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A VARIATIONAL APPROACH TO SOLVING NONLOCAL ELLIPTIC PROBLEMS DRIVEN BY A $p(x)$-BIHARMONIC OPERATOR

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Abstract. This paper analyses nonlocal elliptic problems driven by a $p(x)$-biharmonic operator. The proof of the result is based on variational methods. One example is presented to demonstrate the application of our main result.

Keywords: Nonlocal elliptic system, $p(x)$-biharmonic operator, variational methods.

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1 Introduction

The aim of this paper is to establish the existence of at least one weak solution for the following nonlocal problem

$$\begin{cases} \Delta^2_{p(x)} u - M\left(\int_\Omega \frac{|\nabla u|^{p(x)}\,\mathrm{d}x}{p(x)}\right)\Delta_{p(x)} u + \rho(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ^f}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open bounded domain with smooth boundary, $\Delta^2_{p(x)} u$ is the operator defined as $\Delta(|\Delta u|^{p(x)-2}\Delta u)$, $p(x) \in C(\overline{\Omega})$, $\rho(x) \in L^\infty(\Omega)$, $M\colon [0,+\infty) \to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all

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$t \geq 0$, $\frac{N}{2} \leq p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \infty$, $\lambda > 0$ and $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

Problems like (P_λ^f) are usually called nonlocal problems because of the presence of the integral over the entire domain; this, in particular, implies that the first equation in (P_λ^f) is no longer a pointwise identity. Such kind of problems can be traced back to the work of Kirchhoff. In [27] Kirchhoff proposed the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.1}$$

as an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings. The problem (P_λ^f) is related to the stationary analogue of the problem (1.1). Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process which depends on the average of itself; for example, the population density. In [31] Lions proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions, various equations of Kirchhoff-type have been studied extensively; for instance, see [14, 37, 38].

The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The operator

$$\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$$

is called $p(x)$-biharmonic and becomes p-biharmonic when $p(x) \equiv p$ (a constant). The $p(x)$-biharmonic problem is the general form of the p-biharmonic problem. The $p(x)$-Laplace operator is not homogeneous. The study of various mathematical problems with variable exponent has received considerable attention in recent years. These problems arise in nonlinear elasticity theory, the theory of electrorheological fluids and image processing (see [35, 42]). For background and recent results on $p(x)$-Laplacian and $p(x)$-biharmonic we refer the reader to, respectively, [1, 5, 6, 10, 11, 23, 25] and [13, 16, 17, 21, 28, 29, 33, 39, 40]. For example, in [29] Kong studied the $p(x)$-biharmonic equation with the mountain pass theorem. In [33], Miao obtained the existence of many solutions to the $(p_1(x), \ldots, p_n(x))$-biharmonic problem.

Recently, Kirchhoff-type equations involving the $p(x)$-Laplacian have been investigated; for instance, see [7, 9, 15, 18, 19, 20, 26]. For example, Chung in [7] using the mountain pass theorem combined with the Ekeland variational principle, obtained the existence of at least two distinct non-trivial weak solutions for a class of $p(x)$-Kirchhoff-type equations with combined nonlinearities.

To the best of our knowledge there are a few results concerning the existence and multiplicity of solutions to nonlocal elliptic problems involving the $p(x)$-biharmonic operators. In this regard, in [8] Dai and Hao considered the nonlocal system

$$-M\left(\int_\Omega \frac{|\nabla u|^{p(x)}\,\mathrm{d}x}{p(x)}\right)\Delta_{p(x)}u = \lambda f(x,u) \tag{1.2}$$

with a Dirichlet boundary condition. When the nonlinear term $f(x,u)$ satisfies the Ambrosetti–Rabinowitz condition, using the fountain theorem it can be shown that there are multiple solutions to the problem (1.2). In [32], multiple solutions for the $(p(x), q(x))$ problems of the Kirchhoff-type were obtained using Ricceri's critical point theorem. Very recently, Miao in [34] by means of Ricceri's variational theorem and the definition of a general Lebesgue–Sobolev space obtained sufficient conditions for the existence of infinite solutions for the perturbed problem (P_λ^f). In [24, 30] the authors discussed the existence of solutions for similar problems to (P_λ^f) using variational methods and critical point theory.

Motivated by the above facts, in the present paper we study the existence of at least one non-trivial weak solution for the problem (P_λ^f) under an asymptotical behaviour of the nonlinear datum at zero (see Theorems 3.1). Example 3.2 illustrates Theorem 3.1. We also give some remarks on our results. In Theorem 3.6 we present an application of Theorem 3.1.

Compared to the previous results we give some new assumptions that guarantee the existence of at least one non-trivial weak solution of the problem (P_λ^f). Recent related works are generalized.

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our abstract results.

2 Preliminaries

We shall prove the existence of at least one non-trivial weak solution to the problem (P_λ^f) applying the following version of Ricceri's variational principle (see [36, Theorem 2.1]) as given by Bonanno and Molica Bisci in [4].

Theorem 2.1 *Let X be a reflexive real Banach space and let $\Phi, \Psi \colon X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for every $r > \inf_X \Phi$ let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

We refer the interested reader to the papers [2, 3, 12, 17, 22] in which Theorem 2.1 was successfully employed to prove the existence of at least one non-trivial solution for boundary value problems.

Here and in the sequel, $\operatorname{meas}(\Omega)$ denotes the Lebesgue measure of the set Ω.

Let Ω be a bounded domain of $\mathbb{R}^N$. Set

$$L^{p(x)}(\Omega) := \left\{ u \;\middle|\; u \text{ is measurable and } \int_\Omega |u|^{p(x)}\,\mathrm{d}x < +\infty \right\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \beta > 0 \;\middle|\; \int_\Omega \left|\frac{u}{\beta}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

We define also the generalized Lebesgue–Sobolev space $W^{m,p(x)}(\Omega)$ by putting

$$W^{m,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \;\middle|\; D^{\gamma} u \in L^{p(x)}(\Omega),\ |\gamma| \le m \right\}.$$

It can be equipped with the norm

$$\|u\|_{m,p(x)} := \sum_{|\gamma| \le m} |D^{\gamma} u|_{p(x)}; \tag{2.1}$$

here, γ is the multi-index and $|\gamma|$ is its order. The closure of $C_0^{\infty}(\Omega)$ in $W^{m,p(x)}(\Omega)$ is the space $W_0^{m,p(x)}(\Omega)$. It is well-known (see [11]) that both $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces.

Proposition 2.2 (see [11]) *Suppose that* $\frac{1}{p(x)} + \frac{1}{p^0(x)} = 1$. *Then,* $L^{p^0(x)}(\Omega)$ *and* $L^{p(x)}(\Omega)$ *are conjugate spaces and satisfy the Hölder inequality*

$$\left| \int_{\Omega} uv \,\mathrm{d}x \right| \le \left(\frac{1}{p^-} + \frac{1}{(p^0)^-} \right) |u|_{p(x)} |v|_{p^0(x)}, \quad u \in L^{p(x)}(\Omega),\ v \in L^{p^0(x)}(\Omega).$$

By X we denote the space $W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$ endowed with the norm

$$\|u\| = \inf \left\{ \sigma > 0 \;\middle|\; \int_{\Omega} \left(\left|\frac{u}{\sigma}\right|^{p(x)} + \left|\frac{\nabla u}{\sigma}\right|^{p(x)} + \left|\frac{\Delta u}{\sigma}\right|^{p(x)} \right) \mathrm{d}x \le 1 \right\}.$$

It is known that X is a separable and reflexive Banach space. By [41], $\|\cdot\|$, $\|\cdot\|_{2,p(\cdot)}$ and $\|\Delta u\|_{p(\cdot)}$ are equivalent norms on X.

Proposition 2.3 (see [40]) *When* $p^- > \frac{N}{2}$ *and* $\Omega \subset \mathbb{R}$ *is a bounded region, the embedding* $X \hookrightarrow C(\overline{\Omega})$ *is compact.*

According to Proposition 2.3, there exists a constant $c > 0$ that depends on $p(\cdot)$, N and Ω such that for all $u \in X$ we have

$$\|u\|_{\infty} = \sup_{x \in \Omega} |u(x)| \le c\|u\|. \tag{2.2}$$

Throughout the paper, we also assume that $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, which means that

(a) $t \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$,

(b) $x \mapsto f(x,t)$ is continuous for a.e. $x \in \Omega$,

(c) for every $\varepsilon > 0$ there exists a function $l_{\varepsilon} \in L^1(\Omega)$ such that $\sup_{|t| \le \varepsilon} |f(x,t)| \le l_{\varepsilon}(x)$ for a.e. $x \in \Omega$.

Corresponding to the functions f and M, we introduce the functions $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{M}\colon [0,+\infty) \to \mathbb{R}$, respectively, as follows

$$F(x,t) = \int_0^t f(x,\xi)\,\mathrm{d}\xi \ \text{ for all } (x,t) \in \Omega \times \mathbb{R}$$

and

$$\widetilde{M}(t) = \int_0^t M(\xi)\,\mathrm{d}\xi \ \text{ for all } t \geq 0.$$

We say that $u \in X$ is a weak solution of the problem (P_λ^f) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,\mathrm{d}x + M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,\mathrm{d}x + \int_\Omega \rho(x)|u|^{p(x)-2}uv\,\mathrm{d}x - \lambda\int_\Omega f(x,u)uv\,\mathrm{d}x = 0$$

for every $v \in X$.

Proposition 2.4 (see [11]) *For every $u \in L^{p(x)}(\Omega)$ let $J(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x$. We have*

(a) $\|u\|_{p(x)} < 1 (= 1; > 1) \Longleftrightarrow J(u) < 1 (= 1; > 1)$,

(b) $\|u\|_{p(x)} \geq 1 \Longrightarrow \|u\|_{p(x)}^{p^-} \leq J(u) \leq \|u\|_{p(x)}^{p^+}$,

(c) $\|u\|_{p(x)} \leq 1 \Longrightarrow \|u\|_{p(x)}^{p^+} \leq J(u) \leq \|u\|_{p(x)}^{p^-}$,

(d) $\|u\|_{p(x)} \longrightarrow 0 \Longleftrightarrow J \longrightarrow 0$.

According to Proposition 2.4, for every $u \in X$ we can deduce the following estimates:

$$\|u\|^{p^-} \leq \int_\Omega \left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho(x)|u|^{p(x)}\right)\,\mathrm{d}x \leq \|u\|^{p^+} \quad \text{if } \|u\| \geq 1$$

and

$$\|u\|^{p^+} \leq \int_\Omega \left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho(x)|u|^{p(x)}\right)\,\mathrm{d}x \leq \|u\|^{p^-} \quad \text{if } \|u\| \leq 1.$$

3 Main results

Put $M^- = \min\{1, m_0\}$ and $M^+ = \max\{1, m_1\}$. We state our main result as follows.

Theorem 3.1 *Assume that*

$$\sup_{\gamma>0} \frac{\gamma^{p^-}}{\displaystyle\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x} > \frac{p^+ c^{p^-}}{M^-}, \tag{D_F}$$

where c is the constant defined in (2.2), and that there are non-empty open sets $D \subseteq \Omega$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{\inf_{x\in B} F(x,\xi)}{|\xi|^{p^-}} = +\infty \tag{3.1}$$

and

$$\liminf_{\xi \to 0^+} \frac{\inf_{x\in D} F(x,\xi)}{|\xi|^{p^-}} > -\infty. \tag{3.2}$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{M^-}{p^+ c^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_\Omega \sup_{|t|\le\gamma} F(x,t)\,\mathrm{d}x} \right)$$

the problem (P_λ^f) admits at least one non-trivial weak solution $u_\lambda \in X$. Moreover, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \mapsto \int_\Omega \frac{|\Delta u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x + \widetilde{M}\left(\int_\Omega \frac{|\nabla u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x \right) + \int_\Omega \frac{\rho(x)|u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x - \lambda \int_\Omega F(x,u_\lambda)\,\mathrm{d}x$$

is negative and strictly decreasing in Λ.

Proof. Our aim is to apply Theorem 2.1 to the problem (P_λ^f). Consider the functionals Φ, Ψ defined for every $u \in X$ by the formulas

$$\Phi(u) = \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,\mathrm{d}x + \widetilde{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,\mathrm{d}x \right) + \int_\Omega \frac{\rho(x)|u|^{p(x)}}{p(x)}\,\mathrm{d}x \tag{3.3}$$

and

$$\Psi(u) = \int_\Omega F(x,u)\,\mathrm{d}x, \tag{3.4}$$

and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Let us prove that the functionals Φ and Ψ satisfy the assumptions of Theorem 2.1. It is well-known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega f(x,u)v\,\mathrm{d}x$$

for every $v \in X$. Moreover, Ψ is sequentially weakly upper semicontinuous. Due to Proposition 2.4 we have

$$\Phi(u) \ge \frac{1}{p^+}\int_\Omega |\Delta u|^{p(x)}\,\mathrm{d}x + m_0\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,\mathrm{d}x \right) + \frac{1}{p^+}\int_\Omega \rho(x)|u|^{p(x)}\,\mathrm{d}x \ge \frac{M^-}{p^+}\|u\|^{p^-}$$

for all $u \in X$ such that $\|u\| > 1$, and since $p^- > 1$, it follows that Φ is coercive. Moreover, Φ is continuously differentiable and its differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v \,\mathrm{d}x + M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \,\mathrm{d}x$$
$$+ \int_\Omega \rho(x)|u|^{p(x)-2}uv \,\mathrm{d}x$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1, are satsfied. Note that the critical points of the functional I_λ are the weak solutions of the problem (P_λ^f). We now look at the existence of a critical point of the functional I_λ in X. By the condition (D_F) there exists $\bar{\gamma} > 0$ such that

$$\frac{\bar{\gamma}^{p^-}}{\displaystyle\int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x} > \frac{p^+ c^{p^-}}{M^-}. \tag{3.5}$$

Choose

$$r = \frac{M^-}{p^+}\left(\frac{\bar{\gamma}}{c}\right)^{p^-}.$$

Then, in view of [5, Proposition 2.2], for all $u \in X$ with $\Phi(u) < r$ one has

$$\|u\| \le \max\left\{(p^+ r)^{\frac{1}{p^+}}, (p^+ r)^{\frac{1}{p^-}}\right\}.$$

So, due to the embedding $X \hookrightarrow C^0(\Omega)$ (see (2.2)), one has $\|u\|_\infty \le c\|u\|$. From the definition of r it follows that

$$\Phi^{-1}(-\infty, r) = \{u \in X \mid \Phi(u) < r\} \subseteq \{u \in X \mid |u| \le \bar{\gamma}\}$$

and this ensures that

$$\Psi(u) \le \sup_{u\in\Phi^{-1}(-\infty,r)} \int_\Omega F(x,u)\,\mathrm{d}x \le \int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x$$

for every $u \in X$ such that $\Phi(u) < r$. Then,

$$\sup_{\Phi(u)<r} \Psi(u) \le \int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x.$$

By simple calculations and from the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\varphi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)) - \Psi(u)}{r - \Phi(u)} \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}$$
$$\le \frac{\displaystyle\int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x}{r} \le \frac{\displaystyle\int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x}{\displaystyle\frac{M^-}{p^+}\left(\frac{\bar{\gamma}}{c}\right)^{p^-}}.$$

Hence, putting

$$\lambda^* = \frac{M^-}{p^+ c^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\displaystyle\int_\Omega \sup_{|t|\le\gamma} F(x,t)\,\mathrm{d}x},$$

Theorem 2.1 ensures that for every $\lambda \in (0,\lambda^*) \subseteq \big(0, \frac{1}{\varphi(r)}\big)$, the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$. We will show that the function u_λ cannot be trivial. Let us prove that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.6}$$

In view of to the assumptions (3.1) and (3.2), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that

$$\lim_{n\to+\infty} \frac{\inf_{x\in B} F(x,\xi_n)}{|\xi_n|^{p^-}} = +\infty$$

and

$$\inf_{x\in D} F(x,\xi) \ge \kappa |\xi|^{p^-}$$

for every $\xi \in [0,\sigma]$. We consider a set $\mathcal{G} \subset B$ of positive measure and a function $v \in X$ such that

(k_1) $v(x) \in [0,1]$ for every $x \in \Omega$,

(k_2) $v(x) = 1$ for every $x \in \mathcal{G}$,

(k_3) $v(x) = 0$ for every $x \in \Omega \setminus D$.

Fix $K > 0$ and consider a real positive number η with

$$K < \frac{\eta\,\mathrm{meas}(\mathcal{G}) + \kappa \displaystyle\int_{D\setminus\mathcal{G}} |v|^{p^-}\,\mathrm{d}x}{\dfrac{M^+}{p^-}\|v\|^{p^-}}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$\inf_{x\in B} F(x,\xi_n) \ge \eta |\xi_n|^{p^-}$$

for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is, $0 \le \xi_n v < \sigma$ for n large enough), we have

$$\begin{aligned}\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\displaystyle\int_{\mathcal{G}} F(x,\xi_n)\,\mathrm{d}x + \int_{D\setminus\mathcal{G}} F(x,\xi_n v)\,\mathrm{d}x}{\Phi(\xi_n v)} \\ &> \frac{\eta\,\mathrm{meas}(\mathcal{G}) + \kappa \displaystyle\int_{D\setminus\mathcal{G}} |v|^{p^-}\,\mathrm{d}x}{\dfrac{M^+}{p^-}\|v\|^{p^-}} > K.\end{aligned}$$

Since K could be arbitrarily large, it is concluded that

$$\lim_{n\to\infty}\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)}=+\infty,$$

from which (3.6) clearly follows. Hence, there exists a sequence $\{w_n\}\subset X$ strongly converging to zero such that, for n large enough, $w_n\in\Phi^{-1}(-\infty,r)$ and

$$I_\lambda(w_n)=\Phi(w_n)-\lambda\Psi(w_n)<0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty,r)$, we obtain

$$I_\lambda(u_\lambda)<0. \tag{3.7}$$

Therefore, u_λ is not trivial. From (3.7) we easily observe that the map

$$(0,\lambda^*)\ni\lambda\mapsto I_\lambda(u_\lambda) \tag{3.8}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0.$$

Indeed, bearing in mind that Φ is coercive and for every $\lambda\in(0,\lambda^*)$ the solution $u_\lambda\in\Phi^{-1}(-\infty,r)$, one has that there exists a positive constant L such that $\|u_\lambda\|\le L$ for every $\lambda\in(0,\lambda^*)$. After that, it is easy to see that there exists a positive constant N such that

$$\left|\int_\Omega f(x,u_\lambda)u_\lambda\,\mathrm{d}x\right|\le N\|u_\lambda\|\le NL \tag{3.9}$$

for every $\lambda\in(0,\lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v)=0$ for every $v\in X$ and every $\lambda\in(0,\lambda^*)$. In particular, $I'_\lambda(u_\lambda)(u_\lambda)=0$, that is,

$$\Phi'(u_\lambda)(u_\lambda)=\lambda\int_\Omega f(x,u_\lambda)u_\lambda\,\mathrm{d}x \tag{3.10}$$

for every $\lambda\in(0,\lambda^*)$. For $\|u_\lambda\|\ge 1$, in view of to Proposition 2.4, we have

$$0\le M^-\|u_\lambda\|^{p^-}\le\Phi'(u_\lambda)(u_\lambda),$$

and from (3.10) we have

$$0\le M^-\|u_\lambda\|^{p^-}\le\lambda\int_\Omega f(x,u_\lambda)u_\lambda\,\mathrm{d}x \tag{3.11}$$

for any $\lambda\in(0,\lambda^*)$. Letting $\lambda\to 0^+$, by (3.11) together with (3.9), we get

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0.$$

The proof in the case when $\|u_\lambda\|\le 1$ is similar to the one in the case $\|u_\lambda\|\ge 1$. Thus, we have obviously the desired conclusion.

Finally, we have to show that the map

$$\lambda\mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. For our goal we see that for any $u \in X$ one has

$$I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right). \tag{3.12}$$

Now, let us consider $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for $i = 1, 2$. Clearly, (3.8) together with (3.12) and the positivity of λ imply that

$$m_{\lambda_i} < 0 \text{ for } i = 1, 2. \tag{3.13}$$

Moreover,

$$m_{\lambda_2} \leq m_{\lambda_1}, \tag{3.14}$$

due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.12)–(3.14) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}).$$

So the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. The proof is complete. □

Here we present an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.2 *Let* $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$. *Consider the problem*

$$\begin{cases} \Delta^2_{p(x,y)} u - M\left(\displaystyle\int_\Omega \frac{|\nabla u|^{p(x,y)}}{p(x,y)} \,\mathrm{d}x \,\mathrm{d}y \right) \Delta_{p(x,y)} u + |u|^{p(x,y)-2} u = \lambda f(x, y, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where $M(t) = \frac{5}{2} + \frac{1}{2}\cos(t)$ *for every* $t \in [0, +\infty)$, $p(x, y) = x^2 + y^2 + 4$ *for every* $(x, y) \in \Omega$ *and*

$$f(x, y, t) = \frac{x^2 + y^2}{10^3 c^4 \pi} \left(4t^3 + 2t + e^t + \sinh(t) \right)$$

for every $(x, y, t) \in \Omega \times \mathbb{R}$. *By the definition of* f, *we have*

$$F(x, y, t) = \frac{x^2 + y^2}{10^3 c^4 \pi} \left(t^4 + t^2 + e^t + \cosh(t) - 2 \right)$$

for every $(x, y, t) \in \Omega \times \mathbb{R}$. *By simple calculations, we obtain* $m_0 = 2$, $p^- = 4$ *and* $p^+ = 8$. *Since*

$$\sup_{\gamma > 0} \frac{\gamma^4}{\displaystyle\int_\Omega \sup_{|t| \leq \gamma} F(x, y, t) \,\mathrm{d}x \,\mathrm{d}y} > 8c^4 = \frac{p^+ c^{p^-}}{M^-},$$

we observe that all the assumptions of Theorem 3.1 *are fulfilled. Hence, Theorem* 3.1 *implies that for each* $\lambda \in \left(0, \frac{125}{8}\right)$ *the problem* (3.15) *admits at least one non-trivial weak solution* $u_\lambda \in X$. *Moreover, one has*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \mapsto \int_\Omega \frac{|\Delta u_\lambda|^{p(x,y)}}{p(x,y)}\,\mathrm{d}x\,\mathrm{d}y + \widetilde{M}\left(\int_\Omega \frac{|\nabla u_\lambda|^{p(x,y)}}{p(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right) + \int_\Omega \frac{\rho(x)|u_\lambda|^{p(x,y)}}{p(x,y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$- \lambda \int_\Omega F(x,y,u_\lambda)\,\mathrm{d}x\,\mathrm{d}y$$

is negative and strictly decreasing in $\lambda \in \left(0, \frac{125}{8}\right)$.

Now, we give some remarks concerning our result.

Remark 3.3 *In Theorem* 3.1 *we searched for the critical points of the functional* I_λ *naturally associated with the problem* (P_λ^f). *We note that, in general,* I_λ *can be unbounded from below in* X. *Indeed, for example, if we take* $f(\xi) = 1 + |\xi|^{\varepsilon - p^+}\xi^{p^+-1}$ *for every* $\xi \in \mathbb{R}$ *with* $\varepsilon > p^+$, *for any fixed* $u \in X\backslash\{0\}$ *and* $\iota \in \mathbb{R}$ *we obtain*

$$I_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_\Omega F(\iota u)\,\mathrm{d}x \le \iota^{p^+}\frac{M^+}{p^-}\|u\|^{p^+} - \lambda\iota\|u\|_{L^1(\Omega)} - \lambda\frac{\iota^\varepsilon}{\varepsilon}\|u\|^\varepsilon_{L^\varepsilon(\Omega)} \to -\infty$$

as $\iota \to +\infty$. *Hence, we can not use direct minimization to find critical points of the functional* I_λ. *Furthermore, for fixed* $\bar{\gamma} > 0$ *let*

$$\frac{\bar{\gamma}^{p^-}}{\displaystyle\int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x} > \frac{p^+ c^{p^-}}{M^-}.$$

Then, the result of Theorem 3.1 *holds with* $\|u_\lambda\|_\infty \le \bar{\gamma}$.

Remark 3.4 *We observe that Theorem* 3.1 *is a bifurcation result in the sense that the pair* $(0,0)$ *belongs to the closure of the set*

$$\left\{(u_\lambda, \lambda) \in X \times (0,+\infty) \mid u_\lambda \textit{ is a non-trivial weak solution of } (P_\lambda^f)\right\}$$

in $X \times \mathbb{R}$. *Indeed, by Theorem* 3.1 *we have that*

$$\|u_\lambda\| \to 0 \quad \textit{as} \quad \lambda \to 0.$$

Hence, there exist two sequences $\{u_j\}$ *in* X *and* $\{\lambda_j\}$ *in* $\mathbb{R}^+$ *(here* $u_j = u_{\lambda_j}$*) such that*

$$\lambda_j \to 0^+ \quad \textit{and} \quad \|u_j\| \to 0$$

as $j \to +\infty$. *Moreover, we emphasis that due to the fact that the map*

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$ *with* $\lambda_1 \neq \lambda_2$ *the weak solutions* u_{λ_1} *and* u_{λ_2} *ensured by Theorem* 3.1 *are different.*

Remark 3.5 *If f is non-negative, then the weak solution ensured by Theorem 3.1 is non-negative. Indeed, let u_0 be the weak solution of the problem (P_λ^f) ensured by Theorem 3.1. Arguing by a contradiction, assume that the set $\mathcal{A} = \{x \in \Omega \mid u_0(x) < 0\}$ is non-empty and of positive measure. Put $\bar{u} = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\bar{v} \in X$ and one has*

$$\int_\Omega |\Delta u_0|^{p(x)-2}\Delta u_0 \Delta \bar{v}\,\mathrm{d}x + M\left(\int_\Omega \frac{|\nabla u_0|^{p(x)}}{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla \bar{v}\,\mathrm{d}x + \int_\Omega \rho(x)|u_0|^{p(x)-2}u_0\bar{v}\,\mathrm{d}x - \lambda\int_\Omega f(x,u_0)\bar{v}\,\mathrm{d}x = 0$$

for every $\bar{v} \in X$. Thus, from our sign assumptions on the data, we have

$$M^-\|u\|_{w^{m,p(x)}(\mathcal{A})} \le \int_\mathcal{A} |\Delta u_0|^{p(x)}\,\mathrm{d}x + M\left(\int_\mathcal{A} \frac{|\nabla u_0|^{p(x)}}{p(x)}\,\mathrm{d}x\right)\int_\mathcal{A} |\nabla u_0|^{p(x)}\,\mathrm{d}x + \int_\mathcal{A} \rho(x)|u_0|^{p(x)}\,\mathrm{d}x = \lambda\int_\mathcal{A} f(x,u_0)u_0\,\mathrm{d}x \le 0,$$

i.e.,

$$\|u_0\|_{w^{m,p(x)}(\mathcal{A})} \le 0,$$

which contradicts the fact that u_0 is a non-trivial weak solution. Hence, the set $\mathcal{A}$ is empty, and u_0 is non-negative.

When f does not depend upon x, we obtain the following autonomous version of Theorem 3.1.

Theorem 3.6 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put*

$$F(\xi) = \int_0^\xi f(t)\,\mathrm{d}t$$

for all $\xi \in \mathbb{R}$. Assume that

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{|\xi|^{p^-}} = +\infty.$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{M^-}{p^+ c^{p^-}\operatorname{meas}(\Omega)} \sup_{\gamma > 0} \frac{\gamma^{p^-}}{F(\gamma)}\right),$$

where c is the constant defined in (2.2), the problem

$$\begin{cases} \Delta^2_{p(x)} u - M\left(\int_\Omega \frac{|\nabla u|^{p(x)}\,\mathrm{d}x}{p(x)}\right)\Delta_{p(x)}u + \rho(x)|u|^{p(x)-2}u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in X$ such that

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Moreover, the real function

$$\lambda \mapsto \int_\Omega \frac{|\Delta u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x + \widetilde{M}\left(\int_\Omega \frac{|\nabla u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x\right) + \int_\Omega \frac{\rho(x)|u_\lambda|^{p(x)}}{p(x)}\,\mathrm{d}x - \lambda\int_\Omega F(u_\lambda)\,\mathrm{d}x$$

is negative and strictly decreasing in Λ.

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