Journal of Nonlinear Evolution Equations and Applications
Volume 2022, Number 2, pp. 25–57 (July 2022)
ISSN 2161-3680
http://www.jneea.com

RENORMALIZED SOLUTIONS FOR NONLINEAR ELLIPTIC SYSTEM WITH VARIABLE EXPONENTS AND SINGULAR COEFFICIENT AND WITH DIFFUSE MEASURE DATA

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Received June 27, 2021

Accepted on February 16, 2022

Communicated by Ti-Jun Xiao

Abstract. In this paper we prove the existence of a renormalized solution for a nonlinear elliptic system of the type

$$\begin{cases} -\operatorname{div}\big(A(x,v)\,|\nabla u|^{p(x)-2}\,\nabla u\big) + H_1(x,u,\nabla u) = \mu & \text{in } \Omega, \\ -\operatorname{div}\big(B(x,v)\,|\nabla v|^{p(x)-2}\,\nabla v\big) + H_2(x,v,\nabla v) = \gamma|\nabla u|^{q_0(x)} & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $2 - \frac{1}{N} < p(x) < N$, μ is a diffuse measure, $A(x,s)$ and $H_i(x,s,\xi)$ are Carathéodory functions. The function $B(x,s)$ blows up (uniformly with respect to x) as $s \to m^-$ (with $m > 0$), γ is a positive constant and $q_0(x) \in [1, \frac{N(p(x)-1)}{N-1})$.

Keywords: Nonlinear elliptic system, renormalized solutions, singular coefficient, variable exponents.

2010 Mathematics Subject Classification: Primary: 47A15; secondary: 46A32, 47D20.

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1 Introduction

This paper is devoted to the study of the following nonlinear elliptic system

$$\begin{cases} -\operatorname{div}\big(A(x,v)\,|\nabla u|^{p(x)-2}\,\nabla u\big) + H_1(x,u,\nabla u) = \mu & \text{in } \Omega, \\ -\operatorname{div}\big(B(x,v)\,|\nabla v|^{p(x)-2}\,\nabla v\big) + H_2(x,v,\nabla v) = \gamma|\nabla u|^{q_0(x)} & \text{in } \Omega, \\ u = 0,\ v = 0 & \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$ and $p\colon \overline{\Omega} \to [1,+\infty)$ is a continuous function which for all $x, y \in \overline{\Omega}$ such that $|x-y| < \frac{1}{2}$ satisfies the estimate

$$|p(x)-p(y)| < \frac{1}{-\log|x-y|}.$$

Let $p^- = \min_{x\in\overline{\Omega}} p(x)$, $p^+ = \max_{x\in\overline{\Omega}} p(x)$ and for every $x \in \overline{\Omega}$ let $2-\frac{1}{N} < p^- \leq p(x) \leq p^+ < N$. The function $q_0\colon \overline{\Omega} \to [1,+\infty)$ is continuous and such that $1 \leq q_0(x) < \frac{N(p(x)-1)}{N-1}$ for every $x \in \overline{\Omega}$. Let $A\colon \Omega\times\mathbb{R} \to \mathbb{R}^+$ be a Carathéodory function such that

$$0 < \alpha \leq A(x,s) \tag{1.2}$$

for almost every x in Ω, for every $s \in \mathbb{R}$, where α is a positive constant. Moreover, assume that

$$A \in L^\infty(\Omega\times(-k,k)) \text{ for every } k > 0. \tag{1.3}$$

Let $B\colon \Omega\times(-\infty,m) \to \mathbb{R}^+$ be a Carathéodory function such that

$$\beta \leq b^{p(x)-1}(s) \leq B(x,s) \tag{1.4}$$

for almost every x in Ω, for every $s \in (-\infty,m)$, where m and β are two positive real numbers and b is an increasing function of $C^0((-\infty,m))$ such that

$$\lim_{s\to m^-} b(s) = +\infty, \quad \int_0^m b(s)\,\mathrm{d}s < +\infty \quad \text{and} \quad \frac{B}{b^{p(x)-1}} \in L^\infty(\Omega\times(-\infty,m)). \tag{1.5}$$

For $i = 1,2$ let $H_i\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be Carathéodory functions satisfying the following conditions: for any $k > 0$ there exists $h_{i,k} \in L^1(\Omega)$ such that

$$|H_i(x,s,\xi)| \leq \lambda_i\big(h_{i,k}(x) + |\xi|^{p(x)}\big) \text{ for all } |s| \leq k \text{ with } \lambda_i > 0, \tag{1.6}$$

$$H_i(x,s,\xi)s \geq 0 \tag{1.7}$$

for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. As regards the measure μ we assume that

$$\mu \in \mathcal{M}_0(\Omega). \tag{1.8}$$

We study problem (1.1) in the presence of diffuse measure data μ. By Theorem 2.1 of [11] there exist $f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$ such that

$$\mu = f - \operatorname{div}(F). \tag{1.9}$$

The motivation for studying the above problem comes from the applications in physical settings for which the internal variable v is constrained to remain always smaller than m. (The interested reader may refer to the papers [22, 23] on constrained internal variables, see also [25, 27].)

The study of problems with variable exponent is an interesting topic which has raised many mathematical difficulties. The first difficulty in solving this system is defining the field $B(x,v)|\nabla v|^{p(x)-2}\nabla v$ on the subset $\{x \in \Omega : v(x) = m\}$, since on this set $B(x, v(x)) = +\infty$. In addition, the fields $A(x,u),\ H_1(x,u,\nabla u)$ and $H_2(x,v,\nabla v)$ are not in $\mathcal{D}'(\Omega)$ in general, since $(u,v) \notin (L^\infty(\Omega))^2$. The second difficulty is represented here by the presence of the measure data μ.

To overcome these difficulties we use in this paper the framework of renormalized solutions. This notion was introduced by P.-L. Lions and Di Perna [18] for the study of the Boltzmann equation. A large number of papers were then devoted to the study of renormalized (or entropy) solutions of elliptic and parabolic problems with rough data under various assumptions and in different contexts; in addition to the references already mentioned see [1, 3, 4, 6, 13].

The aim of this paper is to extend our result established in [19] in which $p(x)$ was constant and $H_i(x,s,\xi) = 0$ as well as to extend our result from [20] in which $H_i(x,s,\xi) = |s|^{p(x)-2}s$.

This paper is organized as follows. Section 2 contains some properties of the Lebesgue and Sobolev spaces with variable exponents. In Section 3 we give some notations and the definition of a renormalized solution of problem (1.1). In Section 4 we establish the existence of such a solution.

2 Preliminaries

As the exponent $p(x)$ appearing in (1.1) depends on the variable x, we must work with the Lebesgue and Sobolev spaces with variable exponents. Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$). Set

$$C^+(\overline{\Omega}) = \big\{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for any } x \in \overline{\Omega}\big\}.$$

For every $p \in C^+(\overline{\Omega})$ we define

$$p^- = \min\{p(x) : x \in \overline{\Omega}\} \qquad \text{and} \qquad p^+ = \max\{p(x) : x \in \overline{\Omega}\}.$$

For a fixed $p \in C^+(\overline{\Omega})$ we define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < +\infty\right\}.$$

The space $L^{p(\cdot)}(\Omega)$ endowed with the norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\}$$

is a uniformly convex (and therefore reflexive) Banach space. By $L^{p'(\cdot)}(\Omega)$ we denote the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 2.1 (Generalized Hölder inequality, see [21, 35])

(i) *For any functions* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$ *we have*

$$\left| \int_\Omega uv \, \mathrm{d}x \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

(ii) *For all* $p_1, p_2 \in C^+(\overline{\Omega})$ *such that* $p_1(\cdot) \le p_2(\cdot)$ *a.e. in* Ω*, we have* $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ *and the embedding is continuous.*

Proposition 2.2 (see [21, 35]) *For every* $u \in L^{p(\cdot)}(\Omega)$ *let*

$$\rho(u) = \int_\Omega |u|^{p(x)} \, \mathrm{d}x.$$

Then, the following assertions hold:

(i) $\|u\|_{p(\cdot)} < 1$ (*resp.* $= 1; > 1$) $\Leftrightarrow \rho(u) < 1$ (*resp.* $= 1; > 1$),

(ii) *we have the following implications*

$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \le \rho(u) \le \|u\|_{p(\cdot)}^{p^+}$,

$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \le \rho(u) \le \|u\|_{p(\cdot)}^{p^-}$,

(iii) $\|u\|_{p(\cdot)} \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \rho(u) \to 0$ (*resp.* $\to +\infty$).

Now, we define the variable exponent Sobolev space. We set

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

and consider it with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \text{ for } u \in W^{1,p(\cdot)}(\Omega).$$

By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ with $p(x) < N$.

Proposition 2.3 (see [21, 24])

(i) *Assuming that* $1 < p^- \le p^+ < +\infty$*, the spaces* $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are separable and reflexive Banach spaces.*

(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \Omega$*, then the embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact.*

(iii) *Poincaré inequality*: *there exists a constant* $C_1 > 0$ *such that*

$$\|u\|_{p(\cdot)} \le C_1 \|\nabla u\|_{p(\cdot)} \text{ for every } u \in W_0^{1,p(\cdot)}(\Omega).$$

(iv) *Sobolev–Poincaré inequality*: *there exists a constant* $C_2 > 0$ *such that*

$$\|u\|_{p*(\cdot)} \le C_2 \|\nabla u\|_{p(\cdot)} \text{ for every } u \in W_0^{1,p(\cdot)}(\Omega).$$

Remark 2.4 *By* (iii) *of Proposition* 2.3*, we deduce that the norms* $\|\nabla u\|_{p(\cdot)}$ *and* $\|u\|_{1,p(\cdot)}$ *are equivalent in* $W_0^{1,p(\cdot)}(\Omega)$.

3 Some notation and the definition of a renormalized solution

The following notation will be used throughout the paper. For every $k \geq 0$ by $T_k(s) = \min(k, \max(s, -k))$ we denote the truncation function at level k. For every $n \geq 1$ and every $\delta > 0$ we also introduce the functions as follows

$$\theta_n(s) = T_{n+1}(s) - T_n(s), \quad h_n(s) = 1 - |\theta_n(s)|, \tag{3.1}$$

$$T^n_{m-\frac{1}{n}}(s) = \begin{cases} -n, & \text{if } s \leq -n, \\ s, & \text{if } -n \leq s \leq m - \frac{1}{n}, \\ m - \frac{1}{n}, & \text{if } s \geq m - \frac{1}{n}, \end{cases} \tag{3.2}$$

and

$$z_\delta(s) = \frac{1}{\delta}(T^+_{m-\delta}(s) - T^+_{m-2\delta}(s)). \tag{3.3}$$

We now give the definition of a renormalized solution of problem (1.1).

Definition 3.1 *A couple* (u, v) *is said to be a renormalized solution of system* (1.1) *if the following conditions hold*:

$$(T_k(u), T_k(v)) \in (W_0^{1,p(.)}(\Omega))^2 \text{ for every } k \geq 0, \tag{3.4}$$

$$0 \leq v \leq m \text{ a.e. in } \Omega, \tag{3.5}$$

$$(H_1(x, u, \nabla u), H_2(x, v, \nabla v)) \in (L^1(\Omega))^2, \tag{3.6}$$

$$B(x, v)\nabla T_k(v)\chi_{\{0\leq v<m\}} \in (L^{p(.)}(\Omega))^N \text{ for every } k \geq 0, \tag{3.7}$$

$$\lim_{s\to+\infty} \int_{\{s<|u|<s+1\}} A(x, v)|\nabla u|^{p(x)}\, \mathrm{d}x = 0, \tag{3.8}$$

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{\{m-2\delta<v<m-\delta\}} B(x, v)|\nabla v|^{p(x)}\, \mathrm{d}x = \gamma \int_{\{v=m\}} |\nabla u|^{q_0(x)}\, \mathrm{d}x, \tag{3.9}$$

for any $S \in W^{1,\infty}(\mathbb{R})$ *such that* $\operatorname{supp}(S)$ *is compact we have*

$$\begin{aligned} &-\operatorname{div}\big(A(x, v)\, |\nabla u|^{p(x)-2}\, \nabla u\, S(u)\big) + A(x, v)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla S(u) \\ &\qquad + H_1(x, u, \nabla u)S(u) = fS(u) - \operatorname{div}(FS(u)) + F\nabla S(u) \text{ in } \mathcal{D}'(\Omega), \end{aligned} \tag{3.10}$$

and for any $h \in W^{1,\infty}(\mathbb{R})$ *such that* $\operatorname{supp}(h) \subset [0, m)$ *we have*

$$\begin{aligned} &-\operatorname{div}\big(h(v)B(x, v)\, |\nabla v|^{p(x)-2}\, \nabla v \chi_{\{0\leq v<m\}}\big) + h'(v)B(x, v)|\nabla v|^{p(x)}\chi_{\{0\leq v<m\}} \\ &\qquad + H_2(x, v, \nabla v)h(v) = \gamma|\nabla u|^{q_0(x)}h(v) \text{ in } \mathcal{D}'(\Omega). \end{aligned} \tag{3.11}$$

Remark 3.2

(i) *Note that, in view of* (3.4)–(3.7)*, all terms in* (3.10)–(3.11) *are well-defined.*

(ii) *If we assume that* $\int_0^m b(s)\, \mathrm{d}s = +\infty$*, then it is easy to construct a solution* v *which is strictly less than* m *almost everywhere in* Ω*, that is,* $\operatorname{meas}\{v = m\} = 0$ (*see* [29] *and* [34]).

In the case when $\int_0^m b(s)\, \mathrm{d}s < +\infty$*, in general the measure of the set* $\{x \in \Omega : v(x) = m\}$ *is not equal to 0.*

4 Existence of a renormalized solution

Theorem 4.1 *Assume that the assumptions* (1.2)–(1.6) *hold true. Then, there exists a renormalized solution to problem* (1.1) *in the sense of Definition* 3.1.

Proof. The proof is divided into five steps. In Step 1, we introduce an approximate problem of (1.1). Step 2 is devoted to establishing a few a priori estimates. In this step we also prove that (u, v) satisfies (3.4) and (3.5) of Definition 3.1. Step 3 and Step 4 are devoted to proving the monotonicity estimate and the strong $L^{p(x)}(\Omega)$ convergence of $\nabla T_k(u^n)$ and of $\nabla T_k(v^n)$ as n tends to $+\infty$. At last, Step 5 is devoted to proving the strong $L^1(\Omega)$ convergence of $H_1^n(x, u^n, \nabla u^n)$ and of $H_2^n(x, v^n, \nabla v^n)$. We also prove that (u, v) satisfies (3.6)–(3.11).

Step 1 Let us introduce the following regularization of the data: for a fixed $n \geq 1$ let

$$A_n(x, s) = A(x, T_n(s)), \tag{4.1}$$

$$B_n(x, s) = B(x, T^n_{m-\frac{1}{n}}(s)). \tag{4.2}$$

For $i = 1, 2$ let

$$H_i^n(x, s, \xi) = \frac{H_i(x, s, \xi)}{1 + \frac{1}{n}\,|H_i(x, s, \xi)|}, \tag{4.3}$$

$$f^n = T_n(f) \text{ and } \mu^n \equiv f^n - \operatorname{div}(F), \tag{4.4}$$

$$b_n(s) = b(T^n_{m-\frac{1}{n}}(s)) \quad \text{and} \quad \overline{b_n}(s) = \int_0^s b_n(z)\,\mathrm{d}z. \tag{4.5}$$

According to the hypotheses (1.2)–(1.6) for every $s \in \mathbb{R}$ we have

$$\alpha \leq A_n(x, s) \leq \max_{\{|s| \leq n\}} A(x, s) \in L^\infty(\Omega), \tag{4.6}$$

$$\beta \leq b_n^{p(x)-1}(s) \leq B_n(x, s) \text{ and } \frac{B_n}{b_n^{p(x)-1}} \in L^\infty(\Omega \times \mathbb{R}). \tag{4.7}$$

Let us now consider the following regularized problem

$$-\operatorname{div}\big(A_n(x, v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\big) + H_1^n(x, u^n, \nabla u^n) = f^n - \operatorname{div}(F) \text{ in } \Omega, \tag{4.8}$$

$$-\operatorname{div}\big(B_n(x, v^n)\,|\nabla v^n|^{p(x)-2}\,\nabla v^n\big) + H_2^n(x, v^n, \nabla v^n) = \gamma|\nabla u^n|^{q_0(x)} \text{ in } \Omega, \tag{4.9}$$

$$u^n = 0 \text{ and } v^n = 0 \text{ on } \partial\Omega. \tag{4.10}$$

As a consequence, proving the existence of a weak solution $(u^n, v^n) \in (W_0^{1,p(.)}(\Omega))^2$ of (4.8)–(4.10) such that $v^n \geq 0$ is an easy task; it suffices to apply the Schauder fixed theorem (see e.g. [26]).

Step 2 Now, we establish some a priori estimates. Taking $T_k(u^n)$ as a test function in (4.8) gives

$$\begin{aligned} &\int_\Omega \Big(A_n(x, v^n)|\nabla T_k(u^n)|^{p(x)} + H_1^n(x, u^n, \nabla u^n)T_k(u^n)\Big)\,\mathrm{d}x \\ &\quad = \int_\Omega f^n T_k(u^n)\,\mathrm{d}x + \int_\Omega F\nabla T_k(u^n)\,\mathrm{d}x. \end{aligned} \tag{4.11}$$

Using (1.7), (4.4) and the Young inequality, we deduce that

$$\begin{aligned}
&\int_\Omega A_n(x,v^n)|\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x \\
&\quad\le \int_\Omega \Big(A_n(x,v^n)|\nabla T_k(u^n)|^{p(x)} + H_1^n(x,u^n,\nabla u^n)T_k(u^n)\Big)\,\mathrm{d}x \\
&\quad\le \left|\int_\Omega f^n T_k(u^n)\,\mathrm{d}x + \int_\Omega F\nabla T_k(u^n)\,\mathrm{d}x\right| \\
&\quad\le k\|f^n\|_{L^1(\Omega)} + C\int_\Omega |F|^{p\prime(x)}\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x \\
&\quad\le k\|f\|_{L^1(\Omega)} + C\int_\Omega |F|^{p\prime(x)}\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x.
\end{aligned} \tag{4.12}$$

Using (4.6) and (4.12), since $f\in L^1(\Omega)$ and $F\in(L^{p\prime(\cdot)}(\Omega))^N$, we deduce that

$$\int_\Omega |\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x \le kC. \tag{4.13}$$

Poincaré's inequality and (4.13) lead to

$$\begin{aligned}
k\,\mathrm{meas}\{x\in\Omega : |u^n|>k\} &= \int_{\{x\in\Omega : |u^n|>k\}} |T_k(u^n)|\,\mathrm{d}x \\
&\le \int_\Omega |T_k(u^n)|\,\mathrm{d}x \\
&\le \left(\frac{1}{p^-}+\frac{1}{p\prime^-}\right)\|T_k(u^n)\|_{L^{p(\cdot)}(\Omega)}(\mathrm{meas}(\Omega)+1)^{\frac{1}{p\prime^-}} \\
&\le C'\|T_k(u^n)\|_{L^{p(\cdot)}(\Omega)} \\
&\le Ck^{\frac{1}{\rho_1}},
\end{aligned} \tag{4.14}$$

where C does not depend on n and k and

$$\rho_1 = \begin{cases} p^-, & \text{if } \|\nabla T_k(u^n)\|_{(L^{p(\cdot)}(\Omega))^N} > 1, \\ p^+, & \text{if } \|\nabla T_k(u^n)\|_{(L^{p(\cdot)}(\Omega))^N} \le 1. \end{cases}$$

Then,

$$\mathrm{meas}\{x\in\Omega : |u^n|>k\} \le C\frac{1}{k^{1-\frac{1}{\rho_1}}}. \tag{4.15}$$

Taking now $\theta_k(u^n)$ as a test function in (4.8) and using the Young inequality and (1.2), we obtain

$$\begin{aligned}
&\int_\Omega A_n(x,v^n)\,|\nabla\theta_k(u^n)|^{p(x)}\,\mathrm{d}x \\
&\quad\le \int_\Omega A_n(x,v^n)\,|\nabla\theta_k(u^n)|^{p(x)}\,\mathrm{d}x + \int_\Omega H_1^n(x,u^n,\nabla u^n)\theta_k(u^n)\,\mathrm{d}x \\
&\quad\le \int_\Omega |f^n\theta_k(u^n)|\,\mathrm{d}x + \int_\Omega |F\nabla\theta_k(u^n)|\,\mathrm{d}x \\
&\quad\le \int_{\{|u^n|>k\}} |f^n|\,\mathrm{d}x + C\int_{\{|u^n|>k\}} |F|^{p\prime(x)}\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |\nabla\theta_k(u^n)|^{p(x)}\,\mathrm{d}x
\end{aligned} \tag{4.16}$$

and

$$\frac{\alpha}{2}\int_\Omega |\nabla\theta_k(u^n)|^{p(x)}\,\mathrm{d}x \le \int_{\{|u^n|>k\}} |f^n|\,\mathrm{d}x + C\int_{\{|u^n|>k\}} |F|^{p\prime(x)}\,\mathrm{d}x. \tag{4.17}$$

Using a classical argument (see e.g. [2]), for a subsequence still indexed by n from (4.15) and (4.17) we deduce that

$$u^n \longrightarrow u \text{ a.e. in } \Omega, \text{ strongly in } L^{r(.)}(\Omega) \text{ for any } r(x)\in\left[1, \frac{N(p(x)-1)}{N-p(x)}\right), \tag{4.18}$$

$$u^n \rightharpoonup u \text{ weakly in } W_0^{1,q(.)}(\Omega) \text{ for any } q(x)\in\left[1, \frac{N(p(x)-1)}{N-1}\right) \tag{4.19}$$

and

$$T_k(u^n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p(.)}(\Omega) \tag{4.20}$$

for any $k\ge 0$, where u is a measurable function defined on Ω which is finite a.e. in Ω. From (4.16) and (4.18), since $f\in L^1(\Omega)$ and $F\in (L^{p\prime(.)}(\Omega))^N$, we deduce that

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{\{k\le|u^n|\le k+1\}} A_n(x,v^n)|\nabla u^n|^{p(x)}\,\mathrm{d}x = 0. \tag{4.21}$$

To obtain the analogue of (4.13), (4.15) and (4.17) with v^n in place of u^n, we use $T_k(v^n)$ and $\theta_k(v^n)$ as test functions in (4.9). Indeed,

$$\begin{aligned}\int_\Omega \Big(B_n(x,v^n)|\nabla T_k(v^n)|^{p(x)} + H_2^n(x,v^n,\nabla v^n)T_k(v^n)\Big)\,\mathrm{d}x \\ = \gamma\int_\Omega |\nabla u^n|^{q_0(x)}T_k(v^n)\,\mathrm{d}x\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\int_\Omega \Big(B_n(x,v^n)|\nabla \theta_k(v^n)|^{p(x)} + H_2^n(x,v^n,\nabla v^n)\theta_k(v^n)\Big)\,\mathrm{d}x \\ = \gamma\int_\Omega |\nabla u^n|^{q_0(x)}\theta_k(v^n)\,\mathrm{d}x.\end{aligned} \tag{4.23}$$

It follows that (4.16) (since $1\le q_0(x) < \frac{N(p(x)-1)}{N-1}$) and (1.7) imply that

$$\int_\Omega B_n(x,v^n)|\nabla T_k(v^n)|^{p(x)}\,\mathrm{d}x \le C\gamma k \tag{4.24}$$

and

$$\int_\Omega B_n(x,v^n)|\nabla \theta_k(v^n)|^{p(x)}\,\mathrm{d}x \le C\gamma \tag{4.25}$$

uniformly with respect to n. Poincaré's inequality and (4.23) lead to

$$\operatorname{meas}\{x\in\Omega : |v^n|>k\} \le C\gamma\frac{1}{k^{1-\frac{1}{\rho_2}}}, \tag{4.26}$$

where C does not depend on n and k, and

$$\rho_2 = \begin{cases} p^-, & \text{if } \|\nabla T_k(v^n)\|_{(L^{p(.)}(\Omega))^N} > 1,\\ p^+, & \text{if } \|\nabla T_k(v^n)\|_{(L^{p(.)}(\Omega))^N} \le 1.\end{cases}$$

From (4.24) and (4.25) there exist a subsequence (still indexed by n) and a measurable positive function v such that

$$v^n \longrightarrow v \text{ a.e. in } \Omega, \text{ strongly in } L^{r(.)}(\Omega) \text{ for any } r(x) \in \left[1, \frac{N(p(x)-1)}{N-p(x)}\right), \tag{4.27}$$

$$v^n \rightharpoonup v \text{ weakly in } W_0^{1,q(.)}(\Omega) \text{ for any } q(x) \in \left[1, \frac{N(p(x)-1)}{N-1}\right) \tag{4.28}$$

and

$$T_k(v^n) \rightharpoonup T_k(v) \text{ weakly in } W_0^{1,p(.)}(\Omega). \tag{4.29}$$

The proof that v is less than or equal to m is an easy task. Indeed, using $T_{2m}^+(v^n) - T_m^+(v^n)$ as a test function in (4.9), and by (4.7) and (4.17) we obtain

$$b^{\rho_3}\left(m - \frac{1}{n}\right) \int_\Omega |\nabla(T_{2m}^+(v^n) - T_m^+(v^n))|^{p(x)} \,\mathrm{d}x \le C\gamma m, \tag{4.30}$$

where

$$\rho_3 = \begin{cases} p^-, & \text{if } b(m - \frac{1}{n}) > 1, \\ p^+, & \text{if } b(m - \frac{1}{n}) \le 1. \end{cases}$$

Then, in view of (1.5) and with the help of Poincaré's inequality, we deduce that $T_{2m}^+(v) - T_m^+(v) = 0$ a.e. in Ω, that is,

$$0 \le v \le m \quad \text{a.e. in } \Omega. \tag{4.31}$$

To obtain the analogue of (4.21) for v^n, we use (4.19) and (4.23). Indeed, for $\varepsilon(x) > 0$ such that $q_0(x) + \varepsilon(x) < \frac{N(p(x)-1)}{N-1}$ and with the help of Hölder's inequality we deduce that

$$\begin{aligned} &\int_\Omega B_n(x, v^n)|\nabla\theta_k(v^n)|^{p(x)} \,\mathrm{d}x \\ &\quad \le \gamma \| \, |\nabla u^n|^{q_0(x)} \, \|_{L^{\frac{q_0(.)+\varepsilon(.)}{q_0(.)}}(\Omega)} \left(\int_\Omega |\theta_k(v^n)|^{\frac{q_0(x)+\varepsilon(x)}{\varepsilon(x)}} \,\mathrm{d}x \right)^{\frac{1}{\rho_4}} \\ &\quad \le \gamma C \left(\int_\Omega |\theta_k(v^n)|^{\frac{q_0(x)+\varepsilon(x)}{\varepsilon(x)}} \,\mathrm{d}x \right)^{\frac{1}{\rho_4}}, \end{aligned} \tag{4.32}$$

where

$$\rho_4 = \begin{cases} \dfrac{q_0^+ + \varepsilon^+}{\varepsilon^-}, & \text{if } \|\theta_k(v^n)\|_{L^{\frac{q_0(.)+\varepsilon(.)}{q_0(.)}}(\Omega)} \le 1, \\ \dfrac{q_0^- + \varepsilon^-}{\varepsilon^+}, & \text{if } \|\theta_k(v^n)\|_{L^{\frac{q_0(.)+\varepsilon(.)}{q_0(.)}}(\Omega)} > 1. \end{cases}$$

Since θ_k is a continuous and bounded function ($\|\theta_k\|_{L^\infty(\mathbb{R})} = 1$), from (4.28) we deduce that $\theta_k(v^n)$ converges to $\theta_k(v)$ strongly in $L^{\frac{q_0(.)+\varepsilon(.)}{\varepsilon(.)}}(\Omega)$. Now, (4.31) implies that for all $k > m$ we have $\theta_k(v) = 0$ a.e. in Ω. From (4.32) we conclude that for all $k > m$,

$$\lim_{n \to +\infty} \int_{\{k \le |v^n| \le k+1\}} B_n(x, v^n)|\nabla v^n|^{p(x)} \,\mathrm{d}x = 0. \tag{4.33}$$

Step 3 In this step we prove the strong convergence result for u^n.

Lemma 4.2 *For fixed $k \geq 0$ and $l > m$, we have*

$$\lim_{n\to+\infty} \int_\Omega \chi_{\{0\leq v^n\leq l\}} \Big[|\nabla T_k(u^n)|^{p(x)-2} \nabla T_k(u^n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u)\Big] \times \\ \times [\nabla T_k(u^n) - \nabla T_k(u)]\,\mathrm{d}x = 0 \tag{4.34}$$

and

$$\chi_{\{0\leq v^n\leq l\}} \nabla T_k(u^n) \longrightarrow \nabla T_k(u) \text{ strongly in } (L^{p(.)}(\Omega))^N \tag{4.35}$$

and

$$u^n \longrightarrow u \text{ strongly in } W_0^{1,q(.)}(\Omega) \text{ for any } q(x) \in \Big[1, \frac{N(p(x)-1)}{N-1}\Big), \tag{4.36}$$

as n tends to $+\infty$.

Proof. From now onward we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots$, various real-valued functions which converge to 0 as n tends to infinity. Let $\varphi_1(s) = s\exp(\gamma_1 s^2)$, where $\gamma_1 = (\frac{\lambda_1}{2\alpha})^2$. It is well-known that for every $s \in \mathbb{R}$

$$\varphi_1'(s) - \frac{\lambda_1}{\alpha}|\varphi_1(s)| \geq \frac{1}{2}. \tag{4.37}$$

For $h > k > 0$ we set

$$w^n = T_{2k}\Big(u^n - T_h(u^n) + T_k(u^n) - T_k(u)\Big).$$

When we use in (4.8) the test function $\varphi_1(w^n)h_l(v^n)$, it follows that for any $l > m$,

$$\begin{aligned} &\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n \varphi_1'(w^n)\nabla w^n h_l(v^n)\,\mathrm{d}x \\ &\quad + \int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n h_l'(v^n)\nabla v^n \varphi_1(w^n)\,\mathrm{d}x \\ &\quad + \int_\Omega H_1^n(x,u^n,\nabla u^n)\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x \\ &= \int_\Omega f^n\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x + \int_\Omega F\varphi_1'(w^n)\nabla w^n h_l(v^n)\,\mathrm{d}x \\ &\quad + \int_\Omega F h_l'(v^n)\nabla v^n \varphi_1(w^n)\,\mathrm{d}x. \end{aligned} \tag{4.38}$$

Choosing $M = 4k + h$, we have $\nabla w^n = 0$ on the set $\{|u^n| > M\}$ and $H_1^n(x,u^n,\nabla u^n)\varphi_1(w^n)h_l(v^n) \geq 0$ on the set $\{|u^n| > k\}$. Then,

$$\begin{aligned} &\int_\Omega A_n(x,v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n h_l(v^n)\,\mathrm{d}x \\ &\quad + \int_\Omega A_n(x,v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)h_l'(v^n)\nabla v^n\varphi_1(w^n)\,\mathrm{d}x \\ &\quad + \int_{\{|u^n|\leq k\}} H_1^n(x,u^n,\nabla u^n)\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x \\ &\leq \int_\Omega f^n\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x + \int_\Omega F\varphi_1'(w^n)\nabla w^n h_l(v^n)\,\mathrm{d}x \\ &\quad + \int_\Omega F h_l'(v^n)\nabla v^n\varphi_1(w^n)\,\mathrm{d}x. \end{aligned} \tag{4.39}$$

Now, let us study the first term in (4.39). We can rewrite it as follows:

$$\begin{aligned}
&\int_\Omega A_n(x,v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n h_l(v^n)\,\mathrm{d}x\\
&=\int_{\{|u^n|\le k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)\varphi_1'(w^n)\nabla T_{2k}(u^n-T_k(u))\,\mathrm{d}x\\
&\quad+\int_{\{|u^n|>k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n\,\mathrm{d}x.
\end{aligned}\tag{4.40}$$

Since $|u^n-T_k(u)|\le 2k$ on $\{|u^n|\le k\}$, we infer that

$$\begin{aligned}
&\int_{\{|u^n|\le k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)\varphi_1'(w^n)\nabla T_{2k}(u^n-T_k(u))\,\mathrm{d}x\\
&=\int_{\{|u^n|\le k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)\varphi_1'(w^n)[\nabla T_k(u^n)-\nabla T_k(u)]\,\mathrm{d}x.
\end{aligned}\tag{4.41}$$

For the second term on the right-hand side of (4.40), we take $\rho_1^n=u^n-T_h(u^n)+T_k(u^n)-T_k(u)$. Then,

$$\begin{aligned}
&\int_{\{|u^n|>k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n\,\mathrm{d}x\\
&=\int_{\{|u^n|>k\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla\rho_1^n\,\mathrm{d}x\\
&=-\int_{\{k<|u^n|\le h\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla T_k(u)\,\mathrm{d}x\\
&\quad+\int_{\{|u^n|>h\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla(u^n-T_k(u))\,\mathrm{d}x\\
&=-\int_{\{k<|u^n|\le h\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla T_k(u)\,\mathrm{d}x\\
&\quad+\int_{\{|u^n|>h\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla u^n\,\mathrm{d}x\\
&\quad-\int_{\{|u^n|>h\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla T_k(u)\,\mathrm{d}x\\
&=-\int_{\{k<|u^n|\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla T_k(u)\,\mathrm{d}x\\
&\quad+\int_{\{|u^n|>h\}\cap\{|\rho_1^n|\le 2k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)}\,\varphi_1'(w^n)\,\mathrm{d}x.
\end{aligned}$$

Since $1\le\varphi_1'(w^n)\le\varphi_1'(2k)$, we obtain

$$\begin{aligned}
&\int_{\{|u^n|>k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n\,\mathrm{d}x\\
&\qquad\ge -\varphi_1'(2k)\int_{\{|u^n|>k\}} A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-1}\,|\nabla T_k(u)|\,\mathrm{d}x.
\end{aligned}\tag{4.42}$$

Combining (4.40)–(4.42), we deduce that

$$\begin{aligned} &\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_M(u^n)\right|^{p(x)-2}\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n\,\mathrm{d}x \\ &\quad\geq \int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u^n)\right|^{p(x)-2}\nabla T_k(u^n)\varphi_1'(w^n)[\nabla T_k(u^n)-\nabla T_k(u)]\,\mathrm{d}x \\ &\quad\quad-\varphi_1'(2k)\int_{\{|u^n|>k\}}A(x,T_{l+1}(v^n))h_l(v^n)\left|\nabla T_M(u^n)\right|^{p(x)-1}\left|\nabla T_k(u)\right|\mathrm{d}x. \end{aligned} \tag{4.43}$$

The sequence $\left(A_n(x,T_{l+1}(v^n))h_l(v^n)\left|\nabla T_M(u^n)\right|^{p(x)-2}\nabla T_M(u^n)\right)_n$ is bounded in $(L^{p'(x)}(\Omega))^N$ and $\nabla T_k(u)\chi_{\{|u^n|>k\}}\longrightarrow 0$ in $(L^{p(x)}(\Omega))^N$. Then,

$$\int_\Omega A(x,T_{l+1}(v^n))h_l(v^n)\left|\nabla T_M(u^n)\right|^{p(x)-1}\left|\nabla T_k(u)\right|\chi_{\{|u^n|>k\}}\,\mathrm{d}x=\varepsilon_1(n). \tag{4.44}$$

We conclude that

$$\begin{aligned} &\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_M(u^n)\right|^{p(x)-2}\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n\,\mathrm{d}x \\ &\quad\geq \varepsilon_1(n)+\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u^n)\right|^{p(x)-2}\nabla T_k(u^n)\times \\ &\qquad\qquad\times[\nabla T_k(u^n)-\nabla T_k(u)]\varphi_1'(w^n)\,\mathrm{d}x. \end{aligned} \tag{4.45}$$

On the other hand, the term on the right-hand side of (4.45) can be written as

$$\begin{aligned} &\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u^n)\right|^{p(x)-2}\nabla T_k(u^n)[\nabla T_k(u^n)-\nabla T_k(u)]\varphi_1'(w^n)\,\mathrm{d}x \\ &=\int_\Omega A_n(x,v^n)h_l(v^n)\Big[\left|\nabla T_k(u^n)\right|^{p(x)-2}\nabla T_k(u^n)-\left|\nabla T_k(u)\right|^{p(x)-2}\nabla T_k(u)\Big]\times \\ &\qquad\qquad\times[\nabla T_k(u^n)-\nabla T_k(u)]\varphi_1'(w^n)\,\mathrm{d}x \\ &\quad+\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u)\right|^{p(x)-2}\nabla T_k(u)\nabla T_k(u^n)\varphi_1'(T_k(u^n)-T_k(u))\,\mathrm{d}x \\ &\quad-\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u)\right|^{p(x)}\varphi_1'(w^n)\,\mathrm{d}x. \end{aligned} \tag{4.46}$$

According to (1.2) and using the fact that $\varphi_1'(0)=1$, for any $l>m$ we have

$$\begin{aligned} &A_n(x,T_{l+1}(v^n))h_l(v^n)\left|\nabla T_k(u)\right|^{p(x)-2}\nabla T_k(u)\varphi_1'(T_k(u^n)-T_k(u)) \\ &\quad\longrightarrow A(x,v)\left|\nabla T_k(u)\right|^{p(x)-2}\nabla T_k(u) \end{aligned}$$

strongly in $(L^{p'(x)}(\Omega))^N$. Since $\nabla T_k(u^n)\rightharpoonup\nabla T_k(u)$ weakly in $(L^{p'(x)}(\Omega))^N$, we obtain

$$\begin{aligned} &\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u)\right|^{p(x)-2}\nabla T_k(u)\nabla T_k(u^n)\varphi_1'(T_k(u^n)-T_k(u))\,\mathrm{d}x \\ &\quad=\int_\Omega A(x,v)\left|\nabla T_k(u)\right|^{p(x)}\,\mathrm{d}x+\varepsilon_2(n). \end{aligned} \tag{4.47}$$

On the other hand, we have $A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u)\right|^{p(x)}\varphi_1'(w^n)\longrightarrow A(x,v)\left|\nabla T_k(u)\right|^{p(x)}$. Then,

$$\int_\Omega A_n(x,v^n)h_l(v^n)\left|\nabla T_k(u)\right|^{p(x)}\varphi_1'(w^n)\,\mathrm{d}x=\int_\Omega A(x,v)\left|\nabla T_k(u)\right|^{p(x)}\,\mathrm{d}x+\varepsilon_3(n). \tag{4.48}$$

Combining (4.45)–(4.48), we obtain

$$\begin{aligned}\int_\Omega & A_n(x,v^n)h_l(v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\varphi_1'(w^n)\nabla w^n\,\mathrm{d}x\\ &\geq \int_\Omega A_n(x,v^n)h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)\right]\times\\ &\qquad\qquad \times[\nabla T_k(u^n)-\nabla T_k(u)]\varphi_1'(w^n)\,\mathrm{d}x+\varepsilon_4(n).\end{aligned}\tag{4.49}$$

We will now study the second term in (4.39). Using (1.3) and Hölder's inequality we deduce that

$$\begin{aligned}&\left|\int_\Omega A_n(x,v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\nabla h_l(v^n)\varphi_1(w^n)\,\mathrm{d}x\right|\\ &\quad\leq \varphi_1(2k)\|A\|_{L^\infty(\Omega\times(-(l+1),(l+1)))}\|\,|\nabla T_M(u^n)|^{p(x)-1}\,\|_{L^{p'(\cdot)}(\Omega)}\|\nabla h_l(v^n)\|_{(L^{p(\cdot)}(\Omega))^N}.\end{aligned}\tag{4.50}$$

In order to prove that $\nabla h_l(v^n)$ converges to zero strongly in $(L^{p(\cdot)}(\Omega))^N$ as n tends to $+\infty$, we use (1.4) and (4.33). Then, for any $l>m$ we obtain

$$\lim_{n\to+\infty}\int_\Omega|\nabla\theta_l(v^n)|^{p(x)}\,\mathrm{d}x=0.\tag{4.51}$$

Using now (1.3), (4.50), (4.51), since $|\nabla h_l(v^n)|=|\nabla\theta_l(v^n)|$ a.e. in Ω (see (3.1)), for any $l>m$ we have

$$\lim_{n\to+\infty}\int_\Omega A_n(x,v^n)\,|\nabla T_M(u^n)|^{p(x)-2}\,\nabla T_M(u^n)\nabla h_l(v^n)\varphi_1(w^n)\,\mathrm{d}x=0.\tag{4.52}$$

We can pass now to the study of the third term in (4.39). By (1.2) and (1.6), we can write

$$\begin{aligned}&\left|\int_{\{|u^n|\leq k\}}H_1^n(x,u^n,\nabla u^n)\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x\right|\\ &\quad\leq\int_{\{|u^n|\leq k\}}\lambda_1\left[h_{1,k}(x)+|\nabla T_k(u^n)|^{p(x)}\right]|\varphi_1(w^n)|\,h_l(v^n)\,\mathrm{d}x\\ &\quad\leq\lambda_1\int_{\{|u^n|\leq k\}}h_{1,k}(x)\,|\varphi_1(T_{2k}(T_k(u^n)-T_k(u)))|\,\mathrm{d}x\\ &\qquad+\frac{\lambda_1}{\alpha}\int_\Omega A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)}\,|\varphi_1(w^n)|\,\mathrm{d}x.\end{aligned}\tag{4.53}$$

Since $h_{1,k}(x)$ belongs to $L^1(\Omega)$ and $|\varphi_1(T_{2k}(T_k(u^n)-T_k(u)))|\to 0$ a.e. in Ω and weakly-* in $L^\infty(\Omega)$, we have

$$\lambda_1\int_{\{|u^n|\leq k\}}h_{1,k}(x)\,|\varphi_1(T_{2k}(T_k(u^n)-T_k(u)))|\,\mathrm{d}x=\varepsilon_5(n).\tag{4.54}$$

The second term on the right-hand side of the above inequality (4.53) can be written as

$$\begin{aligned}\int_\Omega & A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)}\,|\varphi_1(w^n)|\,\mathrm{d}x\\ &=\int_\Omega A_n(x,v^n)h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)\right]\times\\ &\qquad\qquad\times[\nabla T_k(u^n)-\nabla T_k(u)]|\varphi_1(w^n)|\,\mathrm{d}x\qquad(4.55)\\ &\quad+\int_\Omega A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)\nabla T_k(u)\,|\varphi_1(w^n)|\,\mathrm{d}x\\ &\quad+\int_\Omega A_n(x,v^n)h_l(v^n)\,|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)\,[\nabla T_k(u^n)-\nabla T_k(u)]\,|\varphi_1(w^n)|\,\mathrm{d}x.\end{aligned}$$

As a consequence of the above convergence results, we can pass to the limit in (4.53) and (4.55) as n tends to $+\infty$. This for any $k > 0$ yields

$$\begin{aligned} &\left|\int_{\{|u^n|\le k\}} H_1^n(x,u^n,\nabla u^n)\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x\right| \\ &\quad\le \frac{\lambda_1}{\alpha}\int_\Omega A_n(x,v^n)h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times \\ &\qquad\qquad \times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]|\varphi_1(w^n)|\,\mathrm{d}x+\varepsilon_6(n). \end{aligned} \tag{4.56}$$

From (4.39), (4.52) and (4.56), we have

$$\begin{aligned} &\int_\Omega A_n(x,v^n)h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times \\ &\qquad\qquad \times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]\left(\varphi_1'(w^n)-\frac{\lambda_1}{\alpha}|\varphi_1(w^n)|\right)\mathrm{d}x \\ &\quad\le \int_\Omega f^n\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x+\int_\Omega F\varphi_1'(w^n)h_l(v^n)\nabla w^n\,\mathrm{d}x \\ &\qquad\qquad +\int_\Omega F h_l'(v^n)\nabla v^n\varphi_1(w^n)\,\mathrm{d}x+\varepsilon_7(n). \end{aligned} \tag{4.57}$$

Now, let us study the terms on the right-hand side of (4.57). We have $w^n \rightharpoonup T_{2k}(u-T_h(u))$ weakly in $W_0^{1,p(x)}(\Omega)$ and weakly-* in $L^\infty(\Omega)$. So,

$$\int_\Omega f^n\varphi_1(w^n)h_l(v^n)\,\mathrm{d}x=\int_\Omega f\varphi_1(T_{2k}(u-T_h(u)))\,\mathrm{d}x+\varepsilon_8(n) \tag{4.58}$$

and

$$\int_\Omega F\varphi_1'(w^n)\nabla w^n h_l(v^n)\,\mathrm{d}x=\int_\Omega F\varphi_1'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))\,\mathrm{d}x+\varepsilon_9(n). \tag{4.59}$$

Due to (4.33) and Hölder's inequality for any $l > m$ we have

$$\begin{aligned} &\lim_{n\to+\infty}\left|\int_\Omega F\nabla h_l(v^n)\varphi_1(w^n)\,\mathrm{d}x\right| \\ &\qquad \le \varphi_1(2k)\|F\|_{(L^{p'(x)}(\Omega))^N}\lim_{n\to+\infty}\left(\int_{\{l\le|v^n|\le l+1\}}|\nabla v^n|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{\rho_5}}=0, \end{aligned} \tag{4.60}$$

where

$$\rho_5=\begin{cases} p^-, & \text{if } \|\nabla h_l(v^n)\|_{(L^{p(\cdot)}(\Omega))^N}>1,\\ p^+, & \text{if } \|\nabla h_l(v^n)\|_{(L^{p(\cdot)}(\Omega))^N}\le 1.\end{cases}$$

We are able to pass to the limit as $n\to+\infty$ in the last inequality (4.57) and we obtain

$$\begin{aligned} &\limsup_{n\to+\infty}\int_\Omega A_n(x,v^n)h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times \\ &\qquad\qquad \times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]\mathrm{d}x \\ &\quad\le 2\int_\Omega f\varphi_1(T_{2k}(u-T_h(u)))\,\mathrm{d}x+\int_\Omega F\varphi_1'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))\,\mathrm{d}x. \end{aligned} \tag{4.61}$$

Finally, we deal with the last term. Let us observe that if we take $\varphi_1(T_{2k}(u^n - T_h(u^n)))$ as a test function in (4.8), we obtain

$$\begin{aligned}
&\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n \varphi_1'(T_{2k}(u^n - T_h(u^n)))\nabla T_{2k}(u^n - T_h(u^n))\,\mathrm{d}x \\
&\quad + \int_\Omega H_1^n(x,u^n,\nabla u^n)\varphi_1(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x \\
&\quad = \int_\Omega f^n\varphi_1(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x + \int_\Omega F\varphi_1'(T_{2k}(u^n - T_h(u^n)))\nabla T_{2k}(u^n - T_h(u^n))\,\mathrm{d}x.
\end{aligned} \tag{4.62}$$

From relations (1.2) and (1.7), we obtain

$$\begin{aligned}
&\alpha\int_{\{h\le|u^n|\le 2k+h\}} |\nabla u^n|^{p(x)}\,\varphi_1'(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x \\
&\le \int_\Omega f^n\varphi_1(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x + \int_{\{h\le|u^n|\le 2k+h\}} F\nabla u^n\varphi_1'(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x.
\end{aligned} \tag{4.63}$$

Then, Young's inequality enables us to get

$$\begin{aligned}
&\int_{\{h\le|u^n|\le 2k+h\}} F\nabla u^n\varphi_1'(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x \\
&\quad \le C\int_{\{h\le|u^n|\le 2k+h\}} |F|^{p'(x)}\,\mathrm{d}x \\
&\qquad + \frac{\alpha}{2}\int_{\{h\le|u^n|\le 2k+h\}} |\nabla u^n|^{p(x)}\varphi_1'(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x.
\end{aligned} \tag{4.64}$$

Therefore, from (4.63) we obtain

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{h\le|u^n|\le 2k+h\}} |\nabla u^n|^{p(x)}\,\varphi_1'(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x \\
&\quad \le \int_\Omega f^n\varphi_1(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x + C\int_{\{h\le|u^n|\}} |F|^{p'(x)}\,\mathrm{d}x.
\end{aligned} \tag{4.65}$$

Using the fact that $\varphi_1' \ge 1$, we have

$$\begin{aligned}
&\int_\Omega |\nabla T_{2k}(u - T_h(u))|^{p(x)}\varphi_1'(T_{2k}(u - T_h(u)))\,\mathrm{d}x \\
&\quad \le C\int_\Omega |\nabla T_{2k}(u - T_h(u))|^{p(x)}\,\mathrm{d}x \\
&\quad \le C\liminf_{n\to+\infty}\int_\Omega |\nabla T_{2k}(u^n - T_h(u^n))|^{p(x)}\,\mathrm{d}x \\
&\quad \le C\liminf_{n\to+\infty}\int_\Omega |\nabla T_{2k}(u^n - T_h(u^n))|^{p(x)}\varphi_1'(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x \\
&\quad \le \frac{2C}{\alpha}\liminf_{n\to+\infty}\int_\Omega f^n\varphi_1(T_{2k}(u^n - T_h(u^n)))\,\mathrm{d}x + C\liminf_{n\to+\infty}\int_{\{h\le|u^n|\}} |F|^{p'(x)}\,\mathrm{d}x.
\end{aligned} \tag{4.66}$$

As a consequence of the above convergence results, we can pass to the limit: first as n tends to $+\infty$ and then as h tends to $+\infty$; we obtain

$$\limsup_{h\to+\infty}\int_{\{h\le|u|\le 2k+h\}} |\nabla u|^{p(x)}\,\varphi_1'(T_{2k}(u - T_h(u)))\,\mathrm{d}x = 0.$$

Hence,

$$\lim_{h\to+\infty}\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi_1'(T_{2k}(u-T_h(u)))\,\mathrm{d}x=0.$$

Therefore, by (4.61) letting h tend to $+\infty$, we deduce that

$$\lim_{n\to+\infty}\int_\Omega A_n(x,v^n)h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times \\ \times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]\mathrm{d}x=0.$$

By (1.2) we have

$$\lim_{n\to+\infty}\int_\Omega h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times \\ \times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]\mathrm{d}x=0. \tag{4.67}$$

Then,

$$\lim_{n\to+\infty}\int_\Omega \chi_{\{0\le v^n\le l\}}\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times \\ \times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]\mathrm{d}x=0. \tag{4.68}$$

To prove (4.35), we will use the following well-known inequalities, which hold for any two real numbers a, b and $p>1$:

$$(a\,|a|^{p-2}-b\,|b|^{p-2})(a-b)\ge c(p)\begin{cases}|a-b|^p\,, & \text{if } p\ge 2,\\ \dfrac{|a-b|^2}{(|a|+|b|)^{2-p}}, & \text{if } 1<p<2,\end{cases} \tag{4.69}$$

where $c(p)=2^{2-p}$ when $p\ge 2$ and $c(p)=p-1$ when $1<p<2$.

Let $X_k^n:=(h_l(v^n))^{\frac{1}{p(x)}}\nabla T_k(u^n)$ and $Y_k^n:=(h_l(v^n))^{\frac{1}{p(x)}}\nabla T_k(u)$. Then, we have

$$\begin{aligned}2^{2-p^+}&\int_{\{x\in\Omega\,:\,p(x)\ge 2\}}\left|X_k^n-Y_k^n\right|^{p(x)}\mathrm{d}x\\ &\le\int_{\{x\in\Omega\,:\,p(x)\ge 2\}}\left(X_k^n\,|X_k^n|^{p(x)-2}-Y_k^n\,|Y_k^n|^{p(x)-2}\right)(X_k^n-Y_k^n)\,\mathrm{d}x\\ &\le\int_\Omega\left(X_k^n\,|X_k^n|^{p(x)-2}-Y_k^n\,|Y_k^n|^{p(x)-2}\right)(X_k^n-Y_k^n)\,\mathrm{d}x\\ &\le\int_\Omega h_l(v^n)\left[|\nabla T_k(u^n)|^{p(x)-2}\nabla T_k(u^n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\right]\times\\ &\qquad\times\left[\nabla T_k(u^n)-\nabla T_k(u)\right]\mathrm{d}x=:I(n).\end{aligned} \tag{4.70}$$

Using (4.68), we obtain

$$\lim_{n\to+\infty}I(n)=0. \tag{4.71}$$

Using (4.70) and (4.71), we obtain

$$\lim_{n\to+\infty}\int_{\{x\in\Omega\,:\,p(x)\ge 2\}}|X_k^n-Y_k^n|^{p(x)}\,\mathrm{d}x=0. \tag{4.72}$$

On the set where $2-\frac{1}{N}<p(x)<2$, we employ (4.69) as follows:

$$\begin{aligned}
&\int_{\{x\in\Omega\,:\,2-\frac{1}{N}<p(x)<2\}}\left|X_k^n-Y_k^n\right|^{p(x)}\,\mathrm{d}x\\
&\leq\int_{\{x\in\Omega\,:\,2-\frac{1}{N}<p(x)<2\}}\frac{\chi_{\{0\leq v^n<l+1\}}\left|X_k^n-Y_k^n\right|^{p(x)}}{\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{\frac{p(x)(2-p(x))}{2}}}\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{\frac{p(x)(2-p(x))}{2}}\,\mathrm{d}x\\
&\leq 2\left\|\frac{\chi_{\{0\leq v^n<l+1\}}\left|X_k^n-Y_k^n\right|^{p(x)}}{\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{\frac{p(x)(2-p(x))}{2}}}\right\|_{(L^{\frac{2}{p(x)}}(\Omega))^N}\left\|\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{\frac{p(x)(2-p(x))}{2}}\right\|_{(L^{\frac{2}{2-p(x)}}(\Omega))^N}\\
&\leq 2\max\left\{\left(\int_\Omega\frac{\chi_{\{0\leq v^n<l+1\}}\left|X_k^n-Y_k^n\right|^2}{\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{2-p(x)}}\,\mathrm{d}x\right)^{\frac{p^-}{2}},\left(\int_\Omega\frac{\chi_{\{0\leq v^n<l+1\}}\left|X_k^n-Y_k^n\right|^2}{\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{2-p(x)}}\,\mathrm{d}x\right)^{\frac{p^+}{2}}\right\}\times\\
&\qquad\times\max\left\{\left(\int_\Omega\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{p(x)}\,\mathrm{d}x\right)^{\frac{2-p^+}{2}},\left(\int_\Omega\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{p(x)}\,\mathrm{d}x\right)^{\frac{2-p^-}{2}}\right\}\\
&\leq 2\max\left\{(p^--1)^{-\frac{p^-}{2}}\left(I(n)\right)^{\frac{p^-}{2}},(p^--1)^{-\frac{p^+}{2}}\left(I(n)\right)^{\frac{p^+}{2}}\right\}\times\\
&\qquad\times\max\left\{\left(\int_\Omega\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{p(x)}\,\mathrm{d}x\right)^{\frac{2-p^+}{2}},\left(\int_\Omega\left(\left|X_k^n\right|+\left|Y_k^n\right|\right)^{p(x)}\,\mathrm{d}x\right)^{\frac{2-p^-}{2}}\right\}.
\end{aligned}\tag{4.73}$$

Since $I(n)\to 0$ as $n\to+\infty$ and $(X_k^n)_n$ is bounded in $(L^{p(x)}(\Omega))^N$, by (4.73) we have

$$\lim_{n\to+\infty}\int_{\{x\in\Omega\,:\,2-\frac{1}{N}<p(x)<2\}}\left|X_k^n-Y_k^n\right|^{p(x)}\,\mathrm{d}x=0.\tag{4.74}$$

Using (4.72) and (4.74) we deduce that

$$\lim_{n\to+\infty}\int_\Omega\left|X_k^n-Y_k^n\right|^{p(x)}\,\mathrm{d}x=0.\tag{4.75}$$

Because $Y_k^n:=(h_l(v^n))^{\frac{1}{p(x)}}\nabla T_k(u)\to\nabla T_k(u)$ strongly in $(L^{p(\cdot)}(\Omega))^N$ (since $l>m$ and $h_l(v)=1$ a.e. in Ω) using (4.75), we have

$$(h_l(v^n))^{\frac{1}{p(x)}}\nabla T_k(u^n)\to\nabla T_k(u)\text{ strongly in }(L^{p(\cdot)}(\Omega))^N.\tag{4.76}$$

Now, we can write

$$\int_\Omega\chi_{\{0\leq v^n\leq l\}}\left|\nabla T_k(u^n)\right|^{p(x)}\,\mathrm{d}x\leq\int_\Omega h_l(v^n)\left|\nabla T_k(u^n)\right|^{p(x)}\,\mathrm{d}x.\tag{4.77}$$

Since $\chi_{\{0\leq v^n\leq l\}}\left|\nabla T_k(u^n)\right|^{p(x)}=\chi_{\{0\leq v^n\leq l\}}\frac{1}{h_l(v^n)}h_l(v^n)\left|\nabla T_k(u^n)\right|^{p(x)}$, by (4.76) we have

$$\chi_{\{0\leq v^n\leq l\}}\left|\nabla T_k(u^n)\right|^{p(x)}\to\left|\nabla T_k(u)\right|^{p(x)}\ \text{ a.e. in }\Omega;$$

note that $h_l(v) = 1$ for any $l > m$. Now, using Fatou's lemma and (4.77), for any $l > m$ we obtain

$$\begin{aligned}\int_\Omega |\nabla T_k(u)|^{p(x)}\,\mathrm{d}x &\le \liminf_{n\to+\infty}\int_\Omega \chi_{\{0\le v^n\le l\}}\,|\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x\\ &\le \limsup_{n\to+\infty}\int_\Omega \chi_{\{0\le v^n\le l\}}\,|\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x\\ &\le \lim_{n\to+\infty}\int_\Omega h_l(v^n)\,|\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x\\ &\le \int_\Omega |\nabla T_k(u)|^{p(x)}\,\mathrm{d}x.\end{aligned} \tag{4.78}$$

Then, for all $l > m$ we have

$$\lim_{n\to+\infty}\int_\Omega \chi_{\{0\le v^n\le l\}}\,|\nabla T_k(u^n)|^{p(x)}\,\mathrm{d}x = \int_\Omega |\nabla T_k(u)|^{p(x)}\,\mathrm{d}x. \tag{4.79}$$

Using Lieb's lemma we obtain (4.35).

For (4.36) we can write

$$\begin{aligned}&\int_\Omega |\nabla u^n|^{q(x)}\,\mathrm{d}x\\ &= \int_{\{|u^n|\le k\}} |\nabla u^n|^{q(x)}\,\mathrm{d}x + \int_{\{|u^n|>k\}} |\nabla u^n|^{q(x)}\,\mathrm{d}x\\ &= \int_\Omega |\nabla T_k(u^n)|^{q(x)}\,\mathrm{d}x + \int_{\{|u^n|>k\}} |\nabla u^n|^{q(x)}\,\mathrm{d}x\\ &= \int_\Omega |\nabla T_k(u^n)|^{q(x)}\,\chi_{\{0\le v^n\le l\}}\,\mathrm{d}x + \int_\Omega |\nabla T_k(u^n)|^{q(x)}\,\chi_{\{v^n>l\}}\,\mathrm{d}x + \int_{\{|u^n|>k\}} |\nabla u^n|^{q(x)}\,\mathrm{d}x.\end{aligned} \tag{4.80}$$

Let $\varepsilon(x) > 0$ be such that $q(x) + \varepsilon(x) < \frac{N(p(x)-1)}{N-1}$. By Hölder's inequality and (4.18) we obtain

$$\int_{\{|u^n|>k\}} |\nabla u^n|^{q(x)}\,\mathrm{d}x \le C\big(\mathrm{meas}\,\{|u^n|>k\}\big)^{\rho_6}, \tag{4.81}$$

where

$$\rho_6 = \begin{cases}\dfrac{\varepsilon^+}{q^- + \varepsilon^-}, & \text{if meas}\,\{|u^n|>k\}\ge 1,\\ \dfrac{\varepsilon^-}{q^+ + \varepsilon^+}, & \text{if meas}\,\{|u^n|>k\}< 1.\end{cases}$$

Using (4.15), passing to the limit in (4.81) as n tends to $+\infty$ and as k tends to $+\infty$, and since $u \in W_0^{1,q(.)}(\Omega)$, we deduce that

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{\{|u^n|>k\}} |\nabla u^n|^{q(x)}\,\mathrm{d}x = 0. \tag{4.82}$$

To obtain the analogue of (4.81) we use (4.26) and the fact that $T_k(u) \in W_0^{1,p(.)}(\Omega) \subset W_0^{1,q(.)}(\Omega)$ (when $q(x) < p(x)$); for all $l > m$ we have

$$\lim_{n\to+\infty}\int_\Omega \chi_{\{v^n>l\}}\,|\nabla T_k(u^n)|^{q(x)}\,\mathrm{d}x = 0. \tag{4.83}$$

Using (4.36), (4.81), (4.82), (4.83) and passing to the limit in (4.80) as n tends to $+\infty$, k tends to $+\infty$ and as l tends to $+\infty$, we obtain

$$\lim_{n\to+\infty}\int_\Omega |\nabla u^n|^{q(x)}\,\mathrm{d}x = \int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x. \tag{4.84}$$

Using Lieb's lemma we obtain (4.36). □

Step 4 In this step we prove the convergence result for v^n.

Lemma 4.3 *For a fixed* $0 \le k < m$, *we have*

$$\lim_{n\to+\infty}\int_\Omega B_n(x,T_k(v^n))\Big[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n) - |\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\Big]\times \\ \times[\nabla T_k(v^n)-\nabla T_k(v)]\,\mathrm{d}x = 0, \tag{4.85}$$

$$T_k(v^n) \longrightarrow T_k(v) \textit{ strongly in } W_0^{1,p(.)}(\Omega) \tag{4.86}$$

and

$$\nabla v^n \longrightarrow \nabla v \textit{ strongly in } L^{q(.)}(\Omega) \textit{ for any } q(x)\in\Big[1,\frac{N(p(x)-1)}{N-1}\Big), \tag{4.87}$$

as n *tends to* $+\infty$.

Proof. In the sequel, we denote by $\epsilon_i(n)$, $i = 1, 2, \ldots$, various real-valued functions which converge to 0 as n tends to infinity. Let $\varphi_2(s) = s\exp(\gamma_2 s^2)$, where $\gamma_2 = (\frac{\lambda_2}{2\beta})^2$. It is well-known that for every $s \in \mathbb{R}$ we have

$$\varphi_2'(s) - \frac{\lambda_2}{\beta}|\varphi_2(s)| \ge \frac{1}{2} \tag{4.88}$$

(see Lemma 1 in [14]). We set $z^n = T_{2k}\big(v^n - T_h(v^n) + T_k(v^n) - T_k(v)\big)$, where $h > k > 0$. Now, we use in (4.9) the test function $\varphi_1(z^n)$. It follows that

$$\int_\Omega B_n(x,v^n)|\nabla v^n|^{p(x)-2}\nabla v^n\varphi_2'(z^n)\nabla z^n\,\mathrm{d}x + \int_\Omega H_2^n(x,v^n,\nabla v^n)\varphi_2(z^n)\,\mathrm{d}x \\ = \gamma\int_\Omega |\nabla u^n|^{q_0(x)}\varphi_2(z^n)\,\mathrm{d}x. \tag{4.89}$$

Choosing $M = 4k + h$, we have $\nabla z^n = 0$ on the set $\{|v^n| > M\}$ and $H_2^n(x,v^n,\nabla v^n)\varphi_2(z^n) \ge 0$ on the set $\{|v^n| > k\}$. Then,

$$\int_\Omega B_n(x,T_M(v^n))|\nabla T_M(v^n)|^{p(x)-2}\nabla T_M(v^n)\varphi_2'(z^n)\nabla z^n\,\mathrm{d}x \\ + \int_{\{|v^n|\le k\}} H_2^n(x,v^n,\nabla v^n)\varphi_2(z^n)\,\mathrm{d}x \le \gamma\int_\Omega |\nabla u^n|^{q_0(x)}\varphi_2(z^n)\,\mathrm{d}x. \tag{4.90}$$

Using the same argument as in the last proof, we have

$$\int_\Omega B_n(x,v^n)|\nabla T_M(v^n)|^{p(x)-2}\nabla T_M(v^n)\varphi_2'(z^n)\nabla z^n\,\mathrm{d}x \\ \ge \int_\Omega B_n(x,T_k(v^n))\Big[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n) - |\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\Big]\times \\ \times[\nabla T_k(v^n)-\nabla T_k(v)]\varphi_2'(z^n)\,\mathrm{d}x + \epsilon_1(n). \tag{4.91}$$

The second term on the left-hand side of (4.90) can be estimated as follows:

$$\begin{aligned}\left|\int_{\{v^n\le k\}} H_2^n(x,v^n,\nabla v^n)\varphi_2(z^n)\,\mathrm{d}x\right| \\ \le \frac{\lambda_2}{\beta}\int_\Omega B_n(x,T_k(v^n))\left[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n)-|\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\right]\times \\ \times\left[\nabla T_k(v^n)-\nabla T_k(v)\right]|\varphi_2(z^n)|\,\mathrm{d}x+\epsilon_2(n).\end{aligned}\tag{4.92}$$

Combining (4.90), (4.91) and (4.92), we obtain

$$\begin{aligned}\int_\Omega B_n(x,T_k(v^n))\left[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n)-|\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\right]\times \\ \times\left[\nabla T_k(v^n)-\nabla T_k(v)\right]\left(\varphi_2'(z^n)-\frac{\lambda_2}{\beta}|\varphi_2(z^n)|\right)\mathrm{d}x \\ \le \gamma\int_\Omega|\nabla u^n|^{q_0(x)}\varphi_2(z^n)\,\mathrm{d}x+\epsilon_3(n).\end{aligned}\tag{4.93}$$

We have $z^n \rightharpoonup T_{2k}(v-T_h(v))$ weakly in $W_0^{1,p(x)}(\Omega)$ and weakly-* in $L^\infty(\Omega)$ and $u^n \longrightarrow u$ strongly in $W_0^{1,q(x)}(\Omega)$ for any $q(x)\in\left[1,\frac{N(p(x)-1)}{N-1}\right)$. Then,

$$\int_\Omega\gamma|\nabla u^n|^{q_0(x)}\varphi_2(z^n)\,\mathrm{d}x=\gamma\int_\Omega|\nabla u|^{q_0(x)}\varphi_2(T_{2k}(v-T_h(v)))\,\mathrm{d}x+\epsilon_4(n).\tag{4.94}$$

We are able to pass to the limit as $n\to+\infty$ in the last inequality (4.93) and we obtain

$$\begin{aligned}\limsup_{n\to+\infty}\int_\Omega B_n(x,T_k(v^n))\left[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n)-|\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\right]\times \\ \times\left[\nabla T_k(v^n)-\nabla T_k(v)\right]\mathrm{d}x \\ \le 2\gamma\int_\Omega|\nabla u|^{q_0(x)}\varphi_2(T_{2k}(v-T_h(v)))\,\mathrm{d}x.\end{aligned}\tag{4.95}$$

Therefore, by (4.95) letting h tend to infinity, we deduce (4.85), that is,

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega B_n(x,T_k(v^n))\left[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n)-|\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\right]\times \\ \times\left[\nabla T_k(v^n)-\nabla T_k(v)\right]\mathrm{d}x=0.\end{aligned}$$

According to (1.4), we obtain

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega\left[|\nabla T_k(v^n)|^{p(x)-2}\nabla T_k(v^n)-|\nabla T_k(v)|^{p(x)-2}\nabla T_k(v)\right]\times \\ \times\left[\nabla T_k(v^n)-\nabla T_k(v)\right]\mathrm{d}x=0.\end{aligned}\tag{4.96}$$

And then, we obtain (4.86).

For (4.87) let $\varepsilon(x)>0$ be such that $q(x)+\varepsilon(x)<\frac{N(p(x)-1)}{N-1}$. By Hölder's inequality and (4.27) we obtain

$$\begin{aligned}\int_\Omega|\nabla v^n|^{q(x)}\,\mathrm{d}x&=\int_{\{|v^n|\le k\}}|\nabla v^n|^{q(x)}\,\mathrm{d}x+\int_{\{|v^n|>k\}}|\nabla v^n|^{q(x)}\,\mathrm{d}x \\ &\le\int_\Omega|\nabla T_k(v^n)|^{q(x)}\,\mathrm{d}x+C\big(\mathrm{meas}\,\{|v^n|>k\}\big)^{\rho_7},\end{aligned}\tag{4.97}$$

where

$$\rho_7 = \begin{cases} \dfrac{\varepsilon^+}{q^- + \varepsilon^-}, & \text{if meas}\,\{|v^n| > k\} \geq 1, \\ \dfrac{\varepsilon^-}{q^+ + \varepsilon^+}, & \text{if meas}\,\{|v^n| > k\} < 1. \end{cases}$$

Using (4.26), (4.86) and passing to the limit in (4.97) as n tends to $+\infty$ and as k tends to $+\infty$, and since $v \in W_0^{1,q(.)}(\Omega)$, we obtain

$$\begin{aligned} &\lim_{k\to+\infty} \lim_{n\to+\infty} \int_\Omega |\nabla v^n|^{q(x)} \, \mathrm{d}x \\ &\quad\leq \lim_{k\to+\infty} \lim_{n\to+\infty} \left[\int_\Omega |\nabla T_k(v^n)|^{q(x)} \, \mathrm{d}x + C\big(\text{meas}\,\{|v^n| > k\}\big)^{\rho_7} \right] \\ &\quad\leq \lim_{k\to+\infty} \lim_{n\to+\infty} \int_\Omega |\nabla T_k(v^n)|^{q(x)} \, \mathrm{d}x \\ &\quad\leq \lim_{k\to+\infty} \int_\Omega |\nabla T_k(v)|^{q(x)} \, \mathrm{d}x \\ &\quad= \lim_{k\to+\infty} \int_\Omega |\nabla v|^{q(x)} \, \chi_{\{|v|\leq k\}} \, \mathrm{d}x \\ &\quad= \int_\Omega |\nabla v|^{q(x)} \, \mathrm{d}x. \end{aligned}$$

Then,

$$\lim_{n\to+\infty} \int_\Omega |\nabla v^n|^{q(x)} \, \mathrm{d}x \leq \int_\Omega |\nabla v|^{q(x)} \, \mathrm{d}x. \tag{4.98}$$

Using Fatou's and Lieb's lemmas we obtain (4.87). □

Step 5 In this step, u and v are shown to satisfy (3.6)–(3.11). To obtain such a result, we need the following lemma.

Lemma 4.4 *We have*

$$H_1^n(x, u^n, \nabla u^n) \longrightarrow H_1(x, u, \nabla u) \textit{ strongly in } L^1(\Omega) \tag{4.99}$$

and

$$H_2^n(x, v^n, \nabla v^n) \longrightarrow H_2(x, v, \nabla v) \textit{ strongly in } L^1(\Omega), \tag{4.100}$$

as n tends to $+\infty$.

Proof. We choose $\theta_k(u^n)$ as a test function in (4.8) and $\theta_k(v^n)$ as a test function in (4.9). Using (4.16) and (4.23), we deduce that

$$\int_\Omega H_1^n(x, u^n, \nabla u^n)\theta_k(u^n) \, \mathrm{d}x \leq \int_{\{|u^n|>k\}} |f^n| \, \mathrm{d}x + C \int_{\{|u^n|>k\}} |F|^{p\prime(x)} \, \mathrm{d}x \tag{4.101}$$

and

$$\int_\Omega H_2^n(x, v^n, \nabla v^n)\theta_k(v^n) \, \mathrm{d}x \leq \gamma \int_{\{|v^n|>k\}} |\nabla u^n|^{q_0(x)} \, \mathrm{d}x. \tag{4.102}$$

Using (4.15), (4.19) and (4.31) we infer that for every $\eta > 0$ there exist $k_1(\eta) > 0$ and $k_2(\eta) > 0$ such that

$$\int_{\{|u^n|>k_1(\eta)\}} H_1^n(x, u^n, \nabla u^n)\theta_k(u^n)\,\mathrm{d}x \le \frac{\eta}{2} \tag{4.103}$$

and

$$\int_{\{|v^n|>k_2(\eta)\}} H_2^n(x, v^n, \nabla v^n)\theta_k(v^n)\,\mathrm{d}x \le \frac{\eta}{2}. \tag{4.104}$$

On the other hand, for any measurable subset $E \subset \Omega$ and for all $l > m$, we have

$$\begin{aligned} &\int_E |H_1^n(x, u^n, \nabla u^n)|\,\mathrm{d}x \\ &\quad\le \lambda_1 \int_{E\bigcap\{|u^n|\le k_1(\eta)\}} \big(|h_{1,k}(x)| + |\nabla u^n|^{p(x)}\chi_{\{0\le v^n\le l\}}\big)\,\mathrm{d}x \\ &\qquad + \lambda_1 \int_{E\bigcap\{|u^n|\le k_1(\eta)\}} |\nabla u^n|^{p(x)}\chi_{\{v^n>l\}}\,\mathrm{d}x \\ &\qquad + \int_{\{|u^n|>k_1(\eta)\}} |H_1^n(x, u^n, \nabla u^n)|\,\mathrm{d}x \end{aligned} \tag{4.105}$$

and

$$\begin{aligned} &\int_E |H_2^n(x, v^n, \nabla v^n)|\,\mathrm{d}x \\ &\le \lambda_2 \int_{E\bigcap\{|v^n|\le k_2(\eta)\}} \big(|h_{2,k}(x)| + |\nabla v^n|^{p(x)}\big)\,\mathrm{d}x + \int_{\{|v^n|>k_2(\eta)\}} |H_2^n(x, v^n, \nabla v^n)|\,\mathrm{d}x. \end{aligned} \tag{4.106}$$

Thanks to (4.15), (4.19) and (4.31) there exist $\beta_1(\eta) > 0$ and $\beta_2(\eta) > 0$ such that

$$\begin{aligned} \lambda_1 \int_{E\bigcap\{|u^n|\le k_1(\eta)\}} &\big(|h_{1,k}(x)| + |\nabla u^n|^{p(x)}\chi_{\{0\le v^n\le l\}}\big)\,\mathrm{d}x \\ &+ \lambda_1 \int_{E\bigcap\{|u^n|\le k_1(\eta)\}} |\nabla u^n|^{p(x)}\chi_{\{v^n>l\}}\,\mathrm{d}x \le \frac{\eta}{2} \end{aligned} \tag{4.107}$$

for $\mathrm{meas}(E) \le \beta_1(\eta)$, and

$$\lambda_2 \int_{E\bigcap\{|v^n|\le k_2(\eta)\}} \big(|h_{2,k}(x)| + |\nabla v^n|^{p(x)}\big)\,\mathrm{d}x \le \frac{\eta}{2} \text{ for } \mathrm{meas}(E) \le \beta_2(\eta). \tag{4.108}$$

Finally, by combining (4.103), (4.104), (4.105) and (4.106), (4.107), (4.108), we obtain

$$\int_E |H_1^n(x, u^n, \nabla u^n)|\,\mathrm{d}x \le \eta \text{ with } \mathrm{meas}(E) \le \beta_1(\eta) \tag{4.109}$$

and

$$\int_E |H_2^n(x, v^n, \nabla v^n)|\,\mathrm{d}x \le \eta \text{ with } \mathrm{meas}(E) \le \beta_2(\eta). \tag{4.110}$$

So, $(H_1^n(x, u^n, \nabla u^n))_n$ and $(H_2^n(x, v^n, \nabla v^n))_n$ are equi-inegrable, and by Vitali's theorem we deduce (4.99) and (4.100). □

Now, we are able to prove (3.8) and (3.9). For (3.8) we choose $\theta_k(u^n)h_s(u^n)h_l(v^n)$ as a test function in (4.8) and we get

$$\begin{aligned}&\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla\theta_k(u^n)h_l(v^n)h_s(u^n)\,\mathrm{d}x\\&\quad+\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla h_l(v^n)h_s(u^n)\theta_k(u^n)\,\mathrm{d}x\\&\quad+\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla h_s(u^n)h_l(v^n)\theta_k(u^n)\,\mathrm{d}x\\&\quad+\int_\Omega H_1^n(x,u^n,\nabla u^n)\theta_k(u^n)h_l(v^n)h_s(u^n)\,\mathrm{d}x\\&=\int_\Omega f^n\theta_k(u^n)h_s(u^n)h_l(v^n)\,\mathrm{d}x+\int_\Omega F\nabla\theta_k(u^n)h_s(u^n)h_l(v^n)\,\mathrm{d}x\\&\quad+\int_\Omega F\nabla h_l(v^n)h_s(u^n)\theta_k(u^n)\,\mathrm{d}x+\int_\Omega F\nabla h_s(u^n)h_l(v^n)\theta_k(u^n)\,\mathrm{d}x.\end{aligned}\tag{4.111}$$

Our aim is to pass to the limit in (4.111) when n tends to $+\infty$ and s tends to $+\infty$. We begin with the study of the first term in (4.111). Using (4.35), for any $l>m$ we have

$$\begin{aligned}&\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla\theta_k(u^n)h_l(v^n)h_s(u^n)\,\mathrm{d}x\\&=\lim_{s\to+\infty}\lim_{n\to+\infty}\Big[\int_\Omega A_n(x,T_{l+1}(v^n))\chi_{\{0\le v^n\le l+1\}}\,|\nabla T_{k+1}(u^n)|^{p(x)}\,h_l(v^n)h_s(u^n)\,\mathrm{d}x\\&\quad-\int_\Omega A_n(x,T_{l+1}(v^n))\chi_{\{0\le v^n\le l+1\}}\,|\nabla T_k(u^n)|^{p(x)}\,h_l(v^n)h_s(u^n)\Big]\,\mathrm{d}x\\&=\lim_{s\to+\infty}\Big[\int_\Omega A(x,v)\,|\nabla T_{k+1}(u)|^{p(x)}\,h_s(u)\,\mathrm{d}x-\int_\Omega A(x,v)\,|\nabla T_k(u)|^{p(x)}\,h_s(u)\,\mathrm{d}x\Big]\\&=\int_\Omega A(x,v)\,|\nabla T_{k+1}(u)|^{p(x)}\,\mathrm{d}x-\int_\Omega A(x,v)\,|\nabla T_k(u)|^{p(x)}\,\mathrm{d}x\\&=\int_{\{k<|u|<k+1\}}A(x,v)\,|\nabla u|^{p(x)}\,\mathrm{d}x.\end{aligned}\tag{4.112}$$

As regards the second term in (4.111), we can use (1.3) and Hölder's inequality to deduce that

$$\begin{aligned}&\left|\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla h_l(v^n)h_s(u^n)\theta_k(u^n)\,\mathrm{d}x\right|\\&\qquad\le 2\|A\|_{L^\infty(\Omega\times(-(l+1),(l+1)))}\|\,|\nabla T_s(u^n)|^{p(x)-1}\,\|_{L^{p'(\cdot)}(\Omega)}\|\nabla h_l(v^n)\|_{(L^{p(\cdot)}(\Omega))^N}.\end{aligned}\tag{4.113}$$

Using now (1.3), (4.12), (4.51), (4.113), since $|\nabla h_l(v^n)|=|\nabla\theta_l(v^n)|$ a.e. in Ω (see (3.1)), for any $l>m$ we obtain

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla h_l(v^n)h_s(u^n)\theta_k(u^n)\,\mathrm{d}x=0.\tag{4.114}$$

Now, let us study the third term in (4.111). According to the definition of h_s, we have

$$\begin{aligned}&\left|\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla h_s(u^n)h_l(v^n)\theta_k(u^n)\,\mathrm{d}x\right|\\&\qquad\le\int_{\{s\le|u^n|\le s+1\}}A_n(x,v^n)|\nabla u^n|^{p(x)}\,\mathrm{d}x.\end{aligned}\tag{4.115}$$

Using (4.21), we obtain

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla h_s(u^n)h_l(v^n)\theta_k(u^n)\,\mathrm{d}x=0. \tag{4.116}$$

Let us pass to the study of the fourth term in (4.111). Since $H_1^n(x,u^n,\nabla u^n)$ converges to $H_1(x,u,\nabla u)$ strongly in $L^1(\Omega)$ and $\theta_k(u^n)h_l(v^n)h_s(u^n)$ is bounded and converges to $\theta_k(u)h_l(v)h_s(u)$ a.e. in Ω as n tends to $+\infty$ and $h_s(u)h_l(v)$ converges to 1 strongly as s tends to $+\infty$ and $l>m$, we have

$$\begin{aligned}&\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega H_1^n(x,u^n,\nabla u^n)\theta_k(u^n)h_s(u^n)h_l(v^n)\,\mathrm{d}x\\&=\lim_{s\to+\infty}\int_\Omega H_1(x,u,\nabla u)\theta_k(u)h_l(v)h_s(u)\,\mathrm{d}x\\&=\lim_{s\to+\infty}\int_\Omega H_1(x,u,\nabla u)\theta_k(u)h_s(u)\,\mathrm{d}x\\&=\int_\Omega H_1(x,u,\nabla u)\theta_k(u)\,\mathrm{d}x.\end{aligned} \tag{4.117}$$

Now, we study the fifth term in (4.111). Recal that f^n converges to f strongly in $L^1(\Omega)$, $\theta_k(u^n)h_l(v^n)h_s(u^n)$ is bounded and converges to $\theta_k(u)h_l(v)h_s(u)$ a.e. in Ω as n tends to $+\infty$ and $h_l(v)h_s(u)$ converges to 1 strongly as s tends to $+\infty$ and $l>m$. Moreover, $f\theta_k(u)\in L^1(\Omega)$. Therefore, we obtain

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega f^n\theta_k(u^n)h_l(v^n)h_s(u^n)\ \mathrm{d}x=\int_\Omega f\theta_k(u)dx. \tag{4.118}$$

As regards the sixth term in (4.111), we have

$$\left|\int_\Omega F\nabla\theta_k(u^n)h_l(v^n)h_s(u^n)\,\mathrm{d}x\right|\le\int_\Omega |F||\nabla\theta_k(u^n)|\,\mathrm{d}x. \tag{4.119}$$

According to (4.16) and (4.17), for any $l>m$ we obtain

$$\lim_{k\to+\infty}\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega F\nabla\theta_k(u^n)h_l(v^n)h_s(u^n)\,\mathrm{d}x=0. \tag{4.120}$$

Finally, we study the seventh term in (4.111). Due to (4.21), (4.33) and to Hölder's inequality for any $l>m$ we have

$$\begin{aligned}&\lim_{s\to+\infty}\lim_{n\to+\infty}\left|\int_\Omega F\nabla h_l(v^n)h_s(u^n)\theta_k(u^n)\,\mathrm{d}x\right|\\&\le 2\|F\|_{(L^{p'(x)}(\Omega))^N}\lim_{s\to+\infty}\lim_{n\to+\infty}\left(\int_{\{l\le|v^n|\le l+1\}}|\nabla v^n|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{\rho_5}}=0\end{aligned} \tag{4.121}$$

and

$$\begin{aligned}&\lim_{s\to+\infty}\lim_{n\to+\infty}\left|\int_\Omega F\nabla h_s(u^n)h_l(v^n)\theta_k(u^n)\,\mathrm{d}x\right|\\&\le 2k\|F\|_{(L^{p'(x)}(\Omega))^N}\lim_{s\to+\infty}\lim_{n\to+\infty}\left(\int_{\{s\le|u^n|\le s+1\}}|\nabla u^n|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{\rho_8}}=0,\end{aligned} \tag{4.122}$$

where

$$\rho_8 = \begin{cases} p^-, & \text{if } \|\nabla h_s(u^n)\|_{(L^{p(.)}(\Omega))^N} > 1, \\ p^+, & \text{if } \|\nabla h_s(u^n)\|_{(L^{p(.)}(\Omega))^N} \leq 1. \end{cases}$$

As a consequence of the above convergence results, we can pass to the limit in (4.111) as k tends to $+\infty$ and we obtain

$$\begin{aligned} &\lim_{k\to+\infty}\left[\int_{\{k<|u|<k+1\}} A(x,v)\,|\nabla u|^{p(x)}\,\mathrm{d}x + \int_\Omega H_1(x,u,\nabla u)\theta_k(u)\,\mathrm{d}x\right] \\ &\quad= \lim_{k\to+\infty}\int_\Omega f\theta_k(u)\,\mathrm{d}x. \end{aligned} \tag{4.123}$$

Then,

$$\lim_{k\to+\infty}\int_{\{k<|u|<k+1\}} A(x,v)|\nabla u|^{p(x)}\,\mathrm{d}x = 0. \tag{4.124}$$

For (3.9) we choose $z_\delta(v^n) = \frac{1}{\delta}(T^+_{m-\delta}(v^n) - T^+_{m-2\delta}(v^n))$ as a test function in (4.9). We obtain

$$\begin{aligned} &\frac{1}{\delta}\int_\Omega B_n(x,v^n)\,|\nabla v^n|^{p(x)-2}\,\nabla v^n\nabla z_\delta(v^n)\,\mathrm{d}x + \int_\Omega H_2^n(x,v^n,\nabla v^n)z_\delta(v^n)\,\mathrm{d}x \\ &\quad= \gamma\int_\Omega |\nabla u^n|^{q_0(x)}z_\delta(v^n)\,\mathrm{d}x. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{\delta}\int_{\{m-2\delta<v^n<m-\delta\}} B_n(x,v^n)\,|\nabla v^n|^{p(x)}\,\mathrm{d}x + \int_\Omega H_2^n(x,v^n,\nabla v^n)z_\delta(v^n)\,\mathrm{d}x \\ &\quad= \gamma\int_\Omega |\nabla u^n|^{q_0(x)}z_\delta(v^n)\,\mathrm{d}x. \end{aligned} \tag{4.125}$$

Since

$$\begin{aligned} &\frac{1}{\delta}\int_{\{m-2\delta<v^n<m-\delta\}} B_n(x,v^n)\,|\nabla v^n|^{p(x)}\,\mathrm{d}x \\ &\quad= \frac{1}{\delta}\left[\int_\Omega B_n(x,v^n)|\nabla T^+_{m-\delta}(v^n)|^{p(x)}\,\mathrm{d}x - \int_\Omega B_n(x,v^n)|\nabla T^+_{m-2\delta}(v^n)|^{p(x)}\,\mathrm{d}x\right], \end{aligned} \tag{4.126}$$

according to (4.86), for a fixed $\delta > 0$ we can pass to the limit in (4.126) as n tends to $+\infty$. We get

$$\begin{aligned} &\lim_{n\to+\infty}\frac{1}{\delta}\int_{\{m-2\delta<v^n<m-\delta\}} B_n(x,v^n)\,|\nabla v^n|^{p(x)}\,\mathrm{d}x \\ &\quad= \frac{1}{\delta}\left[\int_\Omega B(x,v)|\nabla T^+_{m-\delta}(v)|^{p(x)}\,\mathrm{d}x - \int_\Omega B(x,v)|\nabla T^+_{m-2\delta}(v)|^{p(x)}\,\mathrm{d}x\right] \\ &\quad= \frac{1}{\delta}\int_{\{m-2\delta<v<m-\delta\}} B(x,v)\,|\nabla v|^{p(x)}\,\mathrm{d}x. \end{aligned} \tag{4.127}$$

Using (4.27), (4.87), (4.100) and the fact that $z_\delta(v^n)$ is bounded and converges to $z_\delta(v)$ a.e. in Ω and $z_\delta(v)$ converges to 0 strongly as δ tends to 0, we have

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\int_\Omega H_2^n(x,v^n,\nabla v^n)z_\delta(v^n)\,\mathrm{d}x = 0. \tag{4.128}$$

Due to (4.36), we obtain

$$\begin{aligned}\lim_{\delta\to 0}\lim_{n\to+\infty}\gamma\int_\Omega |\nabla u^n|^{q_0(x)} z_\delta(v^n)\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\gamma\int_\Omega |\nabla u|^{q_0(x)} z_\delta(v)\,\mathrm{d}x \\ &= \gamma\int_{\{v=m\}} |\nabla u|^{q_0(x)}\,\mathrm{d}x.\end{aligned}\tag{4.129}$$

Passing to the limit in (4.125) first as n tends to $+\infty$ and then as δ tends to 0, and using (4.127), (4.128), (4.129) shows that v satisfies (3.9).

In this part we prove that u satisfies (3.10). Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support. Let k be a positive real number such that $\operatorname{supp}(S)\subset[-k,\ k]$. Pointwise multiplication of the approximate equation (4.8) by $S(u^n)h_l(v^n)$ for any $l>m$ leads to

$$\begin{aligned}&-\operatorname{div}\big(A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n S(u^n)h_l(v^n)\big)\\ &\quad + A_n(x,v^n)\,|\nabla u^n|^{p(x)-2}\,\nabla u^n\nabla(S(u^n)h_l(v^n)) + H_1^n(x,u^n,\nabla u^n)S(u^n)h_l(v^n)\\ &\quad = f^nS(u^n)h_l(v^n) - \operatorname{div}(FS(u^n)h_l(v^n)) + F\nabla(S(u^n)h_l(v^n)) \text{ in } \mathcal{D}'(\Omega).\end{aligned}\tag{4.130}$$

Our aim is now to pass to the limit as n tends to $+\infty$ in each term of (4.130). Since $\operatorname{supp}(S)\subset[-k,\ k]$ and $\operatorname{supp}(h_l)\subset[-(l+1),\ l+1]$, the functions u^n and v^n can be replaced by $T_k(u^n)$ and $T_{l+1}(v^n)$. The pointwise convergence of $A_n(x,T_{l+1}(v^n))$ to $A(x,T_{l+1}(v))$ and of $S(u^n)h_l(v^n)$ to $S(u)h_l(v)$, the uniform boundedness of $A_n(x,T_{l+1}(s))$ with respect to n and the bounded character of $S(s)h_l(s)$, the weak convergence of $h_l(v^n)$ to $h_l(v)$ in $W^{1,p(.)}(\Omega)$ with $h_l(v)=1$ a.e. in Ω since $0\le v\le m$ and $l>m$ (see (4.29) and (4.31)), and finally the strong convergence of $S(u^n)$ to $S(u)$ in $W^{1,p(.)}(\Omega)$ (see (4.35) and (4.36)) permit us to infer that

$$\begin{aligned}&A_n(x,T_{l+1}(v^n))\chi_{\{0\le v^n\le l+1\}}\,|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)S(u^n)h_l(v^n) \text{ weakly}\\ &\quad\text{converges to } A(x,T_{l+1}(v))\,|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)S(u)h_l(v) \text{ in } (L^{p(x)}(\Omega))^N,\end{aligned}\tag{4.131}$$

$$\begin{aligned}&A_n(x,T_{l+1}(v^n))\chi_{\{0\le v^n\le l+1\}}\,|\nabla T_k(u^n)|^{p(x)-2}\,\nabla T_k(u^n)\nabla(S(u^n)h_l(v^n)) \text{ weakly}\\ &\quad\text{converges to } A(x,T_{l+1}(v))\,|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)\nabla(S(u)h_l(v)) \text{ in } L^1(\Omega)\end{aligned}\tag{4.132}$$

and

$$H_1^n(x,u^n,\nabla u^n)S(u^n)h_l(v^n)\longrightarrow H_1(x,u,\nabla u)S(u)h_l(v) \text{ strongly in } L^1(\Omega),\tag{4.133}$$

as n tends to $+\infty$; note that since $0\le v\le m$, for every $l>m$ we have $h_l(v)=1$ and $T_{l+1}(v)=v$ a.e. in Ω. Now, for $l>m$ we have

$$A(x,T_{l+1}(v))\,|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)S(u)h_l(v) = A(x,v)\,|\nabla u|^{p(x)-2}\,\nabla uS(u) \text{ a.e. in } \Omega,$$
$$A(x,T_{l+1}(v))\,|\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)\nabla(S(u)h_l(v)) = A(x,v)\,|\nabla u|^{p(x)-2}\,\nabla u\nabla S(u) \text{ a.e. in } \Omega$$

and

$$H_1(x,u,\nabla u)S(u)h_l(v) = H_1(x,u,\nabla u)S(u) \text{ a.e. in } \Omega.$$

Since f^n converges to f strongly in $L^1(\Omega)$ and $F \in (L^{p'(x)}(\Omega))^N$, repeating the technique that lead to (4.131) and (4.132), we deduce that for every $l > m$

$$\begin{aligned} &f^n S(u^n)h_l(v^n) \longrightarrow fS(u) \text{ strongly in } L^1(\Omega),\\ &FS(u^n)h_l(v^n) \longrightarrow FS(u) \text{ strongly in } (L^{p'(x)}(\Omega))^N,\\ &F\nabla(S(u^n)h_l(v^n)) \longrightarrow F\nabla(S(u)) \text{ strongly in } L^1(\Omega), \end{aligned} \tag{4.134}$$

as n tends to $+\infty$. As a consequence of the above convergence results, we can pass to the limit in (4.130) as n tends to $+\infty$, and then (3.10) follows.

In this part we prove that v satisfies (3.11). For that we need the following lemma.

Lemma 4.5 *We have*

$$B_n(x,v^n)\,|\nabla T_m(v^n)|^{p(x)-2}\,\nabla T_m(v^n) \rightharpoonup U_m \textit{ weakly in } (L^{p'(x)}(\Omega))^N \tag{4.135}$$

with $U_m = B(x,v)\,|\nabla T_m(v)|^{p(x)-2}\,\nabla T_m(v)$ *a.e. in* $\{x \in \Omega : v(x) < m\}$ *and*

$$B_n(x,v^n)^{\frac{1}{p(x)}}\nabla T_k(v^n) \longrightarrow B(x,v)^{\frac{1}{p(x)}}\nabla T_k(v)\chi_{\{v<m\}} \textit{ strongly in } (L^{p(x)}(\Omega))^N, \tag{4.136}$$

as n tends to $+\infty$.

Proof. For the proof of (4.135) see [20]. For (4.136) we have

$$B_n(x,T_m(v^n))\,|\nabla T_m(v^n)|^{p(x)} = B_n(x,T_m(v^n))\,|\nabla T_m(v^n)|^{p(x)-2}\,\nabla T_m(v^n)\nabla T_m(v^n).$$

Using (4.35) of Lemma 4.2 and (4.135) of Lemma 4.5, we obtain

$$B_n(x,T_m(v^n))\,|\nabla T_m(v^n)|^{p(x)} \rightharpoonup U_m\nabla T_m(v) \text{ weakly in } L^1(\Omega), \tag{4.137}$$

as n tends to $+\infty$. Using (4.24) and the same technique as the one used in [20], we infer that

$$(B_n(x,v^n))^{\frac{1}{p(x)}}\nabla T_m(v^n) \rightharpoonup Y_m \text{ weakly in } (L^{p(x)}(\Omega))^N \text{ as } n \text{ tends to } +\infty, \tag{4.138}$$

where

$$Y_m = (B(x,v))^{\frac{1}{p(x)}}\nabla T_m(v) \;\text{ a.e. in } \{x \in \Omega : 0 \le v(x) < m\}. \tag{4.139}$$

Using (4.69), we can write

$$\begin{aligned} &2^{2-p^+}\int_{\{x\in\Omega : p(x)\ge 2\}} \left|Y_m^n - Y_m\chi_{\{0\le v<m\}}\right|^{p(x)} \mathrm{d}x\\ &\le \int_{\{x\in\Omega : p(x)\ge 2\}} \Big(Y_m^n\,|Y_m^n|^{p(x)-2} - Y_m\,|Y_m|^{p(x)-2}\,\chi_{\{0\le v<m\}}\Big)(Y_m^n - Y_m\chi_{\{0\le v<m\}})\,\mathrm{d}x\\ &\le \int_\Omega \Big(Y_m^n\,|Y_m^n|^{p(x)-2} - Y_m\,|Y_m|^{p(x)-2}\,\chi_{\{0\le v<m\}}\Big)(Y_m^n - Y_m\chi_{\{0\le v<m\}})\,\mathrm{d}x =: J(n). \end{aligned} \tag{4.140}$$

We write

$$\begin{aligned} J(n) = &\int_\Omega |Y_m^n|^{p(x)}\,\mathrm{d}x + \int_\Omega |Y_m|^{p(x)}\,\chi_{\{0\le v<m\}}\,\mathrm{d}x - \int_\Omega |Y_m^n|^{p(x)-2}\,Y_m^nY_m\chi_{\{0\le v<m\}}\,\mathrm{d}x\\ &- \int_\Omega |Y_m|^{p(x)-2}\,Y_mY_m^n\chi_{\{0\le v<m\}}\,\mathrm{d}x. \end{aligned} \tag{4.141}$$

Using (4.137), we obtain

$$\lim_{n\to+\infty}\int_\Omega |Y_m^n|^{p(x)}\,\mathrm{d}x = \lim_{n\to+\infty}\int_\Omega B_n(x,v^n)|\nabla T_m(v^n)|^{p(x)}\,\mathrm{d}x \\ = \int_\Omega U_m\nabla T_m(v)\,\mathrm{d}x. \tag{4.142}$$

From (4.139), we have

$$\int_\Omega |Y_m|^{p(x)}\,\chi_{\{0\le v<m\}}\,\mathrm{d}x = \int_\Omega B(x,v)|\nabla T_m(v)|^{p(x)}\chi_{\{0\le v<m\}}\,\mathrm{d}x. \tag{4.143}$$

Since $B_n(x,v^n)^{\frac{-1}{p(x)}}$ converges to $B(x,v)^{\frac{-1}{p(x)}}$ a.e. in Ω and weakly-* in $L^\infty(\Omega)$ (see [20]), in view of (4.135) of Lemma 4.5, we deduce that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_\Omega |Y_m^n|^{p(x)-2}\,Y_m^nY_m\chi_{\{0\le v<m\}}\,\mathrm{d}x\\
&=\lim_{n\to+\infty}\int_\Omega B_n(x,v^n)^{\frac{1}{p'(x)}}|\nabla T_m(v^n)|^{p(x)-2}\nabla T_m(v^n)Y_m\chi_{\{0\le v<m\}}\,\mathrm{d}x\\
&=\lim_{n\to+\infty}\int_\Omega B_n(x,v^n)^{\frac{-1}{p(x)}}B_n(x,v^n)|\nabla T_m(v^n)|^{p(x)-2}\nabla T_m(v^n)Y_m\chi_{\{0\le v<m\}}\,\mathrm{d}x\\
&=\int_\Omega B(x,v)^{\frac{-1}{p(x)}}(B(x,v))^{\frac{1}{p(x)}}U_m\nabla T_m(v)\chi_{\{0\le v<m\}}\,\mathrm{d}x\\
&=\int_\Omega B(x,v)\,|\nabla T_m(v)|^{p(x)}\,\chi_{\{0\le v<m\}}\,\mathrm{d}x.
\end{aligned} \tag{4.144}$$

Due to (4.138) we have

$$\lim_{n\to+\infty}\int_\Omega |Y_m|^{p(x)-2}\,Y_mY_m^n\chi_{\{0\le v<m\}}\,\mathrm{d}x = \int_\Omega |Y_m|^{p(x)}\,\chi_{\{0\le v<m\}}\,\mathrm{d}x \\ = \int_\Omega B(x,v)|\nabla T_m(v)|^{p(x)}\chi_{\{0\le v<m\}}\,\mathrm{d}x. \tag{4.145}$$

As a consequence of the above convergence results, we can pass to the limit in (4.141) as n tends to $+\infty$ and to conclude that

$$\begin{aligned}
&\lim_{n\to+\infty}J(n)\\
&=\int_\Omega U_m\nabla T_m(v)\,\mathrm{d}x+\int_\Omega B(x,v)|\nabla T_m(v)|^{p(x)}\chi_{\{0\le v<m\}}\,\mathrm{d}x\\
&-\int_\Omega B(x,v)\,|\nabla T_m(v)|^{p(x)}\,\chi_{\{0\le v<m\}}\,\mathrm{d}x-\int_\Omega B(x,v)|\nabla T_m(v)|^{p(x)}\chi_{\{0\le v<m\}}\,\mathrm{d}x=0.
\end{aligned} \tag{4.146}$$

Then, using (4.140) we have

$$\lim_{n\to+\infty}\int_{\{x\in\Omega\,:\,p(x)\ge 2\}}\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^{p(x)}\mathrm{d}x=0. \tag{4.147}$$

On the set where $2-\frac{1}{N}<p(x)<2$, we employ (4.69) as follows:

$$\begin{aligned}
&\int_{\{x\in\Omega\,:\,2-\frac{1}{N}<p(x)<2\}}\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^{p(x)}\mathrm{d}x\\
&\le\int_{\{x\in\Omega\,:\,2-\frac{1}{N}<p(x)<2\}}\frac{\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^{p(x)}}{\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{\frac{p(x)(2-p(x))}{2}}}\times\\
&\qquad\times\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{\frac{p(x)(2-p(x))}{2}}\mathrm{d}x\\
&\le 2\left\|\frac{\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^{p(x)}}{\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{\frac{p(x)(2-p(x))}{2}}}\right\|_{(L^{\frac{2}{p(x)}}(\Omega))^N}\times\\
&\qquad\times\left\|\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{\frac{p(x)(2-p(x))}{2}}\right\|_{(L^{\frac{2}{(2-p(x)}}(\Omega))^N}\\
&\le 2\max\left\{\left(\int_\Omega\frac{\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^2}{\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{2-p(x)}}\mathrm{d}x\right)^{\frac{p^-}{2}},\right.\\
&\qquad\left.\left(\int_\Omega\frac{\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^2}{\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{2-p(x)}}\mathrm{d}x\right)^{\frac{p^+}{2}}\right\}\times\\
&\qquad\times\max\left\{\left(\int_\Omega\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{p(x)}\mathrm{d}x\right)^{\frac{2-p^+}{2}},\right.\\
&\qquad\left.\left(\int_\Omega\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{p(x)}\mathrm{d}x\right)^{\frac{2-p^-}{2}}\right\}\\
&\le 2\max\left\{(p^--1)^{-\frac{p^-}{2}}(J(n))^{\frac{p^-}{2}},(p^--1)^{-\frac{p^+}{2}}(J(n))^{\frac{p^+}{2}}\right\}\times\\
&\qquad\times\max\left\{\left(\int_\Omega\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{p(x)}\mathrm{d}x\right)^{\frac{2-p^+}{2}},\right.\\
&\qquad\left.\left(\int_\Omega\left(\left|Y_m^n\right|+\left|Y_m\chi_{\{0\le v<m\}}\right|\right)^{p(x)}\mathrm{d}x\right)^{\frac{2-p^-}{2}}\right\}.
\end{aligned}\tag{4.148}$$

Since $J(n)\to0$ as $n\to+\infty$ and $(Y_m^n)_n$ is bounded in $(L^{p(x)}(\Omega))^N$, by (4.148) we obtain

$$\lim_{n\to+\infty}\int_{\{x\in\Omega\,:\,2-\frac{1}{N}<p(x)<2\}}\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^{p(x)}\mathrm{d}x=0.\tag{4.149}$$

Using (4.147) and (4.149), we deduce that

$$\lim_{n\to+\infty}\int_\Omega\left|Y_m^n-Y_m\chi_{\{0\le v<m\}}\right|^{p(x)}\mathrm{d}x=0.\tag{4.150}$$

Then,

$$\lim_{n\to+\infty}\int_\Omega \left|B_n(x,v^n)^{\frac{1}{p(x)}}\nabla T_m(v^n)-B(x,v)^{\frac{1}{p(x)}}\nabla T_m(v)\chi_{\{0\le v<m\}}\right|^{p(x)}\mathrm{d}x=0. \tag{4.151}$$

This proves (4.136). □

Now, let h in $W^{1,\infty}(\mathbb{R})$ be such that $\operatorname{supp}(h)\subset[-k,k]$, where $k\le m$. The pointwise multiplication of the approximate equation (4.9) by $h(v^n)$ leads to

$$\begin{aligned}&-\operatorname{div}\Big(B_n(x,v^n)\left|\nabla v^n\right|^{p(x)-2}\nabla v^n h(v^n)\Big)\\&\quad+B_n(x,v^n)\left|\nabla v^n\right|^{p(x)-2}\nabla v^n\nabla h(v^n)+H_2^n(x,v^n,\nabla v^n)h(v^n)\\&\qquad=\gamma|\nabla u^n|^{q_0(x)}h(v^n)\text{ in }\mathcal{D}'(\Omega).\end{aligned} \tag{4.152}$$

We replace v^n by $T_m(v^n)$ and use (4.135) to deduce that

$$\begin{aligned}&B_n(x,T_m(v^n))\left|\nabla T_m(v^n)\right|^{p(x)-2}\nabla T_m(v^n)h(v^n)\\&\qquad\rightharpoonup U_m h(v)\chi_{\{0\le v<m\}}\text{ weakly in }(L^{p'(.)}(\Omega))^N,\end{aligned} \tag{4.153}$$

as n tends to $+\infty$. Since $0\le v\le m$ a.e. in Ω and $h(m)=0$, we infer that

$$U_m h(v)=B(x,v)\left|\nabla v\right|^{p(x)-2}\nabla v h(v)\chi_{\{0\le v<m\}}\text{ a.e. in }\Omega. \tag{4.154}$$

Then, using (4.136) we have

$$\begin{aligned}&B_n(x,v^n)\left|\nabla v^n\right|^{p(x)-2}\nabla v^n\nabla h(v^n)\\&\qquad\longrightarrow B(x,v)\left|\nabla T_m(v)\right|^{p(x)-2}\nabla T_m(v)\nabla h(v)\chi_{\{0\le v<m\}}\end{aligned} \tag{4.155}$$

strongly in $L^1(\Omega)$, as n tends to $+\infty$. Since $0\le v\le m$ a.e. in Ω and $h(m)=0$, we have

$$\begin{aligned}&B(x,T_m(v))\left|\nabla T_m(v)\right|^{p(x)-2}\nabla T_m(v)\nabla h(v)\\&\qquad=B(x,v)\left|\nabla v\right|^{p(x)-2}\nabla v\nabla h(u)\chi_{\{0\le v<m\}}\text{ a.e. in }\Omega.\end{aligned} \tag{4.156}$$

The pointwise convergence of $h(v^n)$ to $h(v)$, the bounded character of h and the strong convergence of u^n to u in $W_0^{1,q(.)}(\Omega)$ for every $1\le q(x)<\frac{N(p(x)-1)}{N-1}$ as n tends to $+\infty$ (see (4.36) of Lemma 4.2) and (4.100) of Lemma 4.4 make it possible to conclude that

$$H_2^n(x,v^n,\nabla v^n)h(v^n)\longrightarrow H_2(x,v,\nabla v)h(v)\text{ strongly in }L^1(\Omega) \tag{4.157}$$

and

$$\gamma|\nabla u^n|^{q_0(x)}h(v^n)\longrightarrow\gamma|\nabla u|^{q_0(x)}h(v)\text{ strongly in }L^1(\Omega). \tag{4.158}$$

As a consequence of the above convergence results, we can pass to the limit in (4.152) as n tends to $+\infty$, which, in turn, implies (3.11). The proof of Theorem 4.1 is complete. □

Acknowledgement The authors would like to thank the anonymous reviewer and the corresponding editor for helpful suggestions that improved the quality of the manuscript.

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