lambda}]}\left\|x^b(\tau)\right\|\right\}.$$

Next, we consider the Banach space

$$\mathbb{F}=\left\{x:\left[a-\lambda,b+\tilde{\lambda}\right]\to\mathbb{R}:x|_{[a-\lambda,a]}\in\mathcal{C},\,x|_{[b,b+\tilde{\lambda}]}\in\tilde{\mathcal{C}}\text{ and }x|_{(a,b]}\in PC_{\xi;\psi}(J)\right\}$$

with the norm

$$\|x\|_{\mathbb{F}}=\max\left\{\|x\|_{\mathcal{C}},\|x\|_{\tilde{\mathcal{C}}},\|x\|_{PC_{\xi;\psi}}\right\}.$$

By $L^1(J)$, we denote the space of Bochner–integrable functions $f:J\longrightarrow E$ with the norm

$$\|f\|_1=\int_a^b\|f(t)\|\,\mathrm{d}t.$$

**Definition 2.1** [9] *The $k$-gamma function is defined by*

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}\,\mathrm{d}t,\alpha>0.$$

*When $k\to 1$ then $\Gamma(\alpha)=\Gamma_k(\alpha)$, we have also some useful following relations $\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right)$, $\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha)$ and $\Gamma_k(k)=\Gamma(1)=1$. Furthermore $k$-beta function is defined as follows*

$$B_k(\alpha,\beta)=\frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\,\mathrm{d}t$$

*so that $B_k(\alpha,\beta)=\frac{1}{k}B\left(\frac{\alpha}{k},\frac{\beta}{k}\right)$ and $B_k(\alpha,\beta)=\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$.*

Now, we give all the definitions to the different fractional operators used throughout this paper.

**Definition 2.2** [18] (*$k$-Generalized $\psi$-fractional Integral*) *Let $[a,b]$ be a finite or infinite interval on the real axis $\mathbb{R}=(-\infty,\infty)$, $\psi(t)>0$ be an increasing function on $(a,b]$ and $\psi'(t)>0$ be continuous on $(a,b)$ and $\vartheta>0$. The generalized $k$-fractional integral operators of a function $g$ (left-sided and right-sided) of order $\vartheta$ are defined by*

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}g(t)=\int_a^t\bar{\Psi}_{\vartheta}^{k,\psi}(t,s)\psi'(s)g(s)\,\mathrm{d}s,$$
$$\mathcal{J}_{b-}^{\vartheta,k;\psi}g(t)=\int_t^b\bar{\Psi}_{\vartheta}^{k,\psi}(s,t)\psi'(s)g(s)\,\mathrm{d}s,$$

*with $k>0$ and $\bar{\Psi}_{\vartheta}^{k,\psi}(t,s)=\frac{(\psi(t)-\psi(s))^{\frac{\vartheta}{k}-1}}{k\Gamma_k(\vartheta)}$.*

Also in [17], Nápoles Valdés gave a more generalized fractional integral operators defined by

$$\mathcal{J}_{G,a+}^{\vartheta,k;\psi}g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)g(s)}{G(\psi(t)-\psi(s),\frac{\vartheta}{k})}\,\mathrm{d}s,$$
$$\mathcal{J}_{G,b-}^{\vartheta,k;\psi}g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_t^b \frac{\psi'(s)g(s)}{G(\psi(s)-\psi(t),\frac{\vartheta}{k})}\,\mathrm{d}s,$$

where $G(z,\vartheta)\in AC[a,b]$; the space of absolutely continuous functions defined on $[a,b]$.

**Theorem 2.3** [17] *Let $g:[a,b]\to\mathbb{R}$ be an integrable function, and take $\vartheta>0$ and $k>0$. Then $\mathcal{J}_{G,a+}^{\vartheta,k;\psi}g$ exists for all $t\in[a,b]$.*

**Theorem 2.4** [17] *Let $g\in L^1(J)$ and take $\vartheta>0$ and $k>0$. Then $\mathcal{J}_{G,a+}^{\vartheta,k;\psi}g\in C([a,b],\mathbb{R})$.*

**Lemma 2.5** *Let $\vartheta>0$, $r>0$ and $k>0$. Then, we have the following semigroup property given by*

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\mathcal{J}_{a+}^{r,k;\psi}f(t) = \mathcal{J}_{a+}^{\vartheta+r,k;\psi}f(t) = \mathcal{J}_{a+}^{r,k;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}f(t)$$

*and*

$$\mathcal{J}_{b-}^{\vartheta,k;\psi}\mathcal{J}_{b-}^{r,k;\psi}f(t) = \mathcal{J}_{b-}^{\vartheta+r,k;\psi}f(t) = \mathcal{J}_{b-}^{r,k;\psi}\mathcal{J}_{b-}^{\vartheta,k;\psi}f(t).$$

*Proof.* By Lemma 1 in [24] and the property of $k$-gamma function, for $\vartheta>0$, $r>0$ and $k>0$, we get

$$\begin{aligned}\mathcal{J}_{a+}^{\vartheta,k;\psi}\mathcal{J}_{a+}^{r,k;\psi}f(t) &= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I_{a+}^{\frac{\vartheta}{k};\psi}I_{a+}^{\frac{r}{k};\psi}f(t)\\ &= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2k^{\frac{\vartheta}{k}-1}\Gamma(\frac{\vartheta}{k})k^{\frac{r}{k}-1}\Gamma(\frac{r}{k})}I_{a+}^{\frac{\vartheta}{k};\psi}I_{a+}^{\frac{r}{k};\psi}f(t)\\ &= \frac{1}{k^{\frac{\vartheta+r}{k}}}I_{a+}^{\frac{\vartheta+r}{k};\psi}f(t)\\ &= \mathcal{J}_{a+}^{\vartheta+r,k;\psi}f(t),\end{aligned}$$

where $I_{a+}^{\vartheta;\psi}$ is the fractional integral defined in [24], we have also,

$$\begin{aligned}\mathcal{J}_{a+}^{\vartheta,k;\psi}\mathcal{J}_{a+}^{r,k;\psi}f(t) &= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I_{a+}^{\frac{\vartheta}{k};\psi}I_{a+}^{\frac{r}{k};\psi}f(t)\\ &= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I_{a+}^{\frac{r}{k};\psi}I_{a+}^{\frac{\vartheta}{k};\psi}f(t)\\ &= \mathcal{J}_{a+}^{r,k;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}f(t).\end{aligned}$$

□

**Lemma 2.6** *Let $\vartheta,r>0$ and $k>0$. Then, we have*

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(t,a)$$

*and*

$$\mathcal{J}_{b-}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(b,t) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(b,t).$$

*Proof.* By Definition 2.2 and using the change of variable

$$\mu = \frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}, \;\; t > a,$$

we get

$$\begin{aligned}\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) &= \int_a^t \bar{\Psi}_\vartheta^{k,\psi}(t,s)\psi'(s)\bar{\Psi}_r^{k,\psi}(s,a)\,\mathrm{d}s \\ &= \int_a^t \bar{\Psi}_\vartheta^{k,\psi}(t,a)\left[1-\frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}\right]^{\frac{\vartheta}{k}-1}\psi'(s)\bar{\Psi}_r^{k,\psi}(s,a)\,\mathrm{d}s \\ &= \bar{\Psi}_\vartheta^{k,\psi}(t,a)\bar{\Psi}_r^{k,\psi}(t,a)\int_0^1 (1-\mu)^{\frac{\vartheta}{k}-1}\mu^{\frac{r}{k}-1}\,\mathrm{d}\mu.\end{aligned}$$

Using Definition 2.1 of $k$-beta function and the relation with gamma function, we have

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(t,a).$$

□

**Theorem 2.7** *Let* $0 < a < b < \infty, \vartheta > 0, 0 \leq \xi < 1$*,* $k > 0$ *and* $x \in C_{\xi,k;\psi}(J)$*. If* $\frac{\vartheta}{k} > 1-\xi$*, then*

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(a) = \lim_{t\to a^+}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = 0.$$

*Proof.* $x \in C_{\xi;\psi}(J)$ means that $\Psi_\xi^\psi(t,a)x(t) \in C(J,E)$, then there exists a positive constant $R$ such that for $t \in (a,b]$ we have

$$\|\Psi_\xi^\psi(t,a)x(t)\| < R,$$

thus,

$$\|x(t)\| < R\Gamma_k(k\xi)|\bar{\Psi}_{k\xi}^{k,\psi}(t,a)|. \tag{2.1}$$

Now, we apply the operator $\mathcal{J}_{a+}^{\vartheta,k;\psi}(\cdot)$ on both sides of equation (2.1) and using Lemma 2.6, so that we have

$$\begin{aligned}\left\|\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t)\right\| &< R\Gamma_k(k\xi)\left|\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_{k\xi}^{k,\psi}(t,a)\right| \\ &= R\Gamma_k(k\xi)\bar{\Psi}_{\vartheta+k\xi}^{k,\psi}(t,a).\end{aligned}$$

Then, we have the right-hand side $\to 0$ as $x \to a$, and

$$\lim_{t\to a^+}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(a) = 0.$$

□

We are now able to define the $k$-generalized $\psi$-Hilfer derivative as follows.

**Definition 2.8** ($k$-*Generalized* $\psi$-*Hilfer Derivative*) *Let* $n-1<\frac{\vartheta}{k}\leq n$ *with* $n\in\mathbb{N}$, $J=[a,b]$ *an interval such that* $-\infty\leq a<b\leq\infty$, $\psi$ *is an increasing function and* $\psi'(t)\neq 0$, *for all* $t\in J$. *The* $k$-*generalized* $\psi$-*Hilfer fractional derivatives* (*left-sided and right-sided*) ${}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+}(\cdot)$ *and* ${}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{b-}(\cdot)$ *of a function* $g$ *of order* $\vartheta$ *and type* $0\leq r\leq 1$, *with* $k>0$ *are defined by*

$$\begin{aligned}{}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+}g\left(t\right)&=\left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+}\left(\frac{1}{\psi'\left(t\right)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\left(k^{n}\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+}g\right)\right)(t)\\&=\left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+}\delta^{n}_{\psi}\left(k^{n}\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+}g\right)\right)(t)\end{aligned}$$

*and*

$$\begin{aligned}{}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{b-}g\left(t\right)&=\left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{b-}\left(-\frac{1}{\psi'\left(t\right)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\left(k^{n}\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{b-}g\right)\right)(t)\\&=\left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{b-}(-1)^{n}\delta^{n}_{\psi}\left(k^{n}\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{b-}g\right)\right)(t)\,,\end{aligned}$$

*where* $\delta^{n}_{\psi}=\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}$.

**Lemma 2.9** *Let* $t>a$, $\vartheta>0, 0\leq r\leq 1, k>0$. *Then for* $0<\xi<1;\xi=\frac{1}{k}(r(k-\vartheta)+\vartheta)$, *we have*

$$\left[{}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+}\left(\Psi^{\psi}_{\xi}(s,a)\right)^{-1}\right](t)=0.$$

*Proof.* From Definitions 2.2 and 2.8, we have

$$\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+}k\left(\Psi^{\psi}_{\xi}(t,a)\right)^{-1}=\int_{a}^{t}k\bar{\Psi}^{k,\psi}_{kX}(t,s)\left(\Psi^{\psi}_{\xi}(s,a)\right)^{-1}\psi'(s)\,\mathrm{d}s,$$

where $X=\frac{1}{k}\left(1-r\right)\left(k-\vartheta\right)$. Now, we make the change of the variable by $\mu=\frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$ to obtain

$$\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+}k\left(\Psi^{\psi}_{\xi}(t,a)\right)^{-1}=\frac{k\left(\Psi^{\psi}_{\xi+X}(t,a)\right)^{-1}}{\Gamma_{k}(kX)}\left[\frac{1}{k}\int_{0}^{1}(1-\mu)^{X-1}\mu^{\xi-1}\,\mathrm{d}\mu\right],$$

then, by the definition of $k$-beta function

$$B_{k}(\alpha,\beta)=\frac{1}{k}\int_{0}^{1}t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\,\mathrm{d}t=\frac{\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}{\Gamma_{k}(\alpha+\beta)},$$

we have

$$\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+}k\left(\Psi^{\psi}_{\xi}(t,a)\right)^{-1}=\frac{k\Gamma_{k}(k\xi)}{\Gamma_{k}(k(X+\xi))}=k\Gamma_{k}(k\xi),$$

then, we have

$$\delta^{1}_{\psi}\left(\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+}k\left(\Psi^{\psi}_{\xi}(t,a)\right)^{-1}\right)=0.$$

□

**Theorem 2.10** *If $f \in C^n_{\xi;\psi}[a,b], n-1 < \frac{\vartheta}{k} < n$, $0 \le r \le 1$, where $n \in \mathbb{N}$ and $k > 0$, then*

$$\left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, {}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+} f\right)(t) = f(t) - \sum_{i=1}^{n} \frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))} \left\{\delta^{n-i}_{\psi}\left(\mathcal{J}^{k(n-\xi),k;\psi}_{a+} f(a)\right)\right\},$$

*where*

$$\xi = \frac{1}{k}\left(r(kn-\vartheta)+\vartheta\right).$$

*In particular, if $n = 1$, we have*

$$\left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, {}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+} f\right)(t) = f(t) - \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(r(k-\vartheta)+\vartheta)} \mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+} f(a).$$

*Proof.* From Definition 2.8 and Lemma 2.5, we have

$$\begin{aligned}
\left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, {}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+} f\right)(t) &= \left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, \mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+} \delta^n_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f\right)\right)(t) \\
&= \left(\mathcal{J}^{r(kn-\vartheta)+\vartheta,k;\psi}_{a+} \delta^n_{\psi}\left(k^n I^{(1-r)(kn-\vartheta),k;\psi}_{a+} f\right)\right)(t) \\
&= \int_a^t \bar{\Psi}^{k,\psi}_{k\xi}(t,s)\psi'(s)\left\{\delta^n_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(s)\right)\right\} \mathrm{d}s.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
\left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, {}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+} f\right)(t) = \; & \frac{-(\psi(t)-\psi(a))^{\xi-1}}{k\Gamma_k(k\xi)} \left\{\delta^{n-1}_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(a)\right)\right\} \\
& + \frac{\xi-1}{k\Gamma_k(k\xi)} \int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{2-\xi}} \left\{\delta^{n-1}_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(s)\right)\right\} \mathrm{d}s.
\end{aligned}$$

Using the propriety of the functions gamma and $k$-gamma, we get

$$\begin{aligned}
\left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, {}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+} f\right)(t) = \; & \frac{-(\psi(t)-\psi(a))^{\xi-1}}{k^{\xi}\Gamma(\xi)} \left\{\delta^{n-1}_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(a)\right)\right\} \\
& + \frac{1}{k^{\xi}\Gamma(\xi-1)} \int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{2-\xi}} \left\{\delta^{n-1}_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(s)\right)\right\} \mathrm{d}s.
\end{aligned}$$

So, with integrating by parts $n$ times, we obtain

$$\begin{aligned}
\left(\mathcal{J}^{\vartheta,k;\psi}_{a+} \, {}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+} f\right)(t) = & -\sum_{i=1}^{n} \frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{\xi}\Gamma(\xi-i+1)} \left\{\delta^{n-i}_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(a)\right)\right\} \\
& + \frac{1}{k^{\xi-n}\Gamma(\xi-n)} \int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{n+1-\xi}} \left(\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(s)\right) \mathrm{d}s, \\
= & -\sum_{i=1}^{n} \frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i}\Gamma_k(k(\xi-i+1))} \left\{\delta^{n-i}_{\psi}\left(k^n \mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(a)\right)\right\} \\
& + \frac{1}{k\Gamma_k(k(\xi-n))} \int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{n+1-\xi}} \left(\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(s)\right) \mathrm{d}s, \\
= & -\sum_{i=1}^{n} \frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))} \left\{\delta^{n-i}_{\psi}\left(\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(a)\right)\right\} \\
& + \mathcal{J}^{k(\xi-n),k;\psi}_{a+} I^{(1-r)(kn-\vartheta),k;\psi}_{a+} f(t),
\end{aligned}$$

then by using Lemma 2.5, we get

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi} f\right)(t) = f(t) - \sum_{i=1}^{n} \frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))} \left\{\delta_{\psi}^{n-i}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi} f(a)\right)\right\}.$$

□

**Lemma 2.11** *Let $\vartheta > 0, 0 \le r \le 1$, and $x \in C^1_{\xi;\psi}(J)$, where $k > 0$, then for $t \in (a,b]$, we have*

$$\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi} x\right)(t) = x(t).$$

*Proof.* We have from Definition 2.8 and Lemma 2.5 that $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$

$$\begin{aligned}\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi} x\right)(t) &= \left(\mathcal{J}_{a+}^{r(k-\vartheta),k;\psi}\delta_{\psi}^{1}\left(k\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)\right)(t)\\ &= \left(\mathcal{J}_{a+}^{k\xi-\vartheta,k;\psi}\delta_{\psi}^{1}\left(k\mathcal{J}_{a+}^{(1-r)(k-\vartheta)+\vartheta,k;\psi}x\right)\right)(t)\\ &= \left(\mathcal{J}_{a+}^{k\xi-\vartheta,k;\psi}\delta_{\psi}^{1}\left(k\mathcal{J}_{a+}^{k-k\xi+\vartheta,k;\psi}x\right)\right)(t),\end{aligned}$$

then, we obtain

$$\begin{aligned}&\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi} x\right)(t)\\ &= \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}}\delta_{\psi}^{1}\left[\int_a^s \frac{\psi'(\tau)x(\tau)\,\mathrm{d}\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}}\right]\mathrm{d}s.\end{aligned} \tag{2.2}$$

On other hand by integrating by parts, we have

$$\int_a^s \frac{\psi'(\tau)x(\tau)\,\mathrm{d}\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}} = \frac{1}{1-\xi+\frac{\vartheta}{k}}\left[x(a)\left(\psi(s)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}} + \int_a^s \frac{x'(\tau)\,\mathrm{d}\tau}{(\psi(s)-\psi(\tau))^{\xi-1-\frac{\vartheta}{k}}}\right],$$

then, by applying $\delta_{\psi}^{1}$ we get

$$\delta_{\psi}^{1}\int_a^s \frac{\psi'(\tau)x(\tau)\,\mathrm{d}\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}} = x(a)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}} + \int_a^s \frac{x'(\tau)\,\mathrm{d}\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}}. \tag{2.3}$$

Now, replacing (2.3) into equation (2.2), and by Dirichlet's formula and the properties of $k$-gamma function, we get

$$\begin{aligned}\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi} x\right)(t) = \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}&\left[\int_a^t \frac{x(a)\psi'(s)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}}\ \mathrm{d}s}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}}\right.\\ &\left.+\int_a^t x'(t)\,\mathrm{d}t\int_s^t \frac{\psi'(s)\,\mathrm{d}\tau}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}\left(\psi(s)-\psi(\tau)\right)^{\xi-\frac{\vartheta}{k}}}\right].\end{aligned}$$

Making the following change of variable $\mu = \frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$ in the integral from $a$ to $t$ and similarly changing the variable in the integral from $s$ to $t$, then we have

$$\begin{aligned}
&\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t)\\
&= \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\int_a^t x(a)\psi'(s)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}}\left(\psi(t)-\psi(s)\right)^{\xi-\frac{\vartheta}{k}-1}\mathrm{d}s\right.\\
&\left.+\int_a^t x'(t)\,\mathrm{d}t\int_s^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\xi-\frac{\vartheta}{k}-1}\left(\psi(s)-\psi(\tau)\right)^{-\xi+\frac{\vartheta}{k}}\mathrm{d}\tau\right]\\
&= \frac{1}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\frac{1}{k}\int_0^1 \mu^{-\xi+\frac{\vartheta}{k}}(1-\mu)^{\xi-\frac{\vartheta}{k}-1}\,\mathrm{d}\mu\right]\left(x(a)+\int_a^t x'(t)\,\mathrm{d}t\right)\\
&= \frac{1}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\frac{1}{k}\int_0^1 \mu^{(1-(\xi-\frac{\vartheta}{k}))-1}(1-\mu)^{\xi-\frac{\vartheta}{k}-1}\,\mathrm{d}\mu\right]\left(x(a)+\int_a^t x'(t)\,\mathrm{d}t\right),
\end{aligned}$$

then by the definition of $k$-beta function, we obtain

$$\begin{aligned}
\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) &= \frac{[\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)]}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left(x(a)+\int_a^t x'(t)\,\mathrm{d}t\right)\\
&= x(a)+\int_a^t x'(t)\,\mathrm{d}t\\
&= x(t).
\end{aligned}$$

□

**Definition 2.12** [5] *Let $X$ be a Banach space and let $\Omega_X$ be the family of bounded subsets of $X$. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \longrightarrow [0, \infty)$ defined by*

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, \mathrm{diam}(M_j) \leq \epsilon\},$$

*where $M \in \Omega_X$. The map $\mu$ satisfies the following Properties :*

- $\mu(M) = 0 \Leftrightarrow \overline{M}$ *is compact ($M$ is relatively compact).*
- $\mu(M) = \mu(\overline{M})$.
- $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
- $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.
- $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
- $\mu(convM) = \mu(M)$.

**Theorem 2.13** [15, Mönch's fixed point Theorem] *Let $D$ be closed, bounded and convex subset of a Banach space $X$ such that $0 \in D$, and let $T$ be a continuous mapping of $D$ into itself. If the implication*

$$V = \overline{conv}T(V), \ \ or \ \ V = T(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{2.4}$$

*holds for every subset $V$ of $D$, then $T$ has a fixed point.*

**Lemma 2.14** *Let the function $\varphi(\cdot) \in C(J,E)$. Then $x \in C_{\xi;\psi}(J_i)$ is a solution of the differential equation:*

$$\left({}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi} x\right)(t) = \varphi(t), \;\; t \in J_i, \; i = 0,\ldots,m, \; 0 < \vartheta < k, \; 0 \leq r \leq 1, \tag{2.5}$$

*if and only if $x$ satisfies the following Volterra integral equation:*

$$x(t) = \frac{\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi} x(t_i)}{\Psi_{\xi}^{\psi}(t,t_i)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t), \tag{2.6}$$

*where $\xi = \frac{r(k-\vartheta)+\vartheta}{k}$, $k > 0$.*

*Proof.* By applying the fractional integral operator $\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}(\cdot)$ on both sides of the fractional equation (2.5) and using Theorem 2.10, we obtain the equation (2.6).

Now, applying the fractional derivative operator ${}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi}(\cdot)$ on both sides of the fractional equation (2.6), then we get

$$\left({}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi} x\right)(t) = {}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi}\left(\frac{\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi} x(t_i)}{\Psi_{\xi}^{\psi}(t,t_i)\Gamma_k(k\xi)}\right) + \left({}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi}\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t).$$

Using Lemma 2.9 and Lemma 2.11, we obtain equation (2.5). □

# 3 Existence of Solutions

We consider the following fractional differential equation

$$\left({}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi} x\right)(t) = \varphi(t), \;\; t \in J_i, \; i = 0,\ldots,m, \tag{3.1}$$

where $0 < \vartheta < k, 0 \leq r \leq 1$, with the conditions

$$\left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi} x\right)(t_i^+) = \left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi),k;\psi} x\right)(t_i^-) + L_i(x(t_i^-)); i = 1,\ldots,m, \tag{3.2}$$

$$\alpha_1\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi} x\right)(a^+) + \alpha_2\left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi} x\right)(b) = \alpha_3, \tag{3.3}$$

$$x(t) = \varpi(t), \;\; t \in [a-\lambda, a], \; \lambda > 0, \tag{3.4}$$

$$x(t) = \tilde{\varpi}(t), \;\; t \in \left[b, b+\tilde{\lambda}\right], \; \tilde{\lambda} > 0, \tag{3.5}$$

where $\xi = \frac{r(k-\vartheta)+\vartheta}{k}$, $k > 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$, $\alpha_3 \in E$, $\varphi(\cdot) \in C(J,E)$, $\varpi(\cdot) \in \mathcal{C}$ and $\tilde{\varpi}(\cdot) \in \tilde{\mathcal{C}}$.

The following theorem shows that the problem (3.1)-(3.5) have a unique solution.

**Theorem 3.1** *The function $x(\cdot)$ satisfies (3.1)-(3.5) if and only if it satisfies*

$$x(t)=\begin{cases}\dfrac{1}{\Gamma_k(k\xi)\Psi_\xi^{\psi}(t,a)}\left[\dfrac{\alpha_3}{\alpha_1+\alpha_2}-\dfrac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-))\right.\\ \left.-\dfrac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\right]+\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t), \quad \text{if } t\in J_0,\\ \dfrac{1}{\Psi_\xi^{\psi}(t,t_i)\Gamma_k(k\xi)}\left[\dfrac{\alpha_3}{\alpha_1+\alpha_2}-\dfrac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-))\right.\\ -\dfrac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)+\sum_{j=1}^{i}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\ \left.+\sum_{j=1}^{i}L_j(x(t_j^-))\right]+\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t), \quad t\in J_i; i=1,\ldots,m,\\ \varpi(t), \ \ t\in[a-\lambda,a],\\ \tilde{\varpi}(t), \ \ t\in\left[b,b+\tilde{\lambda}\right].\end{cases} \tag{3.6}$$

*Proof.* Assume $x$ satisfies (3.1)-(3.5). If $t\in J_0$, then

$$\left({}_k^H\mathcal{D}_{a^+}^{\vartheta,r;\psi}x\right)(t)=\varphi(t),$$

Lemma 2.14 implies that the solution can be written as

$$x(t)=\frac{\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x(a)}{\Psi_\xi^{\psi}(t,a)\Gamma_k(k\xi)}+\left(\mathcal{J}_{a^+}^{\vartheta,k;\psi}\varphi\right)(t). \tag{3.7}$$

If $t\in J_1$, then Lemma 2.14 implies

$$\begin{aligned}x(t)&=\frac{\mathcal{J}_{t_1^+}^{k(1-\xi),k;\psi}x(t_1)}{\Psi_\xi^{\psi}(t,t_1)\Gamma_k(k\xi)}+\left(\mathcal{J}_{t_1^+}^{\vartheta,k;\psi}\varphi\right)(t)\\ &=\frac{\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(t_1^-)+L_1(x(t_1^-))}{\Psi_\xi^{\psi}(t,t_1)\Gamma_k(k\xi)}+\left(\mathcal{J}_{t_1^+}^{\vartheta,k;\psi}\varphi\right)(t)\\ &=\frac{\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a)+\left(\mathcal{J}_{a^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_1)+L_1(x(t_1^-))}{\Psi_\xi^{\psi}(t,t_1)\Gamma_k(k\xi)}+\left(\mathcal{J}_{t_1^+}^{\vartheta,k;\psi}\varphi\right)(t).\end{aligned}$$

If $t \in J_2$, then Lemma 2.14 implies

$$\begin{aligned} x(t) &= \frac{\mathcal{J}_{t_2^+}^{k(1-\xi),k;\psi} x(t_2)}{\Psi_\xi^\psi(t,t_2)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_2^+}^{\vartheta,k;\psi}\varphi\right)(t) \\ &= \frac{\left(\mathcal{J}_{t_1^+}^{k(1-\xi),k;\psi} x\right)(t_2^-) + L_2(x(t_2^-))}{\Psi_\xi^\psi(t,t_2)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_2^+}^{\vartheta,k;\psi}\varphi\right)(t) \\ &= \frac{1}{\Psi_\xi^\psi(t,t_2)\Gamma_k(k\xi)}\Bigg[\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi} x\right)(a) + \left(\mathcal{J}_{t_1^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_2) + L_1(x(t_1^-)) \\ &\quad + \left(\mathcal{J}_{a^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_1) + L_2(x(t_2^-))\Bigg] + \left(\mathcal{J}_{t_2^+}^{\vartheta,k;\psi}\varphi\right)(t). \end{aligned}$$

Repeating the process in this way, the solution $x(t)$ for $t \in J_i$,$i = 1, \ldots, m$, can be written as

$$\begin{aligned} x(t) &= \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\Bigg[\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi} x\right)(a) + \sum_{j=1}^{i} L_j(x(t_j^-)) \\ &\quad + \sum_{j=1}^{i}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\Bigg] + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t). \end{aligned} \tag{3.8}$$

Applying $\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}$ on both sides of $(3.8)$, using Lemma 2.6 and taking $t = b$, we obtain

$$\begin{aligned} \left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi} x\right)(b) &= \left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi} x\right)(a) + \sum_{j=1}^{m} L_j(x(t_j^-)) \\ &\quad + \sum_{j=1}^{m}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j) + \left(\mathcal{J}_{t_m^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b). \end{aligned} \tag{3.9}$$

Multiplying both sides of $(3.9)$ by $\alpha_2$ and using condition $(3.3)$, we obtain

$$\begin{aligned} \alpha_3 - \alpha_1\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi} x\right)(a) &= \alpha_2\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi} x\right)(a) + \alpha_2\sum_{j=1}^{m} L_j(x(t_j^-)) \\ &\quad + \alpha_2\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j), \end{aligned}$$

which implies that

$$\begin{aligned} &\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi} x\right)(a) \\ &= \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m} L_j(x(t_j^-)) - \frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j). \end{aligned} \tag{3.10}$$

Substituting $(3.10)$ into $(3.8)$ and $(3.7)$ we obtain $(3.6)$.

Reciprocally, applying $\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}$ on both sides of (3.6) and using Lemma 2.6 and Lemma 2.5, we get

$$\left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}x\right)(t)=\begin{cases}\frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\ -\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-))+\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t),\quad \text{if } t\in J_0,\\ \frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-))+\sum_{j=1}^{i}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\ -\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)+\sum_{j=1}^{i}L_j(x(t_j^-))\\ +\left(\mathcal{J}_{t_i^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t),\quad t\in J_i;i=1,\dots,m.\end{cases} \tag{3.11}$$

Next, taking the limit $t\to a^+$ of (3.11) and using Theorem 2.7, with $k(1-\xi)<k(1-\xi)+\vartheta$, we obtain

$$\begin{aligned}\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a^+)&=\frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\&\quad-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-)).\end{aligned} \tag{3.12}$$

Now, taking $t=b$ in (3.11), we get

$$\begin{aligned}&\left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}x\right)(b)\\&=\frac{\alpha_3}{\alpha_1+\alpha_2}+\left(1-\frac{\alpha_2}{\alpha_1+\alpha_2}\right)\left(\sum_{j=1}^{m}L_j(x(t_j^-))+\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\right).\end{aligned} \tag{3.13}$$

From $(3.12)$ and $(3.13)$, we find that

$$\alpha_1\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+)+\alpha_2\left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}x\right)(b)=\alpha_3,$$

which shows that the boundary condition $(3.3)$ is satisfied. Next, apply operator ${}_k^H\mathcal{D}_{t_i^+}^{\vartheta,r;\psi}(\cdot)$ on both sides of $(3.6)$, where $i=0,\dots,m$. Then, from Lemma 2.9 and Lemma 2.11 we obtain equation $(3.1)$. Also, we can easily show that $x$ satisfies the equations $(3.2)$, $(3.4)$ and $(3.5)$. This completes the proof. □

As a consequence of Theorem 3.1, we have the following result.

**Lemma 3.2** *Let $\xi=\frac{r(k-\vartheta)+\vartheta}{k}$ where $0<\vartheta<k$ and $0\le r\le 1$, let $f:J\times PC_{\xi;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)\times E\to E$ be a continuous function, $\varpi(\cdot)\in\mathcal{C}$ and $\tilde{\varpi}(\cdot)\in\tilde{\mathcal{C}}$, then $x\in\mathbb{F}$ satisfies the problem*

(1.1)-(1.5) *if and only if $x$ is the fixed point of the operator $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ defined by*

$$(\mathcal{T}x)(t) = \begin{cases} \dfrac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)} \left[ \dfrac{\alpha_3}{\alpha_1+\alpha_2} - \dfrac{\alpha_2}{\alpha_1+\alpha_2} \sum_{j=1}^{m} L_j(x(t_j^-)) \right. \\ -\dfrac{\alpha_2}{\alpha_1+\alpha_2} \sum_{j=1}^{m+1} \left( \mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi} \varphi \right)(t_j) + \sum_{a<t_i<t} \left( \mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi} \varphi \right)(t_i) \\ \left. + \sum_{a<t_i<t} L_i(x(t_i^-)) \right] + \left( \mathcal{J}_{t_i^+}^{\vartheta,k;\psi} \varphi \right)(t), \quad t \in J_i; i = 0,\dots,m, \\ \varpi(t), \ \ t \in [a-\lambda, a], \\ \tilde{\varpi}(t), \ \ t \in \left[ b, b+\tilde{\lambda} \right], \end{cases} \tag{3.14}$$

*where $\varphi$ is a function satisfying the functional equation*

$$\varphi(t) = f(t, x^t(\cdot), \varphi(t)).$$

*By Theorem* 2.4*, we have $\mathcal{T}x \in \mathbb{F}$.*

The following hypotheses will be used in the sequel :

(*Ax1*) The function $f : J \times PC_{\xi;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right) \times E \to E$ is continuous.

(*Ax2*) There exist constants $\zeta_1 > 0$ and $0 < \zeta_2 < 1$ such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le \zeta_1 \|x_1 - x_2\|_{[-\lambda,\tilde{\lambda}]} + \zeta_2 \|y_1 - y_2\|$$

and

$$\|L_i(y_1) - L_i(y_2)\| \le \ell_1 \Psi_\xi^\psi(t_i, t_{i-1}) \|y_1 - y_2\|$$

for any $x_1, x_2 \in PC_{\xi;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right)$, $y_1, y_2 \in E$ and $t \in J$.

(*Ax3*) For each bounded sets $\beta_1 \in PC_{\xi;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right)$ and $\beta_2 \in E$ and for each $t \in J$, we have

$$\mu\left(f(t,\beta_1,\beta_2)\right) \le \zeta_1 \mu(\beta_1) + \zeta_2 \mu(\beta_2)$$

and

$$\mu\left(L_i(\beta_2)\right) \le \ell_1 \Psi_\xi^\psi(t_i, t_{i-1}) \mu(\beta_2),$$

where $i = 1, \dots, m$.

**Remark 3.3** [4] *It is worth noting that the hypotheses* (Ax2) *and* (Ax3) *are equivalent.*

We are now in a position to state and prove our existence result for the problem (1.1)-(1.5) based on Mönch fixed point theorem.

**Theorem 3.4** *Assume* (Ax1) *and* (Ax2) *hold. If*

$$\mathcal{L} = \frac{1}{\Gamma_k(k\xi)}\left[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\ell_1+\frac{m\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(2k-k\xi+\vartheta)}\right)\right.$$
$$\left.+\frac{\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\right] < 1, \qquad (3.15)$$

*then the problem* (1.1)-(1.5) *has at least one solution in* $\mathbb{F}$.

*Proof.* The proof will be given in several steps.

**Step 1:** We show that the operator $\mathcal{T}$ defined in (3.14), transforms the ball $B_M = B(0,M) = \{x \in \mathbb{F} : \|x\|_{\mathbb{F}} \le M\}$ into itself.
Let $M$ be a positive constant such that

$$M \ge \max\left\{\frac{\|\alpha_3\|+\tilde{\ell}|\alpha_1+\alpha_2|\Gamma_k(k\xi)}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\,(1-\ell)}, \|\varpi\|_{\mathcal{C}}, \|\tilde{\varpi}\|_{\tilde{\mathcal{C}}}\right\},$$

where

$$\tilde{\ell} := \frac{1}{\Gamma_k(k\xi)}\left[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\tilde{L}+\frac{mf^*\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(2k-k\xi+\vartheta)}\right)\right.$$
$$\left.+\frac{f^*\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\right],$$

$$\tilde{L} = \max_{1\le i\le m}\{\sup\{\|L_i(x)\|, x \in E\}\},$$

and

$$f^* = \sup_{t\in J}\|f(t,0,0)\|.$$

For each $t \in [a-\lambda, a]$, we have

$$\|\mathcal{T}x(t)\| \le \|\varpi\|_{\mathcal{C}},$$

and for each $t \in \left[b, b+\tilde{\lambda}\right]$, we have

$$\|\mathcal{T}x(t)\| \le \|\tilde{\varpi}\|_{\tilde{\mathcal{C}}}.$$

Further, for each $t \in (a,b]$, (3.14) implies that

$$\|\mathcal{T}x(t)\| \le \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{\|\alpha_3\|}{|\alpha_1+\alpha_2|}+\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m}\|L_j(x(t_j^-))\|\right.$$
$$+\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi(s)\|\right)(t_j)+\sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi(s)\|\right)(t_i)$$
$$\left.+\sum_{a<t_i<t}\|L_i(x(t_i^-))\|\right]+\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\|\varphi(s)\|\right)(t). \qquad (3.16)$$

By the hypothesis ($Ax2$), for $t \in (a, b]$, we have

$$\|\varphi(t)\| = \|f(t, x^t, \varphi(t)) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \leq \zeta_1 \|x^t\|_{[-\lambda, \tilde{\lambda}]} + \zeta_2 \|\varphi(t)\| + f^*,$$

which implies that

$$\|\varphi(t)\| \leq \zeta_1 M + \zeta_2 \|\varphi(t)\| + f^*,$$

then

$$\|\varphi(t)\| \leq \frac{f^* + \zeta_1 M}{1 - \zeta_2} := \Delta.$$

Thus for $t \in (a, b]$, by hypothesis ($Ax2$) and from (3.16) we get

$$\begin{aligned}
&\|\Psi_\xi^\psi(t, t_i) \mathcal{T} x(t)\| \\
&\leq \frac{1}{\Gamma_k(k\xi)} \left[ \frac{\|\alpha_3\|}{|\alpha_1 + \alpha_2|} + \left( \frac{|\alpha_2|}{|\alpha_1 + \alpha_2|} + 1 \right) \left( \sum_{j=1}^{m} \|L_j(x(t_j^-)) - L_j(0)\| + \sum_{j=1}^{m} \|L_j(0)\| \right) \right. \\
&\left. + \frac{\Delta |\alpha_2|}{|\alpha_1 + \alpha_2|} \sum_{j=1}^{m+1} \left( \mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta, k; \psi}(1) \right)(t_j) + \Delta \sum_{j=1}^{m} \left( \mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta, k; \psi}(1) \right)(t_j) \right] \\
&+ \Delta \Psi_\xi^\psi(t, t_i) \left( \mathcal{J}_{t_i^+}^{\vartheta, k; \psi}(1) \right)(t).
\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}
\|\Psi_\xi^\psi(t, t_i) \mathcal{T} x(t)\| \leq \frac{1}{\Gamma_k(k\xi)} &\left[ \frac{\|\alpha_3\|}{|\alpha_1 + \alpha_2|} + \frac{m(\tilde{L} + \ell_1 M)|\alpha_2|}{|\alpha_1 + \alpha_2|} + m(\tilde{L} + \ell_1 M) \right. \\
&+ \frac{(m+1)\Delta |\alpha_2| \left(\psi(b) - \psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{|\alpha_1 + \alpha_2| \Gamma_k(k(1-\xi) + \vartheta + k)} \\
&\left. + \frac{m\Delta \left(\psi(b) - \psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(k(1-\xi) + \vartheta + k)} \right] + \frac{\Delta \left(\psi(t) - \psi(t_i)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\Psi_\xi^\psi(t, t_i) \mathcal{T} x(t)\| \leq \frac{1}{\Gamma_k(k\xi)} &\left[ \left( \frac{|\alpha_2|}{|\alpha_1 + \alpha_2|} + 1 \right) \left( m(\tilde{L} + \ell_1 M) + \frac{m\Delta \left(\psi(b) - \psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(2k - k\xi + \vartheta)} \right) \right. \\
&\left. + \Delta \left(\psi(b) - \psi(a)\right)^{1-\xi+\frac{\vartheta}{k}} \left( \frac{|\alpha_2|}{|\alpha_1 + \alpha_2| \Gamma_k(2k - k\xi + \vartheta)} + \frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta + k)} \right) \right] \\
&+ \frac{\|\alpha_3\|}{|\alpha_1 + \alpha_2| \Gamma_k(k\xi)} \\
&\leq M.
\end{aligned}$$

Then, for each $t \in \left[a - \lambda, b + \tilde{\lambda}\right]$ we obtain

$$\|\mathcal{T} x\|_{\mathbb{F}} \leq M.$$

**Step 2:** $\mathcal{T}: B_M \to B_M$ is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathbb{F}$. For each $t \in [a-\lambda, a] \cup \left[b, b+\tilde{\lambda}\right]$, we have

$$\|\mathcal{T}x_n(t) - \mathcal{T}x(t)\| \longrightarrow 0.$$

And for $t \in (a, b]$, we have

$$\begin{aligned}
&\|\mathcal{T}x_n(t) - \mathcal{T}x(t)\| \\
&\leq \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m}\|L_j(x_n(t_j^-)) - L_j(x(t_j^-))\|\right. \\
&+ \frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi_n(s) - \varphi(s)\|\right)(t_j) + \sum_{a<t_i<t}\|L_i(x_n(t_i^-)) - L_i(x(t_i^-))\| \\
&\left.+ \sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi_n(s) - \varphi(s)\|\right)(t_i)\right] + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\|\varphi_n(s) - \varphi(s)\|\right)(t),
\end{aligned}$$

where $\varphi$ and $\varphi_n$ are functions satisfying the functional equations

$$\begin{aligned}
\varphi(t) &= f(t, x^t(\cdot), \varphi(t)), \\
\varphi_n(t) &= f(t, x_n^t(\cdot), \varphi_n(t)).
\end{aligned}$$

Since $x_n \to x$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in (a, b]$, and since $f$ and $L_i; i = 0, \dots, m$ are continuous, then we have

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{\mathbb{F}} \to 0 \text{ as } n \to \infty.$$

**Step 3:** $\mathcal{T}(B_M)$ is bounded and equicontinuous.
Since $\mathcal{T}(B_M) \subset B_M$ and $B_M$ is bounded , then $\mathcal{T}(B_M)$ is bounded.
Let $\tau_1, \tau_2 \in J_i; i = 0, \dots, m$, $\tau_1 < \tau_2$ and let $x \in B_M$. Then

$$\begin{aligned}
&\left\|\Psi_\xi^\psi(\tau_1,t_i)\mathcal{T}x(\tau_1) - \Psi_\xi^\psi(\tau_2,t_i)\mathcal{T}x(\tau_2)\right\| \\
&\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{\tau_1<t_i<\tau_2}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi(s)\|\right)(t_i) + \sum_{\tau_1<t_i<\tau_2}\|L_i(x(t_i^-))\|\right] \\
&+ \left\|\Psi_\xi^\psi(\tau_1,t_i)\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi(s)\right)(\tau_1) - \Psi_\xi^\psi(\tau_2,t_i)\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi(s)\right)(\tau_2)\right\| \\
&\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{\tau_1<t_i<\tau_2}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi(s)\|\right)(t_i) + \sum_{\tau_1<t_i<\tau_2}\|L_i(x(t_i^-))\|\right] \\
&+ \int_{t_i}^{\tau_1}\left|\Psi_\xi^\psi(\tau_1,t_i)\bar{\Psi}_\vartheta^{k,\psi}(\tau_1,s) - \Psi_\xi^\psi(\tau_2,t_i)\bar{\Psi}_\vartheta^{k,\psi}(\tau_2,s)\right|\|\psi'(s)\varphi(s)\|\,\mathrm{d}s \\
&+ \Psi_\xi^\psi(\tau_2,t_i)\left(\mathcal{J}_{\tau_1^+}^{\vartheta,k;\psi}\|\varphi(s)\|\right)(\tau_2).
\end{aligned}$$

By Lemma 2.6, we get

$$\begin{aligned}
&\left\|\Psi_\xi^\psi(\tau_1,t_i)\mathcal{T}x(\tau_1)-\Psi_\xi^\psi(\tau_2,t_i)\mathcal{T}x(\tau_2)\right\| \\
&\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{\tau_1<t_i<\tau_2}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|\varphi(s)\|\right)(t_i)+\sum_{\tau_1<t_i<\tau_2}\|L_i(x(t_i^-))\|\right] \\
&+\Delta\int_{t_i}^{\tau_1}\left|\Psi_\xi^\psi(\tau_1,t_i)\bar{\Psi}_\vartheta^{k,\psi}(\tau_1,s)-\Psi_\xi^\psi(\tau_2,t_i)\bar{\Psi}_\vartheta^{k,\psi}(\tau_2,s)\right||\psi'(s)|\,\mathrm{d}s \\
&+\frac{\Delta\Psi_\xi^\psi(\tau_2,t_i)\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}.
\end{aligned}$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero. Hence, $\mathcal{T}(B_M)$ is bounded and equicontinuous.

**Step 4:** The implication (2.4) of Theorem 2.13 holds.

Now let $D$ be an equicontinuous subset of $B_M$ such that $D\subset\overline{\mathcal{T}(D)}\cup\{0\}$, therefore the function $t\longrightarrow d(t)=\mu(D(t))$ is continuous on $\left[a-\lambda,b+\tilde{\lambda}\right]$. By *(Ax3)* and the properties of the measure $\mu$, for each $t\in(a,b]$, we have

$$\begin{aligned}
\Psi_\xi^\psi(t,t_i)d(t) &\leq \mu\left(\Psi_\xi^\psi(t,t_i)(\mathcal{T}D)(t)\cup\{0\}\right) \\
&\leq \mu\left(\Psi_\xi^\psi(t,t_i)(\mathcal{T}D)(t)\right) \\
&\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\frac{\zeta_1}{1-\zeta_2}\|d\|_{[-\lambda,\tilde{\lambda}]}\right)(t_i)\right. \\
&\left.+\sum_{a<t_i<t}\ell_1\Psi_\xi^\psi(t_i,t_{i-1})\mu(D(t))\right]+\Psi_\xi^\psi(t,t_i)\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\frac{\zeta_1}{1-\zeta_2}\|d\|_{[-\lambda,\tilde{\lambda}]}\right)(t) \\
&\leq \frac{\|d\|_{\mathbb{F}}}{\Gamma_k(k\xi)}\left[m\ell_1+\frac{\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{1-\zeta_2}\left(\frac{m}{\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\right] \\
&\leq \mathcal{L}\|d\|_{\mathbb{F}},
\end{aligned}$$

and for $t\in[a-\lambda,a]\cup\left[b,b+\tilde{\lambda}\right]$, we have

$$d(t)=\mu(\varpi(t))=\mu(\tilde{\varpi}(t))=0.$$

Thus

$$\|d\|_{\mathbb{F}}\leq\mathcal{L}\|d\|_{\mathbb{F}}.$$

From (3.15), we get $\|d\|_{\mathbb{F}}=0$, that is $d(t)=\mu(D(t))=0$, for each $t\in J$, and then $D(t)$ is relatively compact in $E$. In view of the Ascoli-Arzela Theorem, $D$ is relatively compact in $B_M$. Applying now Theorem 2.13, we conclude that $\mathcal{T}$ has a fixed point, which is a solution to the problem (1.1)-(1.5). □

# 4 An Example

Let

$$E=l^1=\left\{x=(x_1,x_2,\ldots,x_n,\ldots),\sum_{n=1}^{\infty}|x_n|<\infty\right\}$$

be the Banach space with the norm

$$\|x\| = \sum_{n=1}^{\infty} |x_n|.$$

**Example 4.1** *Taking* $r \to 1$, $\vartheta = \frac{1}{2}$, $k = 1$, $J = [1,2]$, $\psi(t) = t$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = 0$, $\lambda = \tilde{\lambda} = 1$ *and* $\xi = 1$, *we obtain an impulsive terminal value problem which is a particular case of problem* (1.1)-(1.5) *with Caputo fractional derivative, given by*

$$\left({}_{1}^{H}\mathcal{D}_{1^+}^{\frac{1}{2},1;\psi} x\right)(t) = \left({}^{C}\mathbb{D}_{1^+}^{\frac{1}{2}} x\right)(t) = f\left(t, x^t(\cdot), \left({}^{C}\mathbb{D}_{1^+}^{\frac{1}{2}} x\right)(t)\right), \quad t \in J_0 \cup J_1, \tag{4.1}$$

$$\left(\mathcal{J}_{2^+}^{0,1;\psi} x\right)(2^+) - \left(\mathcal{J}_{1^+}^{0,1;\psi} x\right)(2^-) = L_1(x(2^-)), \tag{4.2}$$

$$\left(\mathcal{J}_{2^+}^{0,1;\psi} x\right)(3) = x(3) = 0, \tag{4.3}$$

$$x(t) = \varpi(t), \quad t \in [0,1], \tag{4.4}$$

$$x(t) = \tilde{\varpi}(t), \quad t \in [3,4], \tag{4.5}$$

*where* $J_0 = (1,2]$, $J_1 = (2,3]$ *and*

$$x = (x_1, x_2, \ldots, x_n, \ldots),$$

$$f = (f_1, f_2, \ldots, f_n, \ldots),$$

$${}_{1}^{H}\mathcal{D}_{1^+}^{\frac{1}{2},1;\psi} x = \left({}_{1}^{H}\mathcal{D}_{1^+}^{\frac{1}{2},1;\psi} x_1, {}_{1}^{H}\mathcal{D}_{1^+}^{\frac{1}{2},1;\psi} x_2, \ldots, {}_{1}^{H}\mathcal{D}_{1^+}^{\frac{1}{2},1;\psi} x_n, \ldots\right).$$

*Set*

$$f(t, x_1, x_2) = \frac{5 + 3|\cos(t)| + \|x_1\| + \|x_2\|}{205 + 123e^{3-t}},$$

*and*

$$L_1(x_2) = \frac{|\sin(t)| + \|x_2\|}{112e^t},$$

*where* $t \in J$, $x_1 \in PC_{\xi;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right)$ *and* $x_2 \in E$. *Since the function* $f$ *is continuous, then the condition* (Ax1) *is satisfied.*
*For each* $x_1, y_1 \in PC_{\xi;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right)$, $x_2, y_2 \in E$ *and* $t \in J$, *we have*

$$\|f(t, x_1, x_2) - f(t, y_1, y_2)\| \leq \frac{1}{205 + 123e^{3-t}} \left(\|x_1 - y_1\|_{[-\lambda,\tilde{\lambda}]} + \|x_2 - y_2\|\right),$$

*and*

$$\|L_1(x_1) - L_1(y_1)\| \leq \frac{\|x_1 - y_1\|_{[-\lambda,\tilde{\lambda}]}}{112e},$$

*then, the conditions* (Ax2) *and* (Ax3) *are satisfied with*

$$\zeta_1 = \zeta_2 = \frac{1}{328} \text{ and } \ell_1 = \frac{1}{112e}.$$

*Also, we have*

$$\mathcal{L} = \frac{2}{112e} + \frac{8\sqrt{2}}{327\sqrt{\pi}} \approx 0.0260893873144793 < 1.$$

*As all the assumptions of Theorem 3.4 are satisfied, the problem* (4.1)-(4.5) *has at least one solution in* $\mathbb{F}$.

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