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MULTIPLE SOLUTIONS FOR NONLINEAR FOURTH-ORDER ELASTIC BEAM EQUATIONS

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Abstract. In this paper, using variational methods and critical point theory, we establish the existence of multiple generalized solutions for the fourth-order nonlinear differential problem

$$\begin{cases} u^{(4)} + Au'' + Bu = \lambda f(x,u), \quad x \in [0,1], \\ u(0) = u(0) = 0, \\ u''(1) = u''(1) = 0. \end{cases}$$

Some recent results are extended and improved.

Keywords: Elastic beam equation, fourth-order, Palais–Smale condition, variational methods.

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1 Introduction

In the present paper, we consider the existence of multiple non-trivial generalized solutions for the fourth-order nonlinear differential problem

$$\begin{cases} u^{(4)} + Au'' + Bu = \lambda f(x,u), \quad x \in [0,1], \\ u(0) = u(0) = 0, \\ u''(1) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the Ambrosetti–Rabinowitz condition (the AR-condition for short), A and B are real constants and λ is a positive parameter. The

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main results of our paper, stated as Theorems 3.3 and 3.5, say that under suitable conditions problem (1.1) has, respectively, at least two non-trivial and infinitely many solutions.

Equations of fourth-order type are usually called elastic beam equations, which comes from the fact that they describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported.

There are many fourth-order differential equations which are similar to problem (1.1) in engineering, material mechanics and so on. Studying fourth-order differential equations is very important in engineering sciences. Therefore, several results concerning the existence of multiple solutions for fourth-order boundary value problems are known; for example see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In [13], using variational methods and maximum principle, the author proved the existence of at least two positive solutions for the problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \ u'''(1) = \mu g(u(1)), \end{cases}$$

where $\lambda > 0$, $\mu > 0$, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Liu and Li in [10], by means of the idea of the decomposition of operators introduced by Chen, obtained an existence and multiplicity result for the following fourth-order boundary value problem (BVP)

$$\begin{cases} u^{(4)}(t) + \mu u''(t) + \zeta u(t) = \lambda f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $\mu, \zeta \in \mathbb{R}$ and $\lambda > 0$ are parameters and $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. D'Aguì *et al.* in [5], using variational methods, proved the existence of two solutions of problem (1.1). In [8], using variational methods and critical point theory, the existence of multiple solutions for the following fourth-order problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)) + h(u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \ u'''(1) = \mu g(u(1)) \end{cases}$$

was established; here $\lambda > 0$, $\mu > 0$, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a non-negative L^1-Carathéodory function, $g\colon \mathbb{R} \to \mathbb{R}$ is a non-positive continuous function and $h\colon \mathbb{R} \to \mathbb{R}$ is a non-negative Lipschitz continuous function with the Lipschitz constant $0 < L < 1$, *i.e.*, $|h(s_1) - h(s_2)| \leq L|s_1 - s_2|$ for every $s_1, s_2 \in \mathbb{R}$, and $h(0) = 0$.

In the present paper we are interested in ensuring the existence of at least two generalized solutions and infinitely many generalized solutions for problem (1.1). The present paper is organized as follows. In Section 2, we recall some basic definitions and our main tools. In Section 3, we state and prove the main theorems of the paper, and finally we give two examples to show the application of our results.

2 Preliminaries and basic notation

In the present section we introduce some definitions and results used in the next section. First, let A and B be two real constants such that

$$\max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}\right\} < 1. \tag{2.1}$$

For instance, condition (2.1) is satisfied if $A \leq 0$ and $B \geq 0$. Moreover, we set

$$\sigma := \max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}, 0\right\} < 1$$

and $\delta := \sqrt{1-\sigma}$. Let $X := H_0^1(0,1) \cap H^2(0,1)$ be the Hilbert space endowed with the norm

$$\|u\|_X = \left(\int_0^1 (|u''|^2 - A|u'|^2 + B|u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}},$$

which is equivalent to the usual norm (see [2]). In particular, $\|u\|_\infty \leq \frac{1}{2\pi\delta}\|u\|_X$.

Definition 2.1 *A weak solution of* (1.1) *is a function* $u \in X$ *such that*

$$\int_0^1 [u''(x)v''(x) - Au'(x)v'(x) + Bu(x)v(x)]\,\mathrm{d}x - \int_0^1 f(x, u(x))v(x)\,\mathrm{d}x = 0$$

for all $v \in X$.

Moreover, a function $u\colon [0,1] \to \mathbb{R}$ is said to be a generalized solution to problem (1.1) if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u(1) = u''(0) = u''(1) = 0$ and $u^{(4)} + Au'' + Bu = \lambda f(x,u)$ for almost every $x \in [0,1]$. If f is continuous on $[0,1] \times \mathbb{R}$, then each generalized solution u is a classical solution.

Proposition 2.2 ([1, Proposition 2.2]) *Assume that* $u \in X$ *is a weak solution of* (1.1)*. Then,* u *is also a generalized solution of* (1.1).

Now, we introduce the functional $\varphi\colon X \to \mathbb{R}$ defined by $\varphi(u) = \Phi(u) - \lambda\Psi(u)$, where

$$\Phi(u) = \frac{1}{2}\int_0^1 (|u''|^2 - A|u'|^2 + B|u|^2)\,\mathrm{d}x,$$
$$\Psi(u) = \int_0^1 F(x, u(x))\,\mathrm{d}x$$

for each $u \in X$ and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ for each $(x,t) \in [0,1]\times\mathbb{R}$. It is clear that the functionals $\Phi(u)$ and $\Psi(u)$ are well-defined on X, and we have

$$\Phi'(u)(v) = \frac{1}{2}\int_0^1 (u''(x)v''(x) - Au'(x)v'(x) + Bu(x)v(x))\,\mathrm{d}x$$

and

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)\,\mathrm{d}x$$

for all $v \in X$.

Definition 2.3 *Let X be a real reflexive Banach space. If any sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ for which $\{\varphi(u_k)\}_{k\in\mathbb{N}}$ is bounded and $\varphi'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence, then we say that φ satisfies the Palais–Smale condition.*

The proofs of our theorems are based on Theorems 2.4, 2.5 and 2.6 stated below.

Theorem 2.4 ([15, Theorem 4.10]) *Let $\varphi \in C^1(X,\mathbb{R})$ satisfy the Palais–Smale condition. Assume also that there exist $u_0, u_1 \in X$ and a bounded neighbourhood Ω of u_0 such that $u_1 \notin \Omega$ and $\inf_{\nu\in\partial\Omega} \varphi(\nu) > \max\{\varphi(u_0), \varphi(u_1)\}$. Then, there exists a critical point u of φ,* i.e. *$\varphi'(u) = 0$, with $\varphi(u) > \max\{\varphi(u_0), \varphi(u_1)\}$.*

Theorem 2.5 ([16, Theorem 9.12]) *Let X be an infinite dimensional real Banach space. Let $\varphi \in C^1(X,\mathbb{R})$ be an even functional which satisfies the Palais–Smale condition and $\varphi(0) = 0$. Suppose that $X = V \bigoplus E$, where V is finite dimensional and φ satisfies the following conditions*:

(i) *there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*

(ii) *for any finite dimensional subspace $W \subset X$ there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$.*

Then, φ possesses an unbounded sequence of critical values.

Theorem 2.6 ([17, Theorem 38]) *For the functional $F\colon M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u\in M} F(u) = \alpha$ has a solution provided that the following conditions hold*:

(h_1) *X is a real reflexive Banach space,*

(h_2) *M is bounded and weak sequentially closed,*

(h_3) *F is weak sequentially lower semi-continuous on M,* i.e., *by definition, for each sequence $\{u_n\}_{n\in\mathbb{N}}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$ we have $F(u) \leq \liminf_{n\to\infty} F(u_n)$.*

In the papers [4, 19], Theorems 2.4 and 2.5 were applied to establish a multiplicity result for a boundary value problem. Moreover, in the paper [18], Theorem 2.5 was successfully applied in the proof of the existence of infinitely many solutions for a boundary value problem.

3 Main results

Throughout this paper we make the following assumptions.

(A_1) There exist constants $T > 0$ and $\nu > 2$ such that $0 < \nu F(x,t) \leq t f(x,t)$ for $|t| > T$.

(f_0) The function $f\colon [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory conditions and there exists a constant $L > 0$ such that $|f(x,t)| \leq c(1 + |t|^{q-1})$ for $|t| \leq L$ and $x \in [0,1]$, where $q > 2$.

(f_1) $f(x,t) = o(|t|)$ as $t \to 0$, uniformly with respect to $x \in [0,1]$.

To prove our main results we need the following two lemmas.

Lemma 3.1 *Assume that* (A_1) *holds. Then,* $\varphi(u)$ *satisfies the Palais–Smale condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ is an arbitrary sequence such that $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|\varphi(u_n)| \leq c_0$ and $|\varphi'(u_n)| \leq c_0$ for all $n \in \mathbb{N}$. Therefore, from the definition of φ' and the assumption (A_1), for some $c_1 > 0$ we have

$$\begin{aligned} c_0 + c_1\|u_n\|_X &\geq \nu\varphi(u_n) - \varphi'(u_n)(u_n) \\ &\geq \Big(\frac{\nu}{2} - 1\Big)\|u_n\|_X^2 - \lambda\Big(\nu\int_0^1 F(x, u_n(x))\,\mathrm{d}x - \int_0^1 f(x, u_n(x))u_n(x)\,\mathrm{d}x\Big) \\ &\geq \Big(\frac{\nu}{2} - 1\Big)\|u_n\|_X^2. \end{aligned}$$

Since $\nu > 2$, this implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, since X is a reflexive Banach space, up to a subsequence, we have $u_n \rightharpoonup u$ in X. By $\varphi'(u_n) \to 0$ and $u_n \rightharpoonup u$ in X, we obtain $(\varphi'(u_n) - \varphi'(u))(u_n - u) \to 0$. From the continuity of f we have

$$\int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\,\mathrm{d}x \to 0.$$

Moreover, an easy computation shows that

$$\begin{aligned} &(\varphi'(u_n) - \varphi'(u))(u_n - u) \\ &= \frac{1}{2}\int_0^1\Big(u_n''(x)(u_n''(x) - u''(x)) - A(u_n'(x))(u_n''(x) - u''(x)) + Bu_n(x)(u_n(x) - u(x))\Big)\mathrm{d}x \\ &\quad - \frac{1}{2}\int_0^1\Big(u''(x)(u_n''(x) - u''(x)) - A(u'(x))(u_n''(x) - u''(x)) + Bu(x)(u_n(x) - u(x))\Big)\mathrm{d}x \\ &\quad - \lambda\int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\,\mathrm{d}x \\ &\geq \frac{1}{4}\|u_n - u\|^2. \end{aligned}$$

Therefore, $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in X. Consequently, φ satisfies the Palais–Smale condition. □

Lemma 3.2 ([20, Lemma 2.2]) *If condition* (A_0) *holds, then for every* $x \in [0,1]$ *the following inequalities are true*:

(i) $F(x,t) \leq F(x, \frac{t}{|t|})|t|^\nu$ *if* $0 < |t| \leq 1$,

(ii) $F(x,t) \geq F(x, \frac{t}{|t|})|t|^\nu$ *if* $|t| \geq 1$.

In view of Lemma 3.2 and (f_0) for every $x \in [0,1]$ we have

$$F(x,t) \leq a_3|t|^\nu \text{ if } |t| \leq 1 \quad \text{and} \quad F(x,t) \geq a_1|t|^\nu \text{ if } |t| \geq 1, \tag{3.1}$$

where $a_3 = \max_{x\in[0,1],\,|t|=1} F(x,t)$ and $a_1 = \min_{x\in[0,1],\,|t|=1} F(x,t)$. By assumption (f_0), we infer that $a_1, a_3 > 0$. In addition, since $F(x,t) - a_1|t|^\nu$ is continuous on $[0,1] \times [0,T]$, there exists a constant $a_2 > 0$ such that

$$F(x,t) \geq a_1|t|^{\nu} - a_2 \text{ for all } (x,t) \in [0,1] \times [0,T]. \tag{3.2}$$

So, it follows from (3.1) and (3.2) that

$$F(x,t) \geq a_1|t|^{\nu} - a_2 \text{ for all } (x,t) \in [0,1] \times \mathbb{R}. \tag{3.3}$$

In this paper the main results are the following.

Theorem 3.3 *Assume that the assumptions* (A_1), (f_0) *and* (f_1) *hold. Then, if* $f(x,t) \geq 0$ *for all* $(x,t) \in [0,1] \times \mathbb{R}$, *problem* (1.1) *has at least two generalized solutions.*

Proof. In our case it is clear that $\varphi(0) = 0$. Lemma 3.1 shows that φ satisfies the Palais–Smale condition.

Step 1 We will show that there exists $M > 0$ such that the functional φ has a local minimum $u_0 \in B_M = \{u \in X : \|u\|_X < M\}$. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \overline{B}_M$ and $u_n \rightharpoonup u$ as $n \to \infty$. By Mazur's Theorem (see [14]), there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} a_{n_j} u_j, \quad \sum_{j=1}^{n} a_{n_j} = 1, \quad a_{n_j} \geq 0,\ j \in \mathbb{N}$$

such that $v_n \to u$ in X. Since $\overline{B}_M$ is a closed and convex set, we have $\{v_n\}_{n\in\mathbb{N}} \subseteq \overline{B}_M$ and $u \in \overline{B}_M$. Noting that φ is weakly sequentially lower semi-continuous on $\overline{B}_M$ and X is a reflexive Banach space, by Theorem 2.6 we infer that φ has a local minimum $u_0 \in \overline{B}_M$. We assume that $\varphi(u_0) = \min_{u\in\overline{B}_M} \varphi(u)$. Now, we will show that $\varphi(u_0) < \inf_{u\in\partial B_M} \varphi(u)$. From (f_0) and (f_1), there exits $\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon|t|^2 + c|t|^q. \tag{3.4}$$

Let $\varepsilon > 0$ be small enough that $\varepsilon < \pi\delta$. Then,

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon \int_0^1 |u(x)|^2 \,\mathrm{d}x - \lambda c \int_0^1 |u(x)|^q \,\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon\|u\|_\infty^2 - \lambda c \int_0^1 |u(x)|^q \,\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon\frac{1}{2\pi\delta}\|u\|_X^2 - \lambda c\frac{1}{2\pi\delta}\|u\|_X^q \\
&\geq \left(\frac{1}{2} - \lambda\varepsilon\frac{1}{2\pi\delta}\right)\|u\|_X^2 - \lambda c\frac{1}{2\pi\delta}\|u\|_X^q,
\end{aligned}$$

when $\|u\|_X < 1$. Since $q > 2$, there exist $r > 0$, $k > 0$ such that $\varphi(u) \geq k > 0$ for every $\|u\|_X = r$. Choosing $M = r$, we have $\varphi(u) > 0 = \varphi(0) \geq \varphi(u_0)$ for $u \in \partial B_M$. Hence, $u_0 \in B_M$ and $\varphi'(u_0) = 0$.

Step 2 Since u_0 is a minimum point of φ on X, we can consider $M > 0$ sufficiently large that $\varphi(u_0) \leq 0 < \inf_{u\in\partial B_M} \varphi(u)$, where $B_M = \{u \in X : \|u\|_X < M\}$. Now, we will show that there exists u_1 with $\|u_1\|_X > M$ such that $\varphi(u_1) < \inf_{u\in\partial B_M} \varphi(u)$. For this, let $e_1(x) \in X$ and $u_1 = re_1$, where $r > 0$ and $\|e_1\|_X = 1$. By (3.3) there exist constants $a_1, a_2 > 0$ such that $F(x,t) \geq a_1|t|^{\nu} - a_2$ for all $x \in [0,1]$. Thus,

$$\begin{aligned}\varphi(u_1) &= (\Phi - \lambda\Psi)(re_1) \\ &\le \frac{1}{2}\|re_1\|_X^2 - \lambda \int_0^1 F(x, re_1(x))\,\mathrm{d}x \\ &\le \frac{1}{2}r^2\|e_1\|_X^2 - \lambda r^\nu a_1 \int_0^1 |e_1|^\nu\,\mathrm{d}x + \lambda a_2.\end{aligned}$$

Since $\nu > 2$, there exists sufficiently large $r > M > 0$ so that $\varphi(re_1) < 0$. Hence, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{\partial B_M} \varphi(u)$. Then, Theorem 2.4 gives the critical point u^*. Therefore, u_0 and u^* are two critical points of φ, which are two generalized solutions of the problem (1.1). □

Example 3.4 *Let $A = 2$, $B = 1$ and for $(x, t) \in [0, 1] \times \mathbb{R}$ set*

$$f(x,t) = \begin{cases} t^2, & \text{if } |t| \le 1, \\ 3t^4 - 2, & \text{if } |t| > 1. \end{cases}$$

Then, we have

$$F(x,t) = \begin{cases} \frac{1}{3}t^3, & \text{if } |t| \le 1, \\ \frac{3}{5}t^5 - 2t + \frac{26}{15}, & \text{if } |t| > 1. \end{cases}$$

Moreover, $f(x, t) = o(|t|)$ as $t \to 0$. By choosing $L = 1$ and $q = 3$ we have $|f(x, t)| < c(1 + |t|^2)$ for $|t| < 1$, where $c > 0$ is some constant. Since

$$\lim_{|t|\to\infty} \frac{tf(x,t)}{F(x,t)} = \lim_{|t|\to\infty} \frac{3t^5 - 2t}{\frac{3}{5}t^5 - 2t - \frac{26}{15}} = 5,$$

by choosing $\nu = 5$ and $T = 1$, we have $5F(x, t) \le tf(x, t)$. Also, $f(x, t) \ge 0$. So, we see that all conditions (A_1), (f_0) and (f_1) of Theorem 3.3 are satisfied. Therefore, in this case problem (1.1) has at least two generalized solutions.

Theorem 3.5 *Assume that the assumption (A_1) holds. Then, if $f(x, t)$ is odd about t, problem (1.1) has infinitely many generalized solutions.*

Proof. Put $X = H_0^1(0, 1) \cap H^2(0, 1)$. According to the definitions of the functionals Φ and Ψ it is clear that $\varphi(u)$ is even and $\varphi(0) = 0$.

Step 1 Let $0 < \lambda_1 < \lambda_2 < \dots$ be the eigenvalues of the functional φ. Suppose that V is k-dimensional and $V = \operatorname{span}\{e_1, e_2, \dots, e_k\}$, where e_i is the eigenfunction corresponding to the eigenvalue λ_i. Let $E = V^\perp$. Thus, $X = V \oplus E$. We will show that φ satisfies condition (i) of Theorem 2.5. Since φ is coercive and also satisfies the Palais–Smale condition, by the minimization theorem [15, Theorem 4.4], the functional φ has a minimum critical point u with $\varphi(u) \ge \alpha > 0$ and $\|u\|_X = \rho$ for $\rho > 0$ small enough.

Step 2 We will show that φ satisfies condition (ii) of Theorem 2.5. Let W be a finite dimensional subspace of X. By (3.3) there exist constants $a_1, a_2 > 0$ such that $F(x, t) \ge a_1|t|^\nu - a_2$ for all $x \in [0, 1]$. Now, for every $r > 0$ and $u \in W \setminus \{0\}$ with $\|u\|_X = 1$, one has

$$\begin{aligned}\varphi(ru) &= (\Phi - \lambda\Psi)(ru)\\ &\le \frac{1}{2}\|ru\|_X^2 - \lambda\int_0^1 F(x, ru(x))\,\mathrm{d}x\\ &\le \frac{1}{2}r^2\|u\|_X^2 - \lambda r^\nu a_1\int_0^1 |u|^\nu\,\mathrm{d}x + \lambda a_2.\end{aligned}$$

The above inequality implies that there exists r_0 such that $\|ru\|_X > \rho$ and $\varphi(ru) < 0$ for every $r \ge r_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that $\varphi(u) \le 0$ on $W \setminus B_{R(W)}$. According to Theorem 2.5, the functional $\varphi(u)$ possesses infinitely many critical points, *i.e.*, problem (1.1) has infinitely many generalized solutions. □

Example 3.6 *Put* $A = 3$*,* $B = 2$ *and*

$$f(x,t) = \begin{cases} t^5 \sin x, & \text{if } |t| > 1,\\ \sin x \sin(\frac{\pi}{2}t), & \text{if } |t| \le 1\end{cases}$$

for $(x,t) \in [0,1] \times \mathbb{R}$*. We have*

$$F(x,t) = \begin{cases} \frac{t^6}{6} \sin x, & \text{if } |t| > 1,\\ \sin x(\frac{\pi}{2}\cos(\frac{\pi}{2}t) + \frac{1}{6}), & \text{if } |t| \le 1.\end{cases}$$

Since $\lim_{|t|\to\infty} \frac{tf(x,t)}{F(x,t)} = 6$*, by choosing* $\nu = 6$*, we get* $6F(x,t) \le tf(x,t)$*. Also,* $f(x,t)$ *is odd about* t*. We see that all conditions in Theorem* 3.5 *are satisfied. Hence, in this case problem* (1.1) *has infinitely many generalized solutions.*

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