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OPTIMAL CONTROL FOR EVOLUTION PROBLEM OF INCOMPLETE DATA

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Abstract. In this work, we give a characterization of the control for ill-posed problems. We propose a regularization method which consists of improving the data in order to obtain a well-posed problem. The optimal control of the regularized system is discussed and the approximated optimality system is presented. We pass to the limit and we obtain a singular optimality system for the low-regret. We use the convergence of the low-regret control to the no-regret control for which we obtain a characterization of the control for the original problem.

Keywords: Low-regret, no-regret, regularization technique.

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1 Statement of the problem

Let Ω be an open bounded subset of $\mathbb{R}$ with a boundary $\partial\Omega = \Gamma$ of class $\mathbf{C}^2$, $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. In the cylinder $Q = \Omega \times (0, T)$ let us consider the parabolic equation

$$\frac{\partial z}{\partial t} - \triangle z = 0 \tag{1.1}$$

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with z being subject to the initial condition

$$z(x,0) = z_0 \text{ in } \Omega, \tag{1.2}$$

where $z_0 \in L^2(\Omega)$ is given, and the boundary condition of Dirichlet and Neumann type

$$z = v_0,\ \frac{\partial z}{\partial \nu} = v_1 \text{ on } \Sigma_0 = \Gamma_0 \times (0,T). \tag{1.3}$$

Thus, the state z and the control $v = \{v_0, v_1\}$ are linked by the system

$$\begin{cases} \dfrac{\partial z}{\partial t} - \triangle z = 0 & \text{in } Q, \\ z = v_0,\ \dfrac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \\ z(0) = z_0 & \text{in } \Omega; \end{cases} \tag{1.4}$$

$z \in L^2(Q)$ is unknown on $\Sigma_1 = \Gamma_1 \times (0,T)$ and $(v_0, v_1) \in L^2(\Sigma_0) \times L^2(\Sigma_0)$.

Problem (1.4) is a Cauchy problem for a parabolic operator. In general, it does not admit a solution and there is instability of the solution when it exists (see, for instance, [6, 14]). However, it is important to control the problem. Therefore, we consider the space U which consists of all $((v_0, v_1), z) \in (L^2(\Sigma_0))^2 \times L^2(\Omega)$ such that $\frac{\partial z}{\partial t} - \triangle z = 0$ in Q, $z = v_0$ and $\frac{\partial z}{\partial \nu} = v_1$ in Σ_0, and $z(0) = z_0$ in Ω. Assume that $U \neq \emptyset$. (To simplify the notation further we will write (v_0, v_1, z) instead of $((v_0, v_1), z)$, but we will still refer to (v_0, v_1, z) as a couple.) The couples $(v_0, v_1, z) \in U$ are called *admissible couples*.

Let J be a strictly convex cost functional defined for all admissible control-state couples (v_0, v_1, z) by

$$J(v_0, v_1, z) = \|z - z_d\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)} + N_1\|v_1\|^2_{L^2(\Sigma_0)}, \tag{1.5}$$

where $(N_0, N_1) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$ and $z_d \in L^2(Q)$ is the desired state. We are then interested in the problem

$$\inf J(v_0, v_1, z),\ (v_0, v_1, z) \in U. \tag{1.6}$$

According to the properties of J, problem (1.6) admits a unique solution (u_0, u_1, z) that we should characterize. To obtain a singular optimality system (SOS) associated with (u_0, u_1, z), Lions in [6, 7] proposed a method of approximation by penalization. He obtained SOS under the additional hypothesis of slater type which reads:

the admissible set of controls has a non-empty interior. (1.7)

Problem (1.4) is a classical example of an ill-posed problem. So, regularization methods may be considered. Theoretical concepts and also computational implementation related to the Cauchy problem have been discussed by many authors. In the parabolic and hyperbolic cases, we can quote M. Barry and O. Nakoulima (see [1, 2]), J. P. Kernevez (see [5]) and G. Mophou, R. G. Foko Tiomela, A. Seibou (see [9]). In the elliptic case we can cite J. L. Lions (see [6]), J. Velin (see [16]), O. Nakoulima (see [10]), S. Sougalo and O. Nakoulima (see [15]). In [4], C. Kenne, G. Leugering and G. Mophou considered a model of population dynamics with age dependence and spatial structure but unknown birth rate and used the notion of low-regret. They proved that we can bring the state of the system to the desired state by acting on the system via a distributed control.

In this paper, we consider another regularization method of the problem. So we define the function $g = \{g_1, g_2\}$ such that $z = g_1$ and $\frac{\partial z}{\partial \nu} = g_2$ on Σ_1 and we consider the following system

$$\begin{cases} \dfrac{\partial z}{\partial t} - \triangle z = 0 & \text{in } Q, \\ z = v_0,\ \dfrac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \\ z = g_1,\ \dfrac{\partial z}{\partial \nu} = g_2 & \text{on } \Sigma_1, \\ z(0) = z_0 & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $g_1 := g_1(x,t)$ and $g_2 := g_2(x,t)$ belong to G and represent "the pollution" which is unknown (incomplete data); G is a closed subspace of $L^2(\Sigma)$ and $\Sigma = \Sigma_0 \cup \Sigma_1$ with $\Sigma_0 = \Gamma_0 \times (0,T)$ and $\Sigma_1 = \Gamma_1 \times (0,T)$.

We here use a method that we find well adapted: the low-regret control concept introduced by Lions. Lions was the first one to use it to control distributed systems of incomplete data, motivated by a number of applications in economics and ecology. In this paper, we generalize the method to ill-posed problems of parabolic type. Lions in [12] proposed a method of approximation by penalization and obtained a singular optimality system under an additional hypothesis of slater type. In [13], O. Nakoulima and G. M. Mophou used a regularization method which consisted in viewing a singular problem as a limit of a family of well-posed problems. They obtained a singular optimality system for the considered control problem also assuming the slater condition. In [3], A. Berhail and A. Omrane used a regularization approach which generates incomplete information. They got a singular optimality system characterizing the no-regret control for a Cauchy elliptic problem.

In the present paper, we use another approximation method to study the problem of evolution (1.4) which to our knowledge has not been treated.

The rest of this paper is organized as follows. In Section 2, we will give the characterization of the low-regret and no-regret control. So, in the Subsection 2.2 and Subsection 2.3, the optimal control of the regularized system is discussed and the approximated optimality system is presented. In the Subsection 2.4 and Subsection 2.5, we go to the limit respectively when $\varepsilon \to 0$, we obtain a singular optimality system for the low-regret and when $\gamma \to 0$ we obtain no-regret control to the original problem, where γ and ε are strictly positive parameters. In Section 3, we will present some concluding remarks.

2 The low-regret and no-regret control

The problems with incomplete data are impossible to solve directly. That is why we use the regularization technique which consists in transforming the problem (1.4) into a complete data problem. We therefore consider the following regularized problem

$$\begin{cases} \dfrac{\partial z_\varepsilon}{\partial t} - \triangle^2 z_\varepsilon - \varepsilon z_\varepsilon = 0 & \text{in } Q, \\ z_\varepsilon - \dfrac{\partial \triangle z_\varepsilon}{\partial \nu} = v_0,\ \dfrac{\partial z_\varepsilon}{\partial \nu} + \triangle z_\varepsilon = v_1 & \text{on } \Sigma_0, \\ \varepsilon z_\varepsilon - \dfrac{\partial \triangle z_\varepsilon}{\partial \nu} = \varepsilon g_0,\ \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \triangle z_\varepsilon = \varepsilon g_1 & \text{on } \Sigma_1, \\ z_\varepsilon(0) = z_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where ε is a strictly positive parameter, $v = (v_0, v_1) \in (L^2(\Sigma_0))^2$ and $g = (g_0, g_1) \in (L^2(\Sigma_1))^2$.

Remark 2.1 *For every fixed εg_0 and εg_1, we assume the existence of a unique solution to* (2.1). *In the rest of the work, εg_0 and εg_1 are considered as data perturbations.*

If we put $\varepsilon = 0$ and we make a change of variables $\eta = \triangle z$, the system (2.1) becomes

$$\begin{cases} \dfrac{\partial z}{\partial t} - \triangle \eta = 0 & \text{in } Q, \\ z - \dfrac{\partial \eta}{\partial \nu} = v_0, \ \dfrac{\partial z}{\partial \nu} + \eta = v_1 & \text{on } \Sigma_0, \\ \dfrac{\partial \eta}{\partial \nu} = 0, \ \eta = 0 & \text{on } \Sigma_1, \\ z(0) = z_0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

From (2.2) we have $\frac{\partial \eta}{\partial \nu} = \eta = 0$ on Σ_1. If we substitute those equalities in Σ_0, we obtain

$$z = v_0, \ \frac{\partial z}{\partial \nu} = v_1 \text{ on } \Sigma_0, \tag{2.3}$$

that is, the same conditions as in the original problem (1.4).

2.1 Cost function and low-regret control.

Consider the cost functional J_ε define by

$$J_\varepsilon(v, g) = \|z_\varepsilon(v, g) - z_d\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)} + N_1\|v_1\|^2_{L^2(\Sigma_0)}. \tag{2.4}$$

To study the problem (2.1) we consider the problem of minimization

$$\inf_{v \in (L^2(\Sigma_0))^2} \left(\sup_{g \in (L^2(\Sigma_1))^2} (J_\varepsilon(v, g) - J_\varepsilon(0, g)) \right). \tag{2.5}$$

The control $u \in (L^2(\Sigma_0))^2$ which is a solution to the minimization problem (2.5) is called the *no-regret control*. Solving (2.5) is not an easy task in general. To make it simpler, Lions introduces the parameters $-\gamma\|g_0\|^2_{L^2(\Sigma_1)}$ and $-\gamma\|g_1\|^2_{L^2(\Sigma_1)}$, where γ is a positive relaxation parameter. Thus, we obtain

$$\inf_{v \in (L^2(\Sigma_0))^2} \left(\sup_{g \in (L^2(\Sigma_1))^2} (J_\varepsilon(v, g) - J_\varepsilon(0, g) - \gamma\|g_0\|^2_{L^2(\Sigma_1)} - \gamma\|g_1\|^2_{L^2(\Sigma_1)}) \right) \tag{2.6}$$

and the solution u of (2.6) is called a *low-regret control*. Note that the *low-regret control* depends on γ and the norm $\|g\|$. It is interpreted as an approximation of the no-regret control. With the low-regret control, we have the possibility to make a choice of control u slightly worse than the ground state with a margin of error that must not exceed $\gamma\|g\|^2_{L^2(Q)}$.

Lemma 2.2 *Let J_ε be the function defined by* (2.4) *and let z_ε be a solution of* (2.1). *Then, for any $v \in (L^2(\Sigma_0))^2$ and $g \in (L^2(\Sigma_1))^2$, we have*

$$\begin{aligned} &J_\varepsilon(v, g) - J_\varepsilon(0, g) \\ &\quad = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + 2\int_0^T\!\!\int_\Omega (z_\varepsilon(v, 0) - z_\varepsilon(0, 0))(z_\varepsilon(0, g) - z_\varepsilon(0, 0))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.7}$$

Proof. Since $z_\varepsilon(v,g) = z_\varepsilon(v,0) + z_\varepsilon(0,g) - z_\varepsilon(0,0)$, we obtain

$$
\begin{aligned}
&J_\varepsilon(v,g) - J_\varepsilon(0,g)\\
&\quad = \|z_\varepsilon(v,0) + z_\varepsilon(0,g) - z_\varepsilon(0,0) - z_d\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)}\\
&\qquad + N_1\|v_1\|^2_{L^2(\Sigma_0)} - \|z_\varepsilon(0,g) - z_d\|^2_{L^2(Q)}\\
&\quad = \|z_\varepsilon(v,0) - z_d\|^2_{L^2(Q)} + 2\langle z_\varepsilon(v,0) - z_d, z_\varepsilon(0,g) - z_\varepsilon(0,0)\rangle_{L^2(Q)}\\
&\qquad + \|z_\varepsilon(0,g) - z_\varepsilon(0,0)\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)} + N_1\|v_1\|^2_{L^2(\Sigma_0)} - \|z_\varepsilon(0,g) - z_d\|^2_{L^2(Q)}\\
&\quad = \|z_\varepsilon(v,0) - z_d\|^2_{L^2(Q)} + 2\langle z_\varepsilon(v,0) - z_d, z_\varepsilon(0,g) - z_\varepsilon(0,0)\rangle_{L^2(Q)}\\
&\qquad + \|z_\varepsilon(0,g) - z_d - (z_\varepsilon(0,0) - z_d)\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)}\\
&\qquad + N_1\|v_1\|^2_{L^2(\Sigma_0)} - \|z_\varepsilon(0,g) - z_d\|^2_{L^2(Q)}\\
&\quad = \|z_\varepsilon(v,0) - z_d\|^2_{L^2(Q)} + 2\langle z_\varepsilon(v,0) - z_d, z_\varepsilon(0,g) - z_\varepsilon(0,0)\rangle_{L^2(Q)}\\
&\qquad + \|z_\varepsilon(0,g) - z_d\|^2_{L^2(Q)} - 2\langle z_\varepsilon(0,g) - z_d, z_\varepsilon(0,0) - z_d\rangle_{L^2(Q)} + \|z_\varepsilon(0,0) - z_d\|^2_{L^2(Q)}\\
&\qquad + N_0\|v_0\|^2_{L^2(\Sigma_0)} + N_1\|v_1\|^2_{L^2(\Sigma_0)} - \|z_\varepsilon(0,g) - z_d\|^2_{L^2(Q)}\\
&\quad = \|z_\varepsilon(v,0) - z_d\|^2_{L^2(Q)} + 2\langle z_\varepsilon(v,0) - z_d, z_\varepsilon(0,g) - z_\varepsilon(0,0)\rangle_{L^2(Q)}\\
&\qquad - 2\langle z_\varepsilon(0,g) - z_\varepsilon(0,0) + z_\varepsilon(0,0) - z_d, z_\varepsilon(0,0) - z_d\rangle_{L^2(Q)}\\
&\qquad + \|z_\varepsilon(0,0) - z_d\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)} + N_1\|v_1\|^2_{L^2(\Sigma_0)}\\
&\quad = \|z_\varepsilon(v,0) - z_d\|^2_{L^2(Q)} + 2\langle z_\varepsilon(v,0) - z_d, z_\varepsilon(0,g) - z_\varepsilon(0,0)\rangle_{L^2(Q)}\\
&\qquad - 2\langle z_\varepsilon(0,g) - z_\varepsilon(0,0), z_\varepsilon(0,0) - z_d\rangle_{L^2(Q)} - 2\langle z_\varepsilon(0,0) - z_d, z_\varepsilon(0,0) - z_d\rangle_{L^2(Q)}\\
&\qquad + \|z_\varepsilon(0,0) - z_d\|^2_{L^2(Q)} + N_0\|v_0\|^2_{L^2(\Sigma_0)} + N_1\|v_1\|^2_{L^2(\Sigma_0)}\\
&\quad = J_\varepsilon(v,0) + J_\varepsilon(0,0) + 2\langle z_\varepsilon(v,0) - z_d, z_\varepsilon(0,g) - z_\varepsilon(0,0)\rangle_{L^2(Q)} - 2J(0,0)\\
&\qquad - 2\langle z_\varepsilon(0,g) - z_\varepsilon(0,0), z_\varepsilon(0,0) - z_d\rangle_{L^2(Q)}.
\end{aligned}
$$

So,

$$
J_\varepsilon(v,g) - J_\varepsilon(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\langle z_\varepsilon(0,g) - z_\varepsilon(0,0), z_\varepsilon(v,0) - z_\varepsilon(0,0)\rangle_{L^2(Q)}
$$

and the proof is complete. □

Lemma 2.3 *Let J_ε be the function defined by* (2.4). *For any $v \in (L^2(\Sigma_0))^2$ and for any $g \in (L^2(\Sigma_1))^2$, we have*

$$
J_\varepsilon(v,g) - J_\varepsilon(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + \frac{\varepsilon^2}{\gamma}\Big(\|\zeta_\varepsilon\|^2_{L^2(\Sigma_1)} + \|\tfrac{\partial \zeta_\varepsilon}{\partial \nu}\|^2_{L^2(\Sigma_1)}\Big), \qquad (2.8)
$$

where $\zeta_\varepsilon(v) := \zeta_\varepsilon(x,t,v)$ *is the solution of*

$$\begin{cases} -\dfrac{\partial \zeta_\varepsilon}{\partial t} - \triangle^2 \zeta_\varepsilon - \varepsilon\zeta_\varepsilon = -(z_\varepsilon(v,0) - z_\varepsilon(0,0)) & \text{in } Q, \\ \zeta_\varepsilon - \dfrac{\partial \triangle \zeta_\varepsilon}{\partial \nu} = 0,\ \dfrac{\partial \zeta_\varepsilon}{\partial \nu} + \triangle \zeta_\varepsilon = 0 & \text{on } \Sigma_0, \\ \varepsilon\zeta_\varepsilon - \dfrac{\partial \triangle \zeta_\varepsilon}{\partial \nu} = 0,\ \varepsilon\dfrac{\partial \zeta_\varepsilon}{\partial \nu} + \triangle \zeta_\varepsilon = 0 & \text{on } \Sigma_1, \\ \zeta_\varepsilon(T,v) = 0 & \text{in } \Omega. \end{cases} \tag{2.9}$$

Proof. Multiplying the first equation of (2.9) by $z_\varepsilon(0,g) - z_\varepsilon(0,0)$ and integrating over on Q, yields

$$\begin{aligned} &-\int_0^T\int_\Omega (z_\varepsilon(v,0) - z_\varepsilon(0,0))(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_0^T\int_\Omega \left(-\frac{\partial \zeta_\varepsilon}{\partial t} - \triangle^2\zeta_\varepsilon - \varepsilon\zeta_\varepsilon\right)(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= -\int_0^T\int_\Omega \frac{\partial \zeta_\varepsilon(v)}{\partial t}(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \int_0^T\int_\Omega \triangle^2\zeta_\varepsilon.(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \varepsilon\zeta_\varepsilon.(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By integrating by parts we obtain

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial \zeta_\varepsilon(v)}{\partial t}.(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_\Omega \left[\zeta_\varepsilon(v).(z_\varepsilon(0,g) - z_\varepsilon(0,0))\right]_0^T\,\mathrm{d}x - \int_0^T\int_\Omega \zeta_\varepsilon(v)\frac{\partial(z_\varepsilon(0,g) - z_\varepsilon(0,0))}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

And so,

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial \zeta_\varepsilon(v)}{\partial t}(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_\Omega \zeta_\varepsilon(T,v).(z_\varepsilon(T;0,g) - z_\varepsilon(T;0,0))\,\mathrm{d}x - \int_\Omega \zeta_\varepsilon(0,v).(z_\varepsilon(0;0,g) - z_\varepsilon(0;0,0))\,\mathrm{d}x \\ &\qquad - \int_0^T\int_\Omega \zeta_\varepsilon(v)\frac{\partial(z_\varepsilon(0,g) - z_\varepsilon(0,0))}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Using the Green formula we get

$$\begin{aligned} &\int_0^T\int_\Omega \triangle^2\zeta_\varepsilon.(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Gamma \frac{\partial(\triangle\zeta_\varepsilon(v))}{\partial \nu}.(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_\Omega \nabla(\triangle\zeta_\varepsilon(v))\nabla(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Gamma \frac{\partial(\triangle\zeta_\varepsilon(v))}{\partial \nu}(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Gamma \triangle\zeta_\varepsilon(v)\frac{\partial(z_\varepsilon(0,g) - z_\varepsilon(0,0))}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Gamma \frac{\partial \zeta_\varepsilon(v)}{\partial \nu}\triangle(z_\varepsilon(0,g) - z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t \end{aligned}$$

$$- \int_0^T\int_\Gamma \zeta_\varepsilon(v).\frac{\partial\triangle(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Omega \zeta_\varepsilon.\triangle^2(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t.$$

In short, we obtain

$$\begin{aligned}
&-\int_0^T\int_\Omega (z_\varepsilon(v,0)-z_\varepsilon(0,0))(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_\Omega \zeta_\varepsilon(T,v).(z_\varepsilon(T;0,g)-z_\varepsilon(0;0,0))\,\mathrm{d}x\\
&\quad + \int_\Omega \zeta_\varepsilon(0,v).(z_\varepsilon(0;0,g)-z_\varepsilon(0;0,0))\,\mathrm{d}x + \int_0^T\int_\Omega \zeta_\varepsilon(v)\frac{\partial(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^T\int_\Gamma \frac{\partial(\triangle\zeta_\varepsilon(v))}{\partial\nu}(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Gamma \triangle\zeta_\varepsilon(v)\frac{\partial(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad - \int_0^T\int_\Gamma \frac{\partial\zeta_\varepsilon(v)}{\partial\nu}\triangle(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Gamma \zeta_\varepsilon(v).\frac{\partial\triangle(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad - \int_0^T\int_\Omega \zeta_\varepsilon.\triangle^2(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \varepsilon\zeta_\varepsilon(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&-\int_0^T\int_\Omega (z_\varepsilon(v,0)-z_\varepsilon(0,0))(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_\Omega \zeta_\varepsilon(T,v).(z_\varepsilon(T;0,g)-z_\varepsilon(T;0,0))\,\mathrm{d}x + \int_\Omega \zeta_\varepsilon(0,v).(z_\varepsilon(0;0,g)-z_\varepsilon(0;0,0))\,\mathrm{d}x\\
&\quad + \int_0^T\int_\Gamma \triangle\zeta_\varepsilon(v)\frac{\partial(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_\Gamma \frac{\partial(\triangle\zeta_\varepsilon(v))}{\partial\nu}(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad - \int_0^T\int_\Gamma \frac{\partial\zeta_\varepsilon(v)}{\partial\nu}\triangle(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Gamma \zeta_\varepsilon(v).\frac{\partial\triangle(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad + \int_0^T\int_\Omega \bigg(\frac{\partial(z_\varepsilon(0,g)-z_\varepsilon(0,0))}{\partial t}\\
&\qquad\qquad - \triangle^2(z_\varepsilon(0,g)-z_\varepsilon(0,0)) - \varepsilon(z_\varepsilon(0,g)-z_\varepsilon(0,0))\bigg).\zeta_\varepsilon(v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Furthermore, $\zeta_\varepsilon(T;v)=0$ in Ω, $\frac{\partial\triangle\zeta_\varepsilon}{\partial\nu}=\varepsilon\zeta_\varepsilon$ on Σ_1, $\triangle\zeta_\varepsilon=-\varepsilon\frac{\partial\zeta_\varepsilon}{\partial\nu}$ on Σ_1, $\varepsilon z_\varepsilon-\frac{\partial\triangle z_\varepsilon}{\partial\nu}=\varepsilon g_0$ on Σ_1, $\varepsilon\frac{\partial z_\varepsilon}{\partial\nu}+\triangle z_\varepsilon=\varepsilon g_1$ on Σ_1 and $\frac{\partial z_\varepsilon}{\partial t}-\triangle^2 z_\varepsilon-\varepsilon z_\varepsilon=0$ in Q. Therefore, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega (z_\varepsilon(v,0)-z_\varepsilon(0,0))(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = \int_0^T\int_{\Gamma_1} \zeta_\varepsilon(\varepsilon g_0)\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_{\Gamma_1} \frac{\partial\zeta_\varepsilon}{\partial\nu}(\varepsilon g_1)\,\mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned}$$

Thus,

$$\sup_{g\in(L^2(\Sigma_1))^2}\bigg(2\int_0^T\int_\Omega (z_\varepsilon(v,0)-z_\varepsilon(0,0))(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t - \gamma\|g_0\|^2_{L^2(\Sigma_1)} - \gamma\|g_1\|^2_{L^2(\Sigma_1)}\bigg)$$

$$= \sup_{g\in(L^2(\Sigma_1))^2}\left(2\int_0^T\int_{\Gamma_1}\zeta_\varepsilon(\varepsilon g_0)\,\mathrm{d}\sigma\,\mathrm{d}t + 2\int_0^T\int_{\Gamma_1}\frac{\partial\zeta_\varepsilon}{\partial\nu}(\varepsilon g_1)\,\mathrm{d}\sigma\,\mathrm{d}t - \gamma\|g_0\|^2_{L^2(\Sigma_1)} - \gamma\|g_1\|^2_{L^2(\Sigma_1)}\right).$$

As a result,

$$\sup_{g\in(L^2(\Sigma_1))^2}\left(2\int_0^T\int_\Omega(z_\varepsilon(v,0)-z_\varepsilon(0,0))(z_\varepsilon(0,g)-z_\varepsilon(0,0))\,\mathrm{d}x\,\mathrm{d}t - \gamma\|g_0\|^2_{L^2(\Sigma_1)} - \gamma\|g_1\|^2_{L^2(\Sigma_1)}\right)$$
$$= \frac{\varepsilon^2}{\gamma}\|\zeta_\varepsilon\|^2_{L^2(\Sigma_1)} + \frac{\varepsilon^2}{\gamma}\left\|\frac{\partial\zeta_\varepsilon}{\partial\nu}\right\|^2_{L^2(\Sigma_1)}.$$

Now, it suffices to apply (2.7) to obtain the result. □

Finally, we can reformulate the problem (2.6) as follows: for all $\gamma > 0$ find $u_\varepsilon^\gamma \in (L^2(\Sigma_0))^2$ such that

$$J_\varepsilon^\gamma(u_\varepsilon^\gamma) = \inf_{v\in(L^2(\Sigma_0))^2} J_\varepsilon^\gamma(v), \tag{2.10}$$

where

$$J_\varepsilon^\gamma(v) = \sup_{g\in(L^2(\Sigma_1))^2}\Big[J_\varepsilon(v,g) - J_\varepsilon(0,g) - \gamma\|g_0\|^2_{L^2(\Sigma_1)} - \gamma\|g_1\|^2_{L^2(\Sigma_1)}\Big].$$

The problem (2.10) is a low-regret problem and its solution, if it exists, will be the low-regret control.

In the rest of our work, we will show that the low-regret of the problem (2.1) admits a unique solution which converges to the no-regret control unique solution of (1.4). Moreover, we will characterize the low-regret control and the no-regret control.

2.2 Uniqueness and existence of the low-regret control

The following proposition shows the existence and uniqueness of the low-regret control.

Proposition 2.4 *There exists a unique low-regret control* $u_\varepsilon^\gamma \in (L^2(\Sigma_0))^2$ *solution of* (2.10)*.*

Proof. From the definition of J_ε^γ for every $v \in (L^2(\Sigma_0))^2$ we have $J_\varepsilon^\gamma(v) \geq -J_\varepsilon(0,0)$. Thus,

$$-J_\varepsilon(0,0) \leq J_\varepsilon(v,g) - J_\varepsilon(0,0) + \frac{\varepsilon^2}{\gamma}\left(\|\zeta_\varepsilon\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial\zeta_\varepsilon}{\partial\nu}\right\|^2_{L^2(\Sigma_1)}\right).$$

Let us define J_ε^γ by

$$J_\varepsilon^\gamma : (L^2(\Sigma_0))^2 \longrightarrow \mathbb{R}$$
$$v \longmapsto J_\varepsilon^\gamma(v).$$

Set

$$A = \{v \in (L^2(\Sigma_0))^2 : J_\varepsilon^\gamma(v) \geq -J_\varepsilon(0,0)\}.$$

We have

$$J_\varepsilon^\gamma(0) = \frac{\varepsilon^2}{\gamma}\|\zeta_\varepsilon(0)\|^2_{L^2(\Sigma_1)} + \frac{\varepsilon^2}{\gamma}\left\|\frac{\partial\zeta_\varepsilon(0)}{\partial\nu}\right\|^2_{L^2(\Sigma_1)} \geq -J_\varepsilon(0,0).$$

We assume that $A \neq \emptyset$. Therefore,

$$d_\varepsilon^\gamma = \inf_{v \in (L^2(\Sigma_0))^2} J_\varepsilon^\gamma(v)$$

exists. Let $v_n = v_n(\varepsilon, \gamma)$ be a minimizing sequence such that

$$d_\varepsilon^\gamma = \lim_{n \longrightarrow \infty} J_\varepsilon^\gamma(v_n),$$

$$- J_\varepsilon(0,0) \leq J_\varepsilon(v_n, 0) - J_\varepsilon(0,0) + \frac{\varepsilon^2}{\gamma}\left(\|\zeta_\varepsilon(v_n)\|^2_{L^2(\Sigma_1)} + \left\| \frac{\partial \zeta_\varepsilon(v_n)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \right) \leq d_\varepsilon^\gamma + 1. \tag{2.11}$$

Consequently, the following estimate holds

$$J_\varepsilon(v_n, 0) + \frac{\varepsilon^2}{\gamma}\left(\|\zeta_\varepsilon(v_n)\|^2_{L^2(\Sigma_1)} + \left\| \frac{\partial \zeta_\varepsilon(v_n)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \right) \leq C_\varepsilon^\gamma,$$

where C_ε^γ is a positive constant independent of n. In particular, with possibly different (yet still independent of n) constants C_ε^γ, we have

$$\begin{cases} J_\varepsilon(v_n, 0) \leq C_\varepsilon^\gamma, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \|\zeta_\varepsilon(v_n)\|^2_{L^2(\Sigma_1)} \leq C_\varepsilon^\gamma, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \left\| \dfrac{\partial \zeta_\varepsilon(v_n)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \leq C_\varepsilon^\gamma. \end{cases} \tag{2.12}$$

The terms of $J_\varepsilon(v_n, 0)$ are positive, so we obtain

$$\begin{cases} \|z_\varepsilon(v_n, 0) - z_d\|^2_{L^2(Q)} \leq C_\varepsilon^\gamma, \\ \|v_n\|_{L^2(\Gamma_0)} \leq C_\varepsilon^\gamma, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \|\zeta_\varepsilon(v_n)\|^2_{L^2(\Sigma_1)} \leq C_\varepsilon^\gamma, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \left\| \dfrac{\partial \zeta_\varepsilon(v_n)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \leq C_\varepsilon^\gamma. \end{cases} \tag{2.13}$$

We have $\|\frac{\partial z_\varepsilon}{\partial t} - \triangle^2 z_\varepsilon - \varepsilon z_\varepsilon\| = 0$. Hence, there exists $C > 0$ such that

$$\left\| \frac{\partial z_\varepsilon}{\partial t} - \triangle^2 z_\varepsilon - \varepsilon z_\varepsilon \right\| \leq C.$$

From (2.10)–(2.13) there exists $C > 0$ such that $\|z_\varepsilon\| \leq C$. Hence, there exist $u_\varepsilon^\gamma = (u_{0\varepsilon}^\gamma, u_{1\varepsilon}^\gamma) \in (L^2(\Sigma_0))^2$, $z_\varepsilon^\gamma \in L^2(Q)$, $\delta \in L^2(Q)$ and sequences v_n and z_n such that

$$\begin{cases} \dfrac{\partial z_\varepsilon}{\partial t} - \triangle^2 z_\varepsilon - \varepsilon z_\varepsilon \rightharpoonup \delta & \text{in } L^2(Q), \\ v_n(\varepsilon, \gamma) \rightharpoonup u_\varepsilon^\gamma & \text{in } L^2(\Sigma_0), \\ z_n \rightharpoonup z_\varepsilon^\gamma & \text{in } L^2(Q). \end{cases}$$

As $v \mapsto J_\varepsilon^\gamma(v)$ is semicontinuous and because $v_n(\varepsilon, \gamma) \rightharpoonup u_\varepsilon^\gamma$ in $(L^2(\Sigma_0))^2$, we obtain $J_\gamma(u^\gamma) \leq \liminf_{n \to \infty} J_\gamma(v_n)$. This implies that there exists m_γ such that $J_\varepsilon^\gamma(u_\varepsilon^\gamma) \leq m_\gamma$. As $m_\gamma \in L^2(Q)$, we conclude that $J_\gamma(u^\gamma) = m_\gamma$. From the strict convexity of the cost function J_ε^γ we also deduce that u_ε^γ is unique. □

2.3 Characterization of the low-regret control

Proposition 2.5 *The approximate low-regret control $u_\varepsilon^\gamma = (u_{0\varepsilon}^\gamma, u_{1\varepsilon}^\gamma)$ to (2.10) is characterized by the unique solution $\{\zeta_\varepsilon^\gamma, z_\varepsilon^\gamma, \beta_\varepsilon^\gamma, \phi_\varepsilon^\gamma\}$ of the system*

$$\begin{cases} -\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial t} - \triangle^2 \zeta_\varepsilon^\gamma - \varepsilon \zeta_\varepsilon^\gamma = -(z_\varepsilon^\gamma - z_\varepsilon(0,0)) & \text{in } Q, \\ \zeta_\varepsilon^\gamma - \dfrac{\partial \triangle \zeta_\varepsilon^\gamma}{\partial \nu} = 0, \ \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} + \triangle \zeta_\varepsilon^\gamma = 0 & \text{on } \Sigma_0, \\ \varepsilon \zeta_\varepsilon^\gamma - \dfrac{\partial \triangle \zeta_\varepsilon^\gamma}{\partial \nu} = 0, \ \varepsilon \dfrac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} + \triangle \zeta_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\ \zeta_\varepsilon^\gamma(T, v) = 0 & \text{in } \Omega, \end{cases} \tag{2.14}$$

$$\begin{cases} \dfrac{\partial z_\varepsilon^\gamma}{\partial t} - \triangle^2 z_\varepsilon^\gamma - \varepsilon z_\varepsilon^\gamma = 0 & \text{in } Q, \\ z_\varepsilon^\gamma - \dfrac{\partial \triangle z_\varepsilon^\gamma}{\partial \nu} = u_{0\varepsilon}^\gamma, \ \dfrac{\partial z_\varepsilon^\gamma}{\partial \nu} + \triangle z_\varepsilon^\gamma = u_{1\varepsilon}^\gamma & \text{on } \Sigma_0, \\ \varepsilon z_\varepsilon^\gamma - \dfrac{\partial \triangle z_\varepsilon^\gamma}{\partial \nu} = 0, \ \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \triangle z_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\ z_\varepsilon^\gamma(0) = z_0^\gamma & \text{in } \Omega, \end{cases} \tag{2.15}$$

$$\begin{cases} \dfrac{\partial \beta_\varepsilon^\gamma}{\partial t} - \triangle^2 \beta_\varepsilon^\gamma - \varepsilon \beta_\varepsilon^\gamma = 0 & \text{in } Q, \\ \beta_\varepsilon^\gamma - \dfrac{\partial \triangle \beta_\varepsilon^\gamma}{\partial \nu} = 0, \ \dfrac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \triangle \beta_\varepsilon^\gamma = 0 & \text{on } \Sigma_0, \\ \varepsilon \beta_\varepsilon^\gamma - \dfrac{\partial \triangle \beta_\varepsilon^\gamma}{\partial \nu} = -\dfrac{\varepsilon^2}{\gamma} \zeta_\varepsilon, \ \varepsilon \dfrac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \triangle \beta_\varepsilon^\gamma = -\dfrac{\varepsilon^2}{\gamma} \dfrac{\partial \zeta_\varepsilon}{\partial \nu} & \text{on } \Sigma_1, \\ \beta_\varepsilon^\gamma(0) = z_0^\gamma & \text{in } \Omega, \end{cases} \tag{2.16}$$

$$\begin{cases} -\dfrac{\partial \phi_\varepsilon^\gamma}{\partial t} - \triangle^2 \phi_\varepsilon^\gamma - \varepsilon \phi_\varepsilon^\gamma = z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma & \text{in } Q, \\ \phi_\varepsilon^\gamma - \dfrac{\partial \triangle \phi_\varepsilon^\gamma}{\partial \nu} = 0, \ \dfrac{\partial \phi_\varepsilon^\gamma}{\partial \nu} + \triangle \phi_\varepsilon^\gamma = 0 & \text{on } \Sigma_0, \\ \varepsilon \phi_\varepsilon^\gamma - \dfrac{\partial \triangle \phi_\varepsilon^\gamma}{\partial \nu} = 0, \ \varepsilon \dfrac{\partial \phi_\varepsilon^\gamma}{\partial \nu} + \triangle \phi_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\ \phi_\varepsilon^\gamma(T, v) = 0 & \text{in } \Omega, \end{cases} \tag{2.17}$$

$$\phi_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma = 0 \text{ in } L^2(\Sigma_0), \tag{2.18}$$

where $z_\varepsilon^\gamma := z_\varepsilon(u_\varepsilon^\gamma, 0)$.

Proof. Let u_ε^γ be the solution of (2.8)–(2.10) in $L^2(\Sigma_0)$. The Euler–Lagrange necessary condition implies that for every $w = (w_0, w_1) \in (L^2(\Sigma_0))^2$ we have

$$\begin{aligned} &\langle z_\varepsilon^\gamma - z_d, z_\varepsilon(w, 0)\rangle + N_0 \langle u_{0\varepsilon}^\gamma, w_0\rangle_{\Sigma_0} + N_1 \langle u_{1\varepsilon}^\gamma, w_1\rangle_{\Sigma_0} \\ &\qquad + \left\langle \frac{\varepsilon^2}{\gamma} \zeta_\varepsilon^\gamma, \zeta_\varepsilon(w) \right\rangle_{\Sigma_1} + \left\langle \frac{\varepsilon^2}{\gamma} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu}, \frac{\partial \zeta_\varepsilon(w)}{\partial \nu} \right\rangle_{\Sigma_1} = 0. \end{aligned} \tag{2.19}$$

Let $\beta_\varepsilon^\gamma := \beta^\gamma(u_\varepsilon^\gamma, 0)$ be a solution of the following system

$$\begin{cases} \dfrac{\partial \beta_\varepsilon^\gamma}{\partial t} - \triangle^2 \beta_\varepsilon^\gamma - \varepsilon \beta_\varepsilon^\gamma = 0 & \text{in } Q, \\ \beta_\varepsilon^\gamma - \dfrac{\partial \triangle \beta_\varepsilon^\gamma}{\partial \nu} = 0, \ \dfrac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \triangle \beta_\varepsilon^\gamma = 0 & \text{on } \Sigma_0, \\ \varepsilon \beta_\varepsilon^\gamma - \dfrac{\partial \triangle \beta_\varepsilon^\gamma}{\partial \nu} = -\dfrac{\varepsilon^2}{\gamma} \zeta_\varepsilon, \ \varepsilon \dfrac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \triangle \beta_\varepsilon^\gamma = -\dfrac{\varepsilon^2}{\gamma} \dfrac{\partial \zeta_\varepsilon}{\partial \nu} & \text{on } \Sigma_1, \\ \beta_\varepsilon^\gamma(0) = z_0^\gamma & \text{on } \Omega. \end{cases} \tag{2.20}$$

From Lemma 2.2 we know that $-\frac{\varepsilon^2}{\gamma}\zeta_\varepsilon, -\frac{\varepsilon^2}{\gamma}\frac{\partial \zeta_\varepsilon}{\partial \nu} \in L^2(\Sigma_1)$ and from Lemma 2.3 we infer that the solution $\beta_\varepsilon^\gamma \in L^2(Q)$ of (2.20) is unique. By multiplying the first equation of (2.20) by $\zeta_\varepsilon(w, 0)$ and integrating over Q, we obtain

$$\int_0^T\!\!\int_\Omega \frac{\partial \beta_\varepsilon^\gamma}{\partial t} \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega \left(\triangle^2 \beta_\varepsilon^\gamma + \varepsilon \beta_\varepsilon^\gamma\right) \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Now, we follow a similar approach to the one we used in the proof of Lemma 2.3. Integrating by parts we obtain

$$\int_0^T\!\!\int_\Omega \frac{\partial \beta_\varepsilon^\gamma}{\partial t} . \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \left[\beta_\varepsilon^\gamma . \zeta_\varepsilon(w,0)\right]_0^T \,\mathrm{d}x - \int_0^T\!\!\int_\Omega \beta_\varepsilon^\gamma \frac{\partial \zeta_\varepsilon(w,0)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\begin{aligned} \int_0^T\!\!\int_\Omega \frac{\partial \beta_\varepsilon^\gamma}{\partial t} \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \beta_\varepsilon^\gamma(T,v).\zeta_\varepsilon(T;w,0)\,\mathrm{d}x - \int_\Omega \beta_\varepsilon^\gamma(0,v).\zeta_\varepsilon(0;w,0)\,\mathrm{d}x \\ &\quad - \int_0^T\!\!\int_\Omega \beta_\varepsilon^\gamma(v) \frac{\partial \zeta_\varepsilon(w,0)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &= - \int_0^T\!\!\int_\Omega \beta_\varepsilon^\gamma(v) \frac{\partial \zeta_\varepsilon(w,0)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Using the Green formula we get

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \triangle^2 \beta_\varepsilon^\gamma . \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\!\!\int_\Gamma \frac{\partial(\triangle \beta_\varepsilon^\gamma(v))}{\partial \nu} . \zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\!\!\int_\Omega \nabla(\triangle \beta_\varepsilon^\gamma(v)) \nabla \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\!\!\int_\Gamma \frac{\partial(\triangle \beta_\varepsilon^\gamma(v))}{\partial \nu} \zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\!\!\int_\Gamma \triangle \beta_\varepsilon^\gamma(v) \frac{\partial \zeta_\varepsilon(w,0)}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t \\ &\quad + \int_0^T\!\!\int_\Gamma \frac{\partial \beta_\varepsilon^\gamma(v)}{\partial \nu} \triangle \zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\!\!\int_\Gamma \beta_\varepsilon^\gamma(v). \frac{\partial \triangle \zeta_\varepsilon(w,0)}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t \\ &\quad + \int_0^T\!\!\int_\Omega \beta_\varepsilon^\gamma(v). \triangle^2 \zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\!\!\int_{\Gamma_1} \frac{\partial(\triangle \beta_\varepsilon^\gamma(v))}{\partial \nu} \zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\!\!\int_{\Gamma_1} \triangle \beta_\varepsilon^\gamma(v) \frac{\partial \zeta_\varepsilon(w,0)}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&- \int_0^T\int_{\Gamma_1} \frac{\partial \beta_\varepsilon^\gamma(v)}{\partial \nu}.\varepsilon \frac{\partial \zeta_\varepsilon(w,0)}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_{\Gamma_1} \beta_\varepsilon^\gamma(v).\varepsilon\zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t \\
&+ \int_0^T\int_\Omega \beta_\varepsilon^\gamma(v).\triangle^2\zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\
= &\int_0^T\int_{\Gamma_1} \left(\frac{\partial(\triangle\beta_\varepsilon^\gamma(v))}{\partial\nu} - \varepsilon\beta_\varepsilon^\gamma(v)\right)\zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t \\
&- \int_0^T\int_{\Gamma_1} \left(\triangle\beta_\varepsilon^\gamma(v) + \varepsilon\frac{\partial\beta_\varepsilon^\gamma(v)}{\partial\nu}\right)\frac{\partial\zeta_\varepsilon(w,0)}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Omega \beta_\varepsilon^\gamma(v).\triangle^2\zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\
= &\int_0^T\int_{\Gamma_1} \frac{\varepsilon^2}{\gamma}\zeta_\varepsilon^\gamma.\zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_{\Gamma_1} \frac{\varepsilon^2}{\gamma}\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}.\frac{\partial\zeta_\varepsilon}{\partial\nu}(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t \\
&+ \int_0^T\int_\Omega \beta_\varepsilon^\gamma(v).\triangle^2\zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial\beta_\varepsilon^\gamma}{\partial t}\zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \left(\triangle^2\beta_\varepsilon^\gamma + \varepsilon\beta_\varepsilon^\gamma\right)\zeta_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_0^T\int_{\Gamma_1} \frac{\varepsilon^2}{\gamma}\zeta_\varepsilon^\gamma.\zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_{\Gamma_1} \frac{\varepsilon^2}{\gamma}\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}.\frac{\partial\zeta_\varepsilon}{\partial\nu}(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad + \int_0^T\int_\Omega \left(-\frac{\partial\zeta_\varepsilon(w,0)}{\partial t} - \triangle^2\zeta_\varepsilon(w,0) - \varepsilon\zeta_\varepsilon(w,0)\right)\beta_\varepsilon^\gamma(v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

And so we obtain

$$
\int_0^T\int_\Omega \beta_\varepsilon^\gamma.z_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_{\Gamma_1} \frac{\varepsilon^2}{\gamma}\zeta_\varepsilon^\gamma.\zeta_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_{\Gamma_1} \frac{\varepsilon^2}{\gamma}\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}.\frac{\partial\zeta_\varepsilon}{\partial\nu}(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t = 0.
$$

Thus, we introduce the adjoint ϕ_ε^γ defined by

$$
\begin{cases}
-\dfrac{\partial\phi_\varepsilon^\gamma}{\partial t} - \triangle^2\phi_\varepsilon^\gamma - \varepsilon\phi_\varepsilon^\gamma = z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma & \text{in } Q, \\
\phi_\varepsilon^\gamma - \dfrac{\partial\triangle\phi_\varepsilon^\gamma}{\partial\nu} = 0,\ \dfrac{\partial\phi_\varepsilon^\gamma}{\partial\nu} + \triangle\phi_\varepsilon^\gamma = 0 & \text{on } \Sigma_0, \\
\varepsilon\phi_\varepsilon^\gamma - \dfrac{\partial\triangle\phi_\varepsilon^\gamma}{\partial\nu} = 0,\ \varepsilon\dfrac{\partial\phi_\varepsilon^\gamma}{\partial\nu} + \triangle\phi_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\
\phi_\varepsilon^\gamma(T,v) = 0 & \text{in } \Omega.
\end{cases}
$$

By multiplying the first equation of the above system by $z_\varepsilon(w,0)$ and integrating over Q, we obtain

$$
\begin{aligned}
&-\int_0^T\int_\Omega \frac{\partial\phi_\varepsilon^\gamma}{\partial t}z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \left(\triangle^2\phi_\varepsilon^\gamma + \varepsilon\phi_\varepsilon^\gamma\right)z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad = \int_0^T\int_\Omega \left(z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma\right)z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

One again we use a similar approach to that used in the proof of Lemma 2.6. Integrating by parts yield

$$\int_0^T\int_\Omega \frac{\partial\phi_\varepsilon^\gamma}{\partial t}.z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \left[\phi_\varepsilon^\gamma.z_\varepsilon(w,0)\right]_0^T\,\mathrm{d}x - \int_0^T\int_\Omega \phi_\varepsilon^\gamma \frac{\partial z_\varepsilon(w,0)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial\phi_\varepsilon^\gamma}{\partial t} z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \phi_\varepsilon^\gamma(T,v).z_\varepsilon(T;w,0)\,\mathrm{d}x - \int_\Omega \phi_\varepsilon^\gamma(0,v).z_\varepsilon(0;w,0)\,\mathrm{d}x\\ &\quad - \int_0^T\int_\Omega \phi_\varepsilon^\gamma(v)\frac{\partial z_\varepsilon(w,0)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_0^T\int_\Omega \phi_\varepsilon^\gamma(v)\frac{\partial z_\varepsilon(w,0)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using the Green formula we get

$$\begin{aligned}&\int_0^T\int_\Omega \triangle^2\phi_\varepsilon^\gamma.z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Gamma \frac{\partial(\triangle\phi_\varepsilon^\gamma(v))}{\partial\nu}.z_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_\Omega \nabla(\triangle\phi_\varepsilon^\gamma(v))\nabla z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Gamma \frac{\partial(\triangle\phi_\varepsilon^\gamma(v))}{\partial\nu} z_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_\Gamma \triangle\phi_\varepsilon^\gamma(v)\frac{\partial z_\varepsilon(w,0)}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t\\ &\quad + \int_0^T\int_\Gamma \frac{\partial\phi_\varepsilon^\gamma(v)}{\partial\nu}\triangle z_\varepsilon(w,0)\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_\Gamma \phi_\varepsilon^\gamma(v).\frac{\partial\triangle z_\varepsilon(w,0)}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t\\ &\quad + \int_0^T\int_\Omega \phi_\varepsilon^\gamma(v).\triangle^2 z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_0^T\int_{\Gamma_0} \phi_\varepsilon^\gamma(v).w\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Omega \phi_\varepsilon^\gamma(v).\triangle^2 z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}&-\int_0^T\int_\Omega \frac{\partial\phi_\varepsilon^\gamma}{\partial t} z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \left(\triangle^2\phi_\varepsilon^\gamma + \varepsilon\phi_\varepsilon^\gamma\right) z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_{\Gamma_0} \phi_\varepsilon^\gamma(v).w\,\mathrm{d}\sigma\,\mathrm{d}t\\ &\quad + \int_0^T\int_\Omega \left(\phi_\varepsilon^\gamma(v)\frac{\partial z_\varepsilon(w,0)}{\partial t} - \phi_\varepsilon^\gamma(v).\triangle^2 z_\varepsilon(w,0) - \varepsilon\phi_\varepsilon^\gamma(v) z_\varepsilon(w,0)\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

And so,

$$-\int_0^T\int_\Omega \frac{\partial\phi_\varepsilon^\gamma}{\partial t} z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \left(\triangle^2\phi_\varepsilon^\gamma + \varepsilon\phi_\varepsilon^\gamma\right) z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Gamma_0} \phi_\varepsilon^\gamma(v).w\,\mathrm{d}\sigma\,\mathrm{d}t.$$

In short, we obtain

$$\int_0^T\int_{\Gamma_0} \phi_\varepsilon^\gamma(v).w\,\mathrm{d}\sigma\,\mathrm{d}t = \int_0^T\int_\Omega \left(z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma\right) z_\varepsilon(w,0)\,\mathrm{d}x\,\mathrm{d}t.$$

And then (2.19) becomes

$$\int_0^T\int_{\Gamma_0} (\phi_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma) .w \,\mathrm{d}\sigma\,\mathrm{d}t = 0 \text{ for every } w \in (L^2(\Sigma_0))^2.$$

Thus, $\phi_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma = 0$ in Q. Therefore,

$$\phi_\varepsilon^\gamma = -N_0 u_{0\varepsilon}^\gamma - N_1 u_{1\varepsilon}^\gamma \text{ in } L^2(\Sigma_0). \tag{2.21}$$

This ends the proof. □

2.4 Singular optimality system (SOS).

In this section, we describe the SOS for low-regret control for the problem (1.4)

Lemma 2.6 *There exists a constant $C > 0$ such that*

$$\begin{cases} \|u_{0\varepsilon}^\gamma\|_{L^2(\Sigma_0)} \le C, \\ \|u_{1\varepsilon}^\gamma\|_{L^2(\Sigma_0)} \le C, \\ \|z_\varepsilon^\gamma\|_{L^2(Q)}^2 \le C, \\ \dfrac{\varepsilon}{\sqrt{\gamma}}\|\zeta_\varepsilon^\gamma\|_{L^2(\Sigma_1)}^2 \le C, \\ \dfrac{\varepsilon}{\sqrt{\gamma}}\|\frac{\partial \zeta_\varepsilon^\gamma}{\partial\nu}\|_{L^2(\Sigma_1)}^2 \le C. \end{cases} \tag{2.22}$$

Proof. As u_ε^γ is a solution of (2.10), for every $v \in (L^2(\Sigma_0))^2$ we have $J_\varepsilon^\gamma(u_\varepsilon^\gamma) \le J_\varepsilon^\gamma(v)$. In particular, when $v = 0$ we obtain

$$\begin{aligned} &\|z_\varepsilon^\gamma - z_d\|_{L^2(Q)}^2 + N_0\|u_{0\varepsilon}^\gamma\|_{L^2(\Sigma_0)}^2 + N_1\|u_{1\varepsilon}^\gamma\|_{L^2(\Sigma_0)}^2 + \frac{\varepsilon^2}{\gamma}\left(\|\zeta_\varepsilon^\gamma\|_{L^2(\Sigma_1)}^2 + \left\|\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}\right\|_{L^2(\Sigma_1)}^2\right) \\ &\quad\le \|z_\varepsilon(0,0) - z_d\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{\gamma}\left(\|\zeta_\varepsilon(0,0)\|_{L^2(\Sigma_1)}^2 + \left\|\frac{\partial\zeta_\varepsilon}{\partial\nu}(0,0)\right\|_{L^2(\Sigma_1)}^2\right). \end{aligned}$$

But $z_\varepsilon(0,0) = 0$ in Q, $\zeta_\varepsilon(0,0) = 0$ and $\frac{\partial\zeta_\varepsilon}{\partial\nu}(0,0) = 0$ on Σ_1. Then,

$$\begin{aligned} &\|z_\varepsilon^\gamma - z_d\|_{L^2(Q)}^2 + N_0\|u_{0\varepsilon}^\gamma\|_{L^2(\Sigma_0)}^2 + N_1\|u_{1\varepsilon}^\gamma\|_{L^2(\Sigma_0)}^2 + \frac{\varepsilon^2}{\gamma}\left(\|\zeta_\varepsilon^\gamma\|_{L^2(\Sigma_1)}^2 + \left\|\frac{\partial\zeta_\varepsilon^\gamma}{\partial\nu}\right\|_{L^2(\Sigma_1)}^2\right) \\ &\le \|z_d\|_{L^2(Q)}^2 = C. \end{aligned}$$

This implies the result. □

Theorem 2.7 *The low-regret control u^γ for problem* (1.4) *is characterized by $\{\zeta^\gamma, z^\gamma, \beta^\gamma, \phi^\gamma\}$ satisfying*

$$\begin{cases} \dfrac{\partial\zeta^\gamma}{\partial t} - \triangle\zeta^\gamma = 0 & \text{in } Q, \\ \zeta^\gamma = 0,\ \dfrac{\partial\zeta^\gamma}{\partial\nu} = 0 & \text{on } \Sigma_0, \\ \zeta^\gamma(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.23}$$

$$\begin{cases} \dfrac{\partial z^\gamma}{\partial t} - \triangle z^\gamma = 0 & \textit{in } Q, \\ z^\gamma = u_0^\gamma, \ \dfrac{\partial z^\gamma}{\partial \nu} = u_1^\gamma & \textit{on } \Sigma_0, \\ z^\gamma(0) = z_0^\gamma & \textit{in } \Omega, \end{cases} \tag{2.24}$$

$$\begin{cases} \dfrac{\partial \beta^\gamma}{\partial t} - \triangle \beta^\gamma = 0 & \textit{in } Q, \\ \beta^\gamma = 0, \ \dfrac{\partial \beta^\gamma}{\partial \nu} = 0 & \textit{on } \Sigma_0, \\ \beta^\gamma(T) = 0 & \textit{in } \Omega, \end{cases} \tag{2.25}$$

$$\begin{cases} \dfrac{\partial \phi^\gamma}{\partial t} - \triangle \phi_\varepsilon^\gamma = z^\gamma - z_d - \beta^\gamma & \textit{in } Q, \\ \phi^\gamma = 0, \ \dfrac{\partial \phi^\gamma}{\partial \nu} = 0 & \textit{on } \Sigma_0, \\ \phi(T) = 0 & \textit{in } \Omega, \end{cases} \tag{2.26}$$

$$\phi^\gamma + N_0 u_0^\gamma + N_1 u_1^\gamma = 0 \textit{ in } L^2(\Sigma_0). \tag{2.27}$$

Proof. From Proposition 2.5, we deduce that z_ε^γ is a solution of the system

$$\begin{cases} \dfrac{\partial z_\varepsilon^\gamma}{\partial t} - \triangle^2 z_\varepsilon^\gamma - \varepsilon z_\varepsilon^\gamma = 0 & \text{in } Q, \\ z_\varepsilon^\gamma - \dfrac{\partial \triangle z_\varepsilon^\gamma}{\partial \nu} = u_{0\varepsilon}^\gamma, \ \dfrac{\partial z_\varepsilon^\gamma}{\partial \nu} + \triangle z_\varepsilon^\gamma = u_{1\varepsilon}^\gamma & \text{on } \Sigma_0, \\ \varepsilon z_\varepsilon^\gamma - \dfrac{\partial \triangle z_\varepsilon^\gamma}{\partial \nu} = 0, \ \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \triangle z_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\ z_\varepsilon^\gamma(0) = z_0^\gamma & \text{in } \Omega. \end{cases} \tag{2.28}$$

Let $\eta_\varepsilon^\gamma = \triangle z_\varepsilon^\gamma$. Then, the system (2.28) becomes

$$\begin{cases} \dfrac{\partial z_\varepsilon^\gamma}{\partial t} - \triangle \eta_\varepsilon^\gamma - \varepsilon z_\varepsilon^\gamma = 0 & \text{in } Q, \\ z_\varepsilon^\gamma - \dfrac{\partial \eta_\varepsilon^\gamma}{\partial \nu} = u_{0\varepsilon}^\gamma, \ \dfrac{\partial z_\varepsilon^\gamma}{\partial \nu} + \eta_\varepsilon^\gamma = u_{1\varepsilon}^\gamma & \text{on } \Sigma_0, \\ \varepsilon z_\varepsilon^\gamma - \dfrac{\partial \eta_\varepsilon^\gamma}{\partial \nu} = 0, \ \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \eta_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\ z_\varepsilon^\gamma(0) = z_0^\gamma & \text{in } \Omega. \end{cases} \tag{2.29}$$

From Lemma 2.6 we know that $\|z_\varepsilon^\gamma\|_{L^2(Q)}^2 \leq C$. Therefore, z_ε^γ converges weakly in $L^2(Q)$ and tends to z^γ when $\varepsilon \longrightarrow 0$ ($z_\varepsilon^\gamma \longrightarrow z^\gamma$). Thus, from (2.29) we obtain $\|\triangle \eta_\varepsilon^\gamma\| = \|\frac{\partial z_\varepsilon^\gamma}{\partial t} - \varepsilon z_\varepsilon^\gamma\| \leq \varepsilon C \longrightarrow 0$. Therefore,

$$\begin{cases} \triangle \eta_\varepsilon^\gamma \longrightarrow 0 & \text{in } L^2(Q), \\ \dfrac{\partial \eta_\varepsilon^\gamma}{\partial \nu} \longrightarrow 0 & \text{in } L^2(\Sigma_1), \\ \eta_\varepsilon^\gamma \longrightarrow 0 & \text{in } L^2(\Sigma_1), \end{cases} \tag{2.30}$$

when $\varepsilon \longrightarrow 0$. We recap that

$$\begin{cases} \dfrac{\partial \eta^\gamma}{\partial t} - \triangle \eta_\varepsilon^\gamma = 0 & \text{in } Q, \\ \dfrac{\partial \eta_\varepsilon^\gamma}{\partial \nu} = 0 & \text{on } \Sigma_1, \\ \eta_\varepsilon^\gamma = 0 & \text{on } \Sigma_1, \\ \eta^\gamma(T) = 0 & \text{in } \Omega. \end{cases} \tag{2.31}$$

By using the unique continuation theorem of Mizohata [8], from (2.31) we deduce that we also have $\eta_\varepsilon^\gamma \equiv 0$ in $\bar{Q}$. Then,

$$\frac{\partial \eta_\varepsilon^\gamma}{\partial \nu} = \eta_\varepsilon^\gamma = 0 \text{ on } \Sigma_0. \tag{2.32}$$

On the other hand, Lemma 2.6 also gives $\|u_{0\varepsilon}^\gamma\|_{L^2(\Sigma_0)} \leq C$ and $\|u_{1\varepsilon}^\gamma\|_{L^2(\Sigma_0)} \leq C$. Therefore,

$$(u_{0\varepsilon}^\gamma, u_{1\varepsilon}^\gamma) \rightharpoonup (u_0^\gamma, u_1^\gamma) \text{ weakly in } L^2(\Sigma_0) \times L^2(\Sigma_0). \tag{2.33}$$

From (2.29)–(2.30)–(2.33) we obtain

$$\begin{cases} \dfrac{\partial z^\gamma}{\partial t} - \triangle z^\gamma = 0 & \text{in } Q, \\ z^\gamma = u_0^\gamma, \ \dfrac{\partial z^\gamma}{\partial \nu} = u_1^\gamma & \text{on } \Sigma_0, \\ z(0) = z_0 & \text{in } \Omega. \end{cases} \tag{2.34}$$

Again, we use the estimate of Lemma 2.6 and deduce that $\frac{\varepsilon}{\sqrt{\gamma}} \zeta_\varepsilon^\gamma \rightharpoonup \lambda_0^\gamma$ weakly in $L^2(\Sigma_1)$ and $\frac{\varepsilon}{\sqrt{\gamma}} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \rightharpoonup \lambda_1^\gamma$ weakly in $L^2(\Sigma_1)$. Thus, $\frac{\varepsilon^2}{\gamma} \zeta_\varepsilon^\gamma \longrightarrow 0$ and $\frac{\varepsilon^2}{\gamma} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \longrightarrow 0$ when $\varepsilon \longrightarrow 0$. We obtain

$$\begin{cases} \dfrac{\partial \zeta^\gamma}{\partial t} - \triangle \zeta^\gamma = z^\gamma - z(0,0) & \text{in } Q, \\ \zeta^\gamma = 0, \ \dfrac{\partial \zeta^\gamma}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \zeta^\gamma(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.35}$$

and $\phi_\varepsilon^\gamma = -N_0 u_{0\varepsilon}^\gamma - N_1 u_{1\varepsilon}^\gamma$ in $L^2(\Sigma_0)$. Finally, from (2.22) and (2.33) we conclude that

$$\phi_\varepsilon^\gamma \rightharpoonup \phi^\gamma = -N_0 u_0^\gamma - N_1 u_1^\gamma \text{ weakly in } L^2(\Sigma_0). \tag{2.36}$$

The proof is complete. □

Remark 2.8 *Note that $\zeta(u^\gamma)$ converges to $\zeta := \zeta(u)$ and (ζ, z) satisfies* (2.9)*. In addition to* (2.22)*, $\frac{\partial \zeta(u^\gamma)}{\partial t} \longrightarrow 0$ strongly. Furthermore, $\frac{\partial \zeta(u^\gamma)}{\partial t} = \frac{\partial \zeta(u)}{\partial t} = 0$ and $u \in \mathcal{A}$. We can therefore conclude that the low-regret control u^γ converges to the no-regret control u.*

2.5 Characterization of the no-regret control

Now, we describe the optimality system for no-regret control.

Theorem 2.9 *The no-regret control $u = (u_0, u_1)$ for problem* (1.4) *is characterized by the unique solution $\{\zeta, z, \beta, \phi\}$ to*

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} - \triangle \zeta = 0 & \text{in } Q, \\ \zeta = 0, \ \dfrac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \zeta(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.37}$$

$$\begin{cases} \dfrac{\partial z}{\partial t} - \triangle z = 0 & \text{in } Q, \\ z = u_0, \ \dfrac{\partial z}{\partial \nu} = u_1 & \text{on } \Sigma_0, \\ z(0) = z_0 & \text{in } \Omega, \end{cases} \tag{2.38}$$

$$\begin{cases} \dfrac{\partial \beta}{\partial t} - \triangle \beta = 0 & \text{in } Q, \\ \beta = 0, \ \dfrac{\partial \beta}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \beta(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.39}$$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \triangle \phi = z - z_d - \beta & \text{in } Q, \\ \phi = 0, \ \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \phi(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.40}$$

$$\phi + N_0 u_0 + N_1 u_1 = 0 \text{ in } L^2(\Sigma_0). \tag{2.41}$$

Proof. From Theorem 2.7, passing to the limit with $\gamma \longrightarrow 0$, we obtain:

$$\begin{cases} \zeta^\gamma \rightharpoonup \zeta = 0 & \text{on } \Sigma_0, \\ \beta^\gamma \rightharpoonup \beta = 0 & \text{on } \Sigma_0, \\ \phi^\gamma \rightharpoonup \phi = 0 & \text{on } \Sigma_0. \end{cases} \tag{2.42}$$

From (2.27), we infer that

$$(u_0^\gamma, u_1^\gamma) \rightharpoonup (u_0, u_1) \text{ weakly in } L^2(\Sigma_0) \times L^2(\Sigma_0). \tag{2.43}$$

Therefore,

$$\begin{cases} z^\gamma \rightharpoonup z = u_0 & \text{on } \Sigma_0, \\ \dfrac{\partial z^\gamma}{\partial \nu} \rightharpoonup \dfrac{\partial z}{\partial \nu} = u_1 & \text{on } \Sigma_0. \end{cases} \tag{2.44}$$

From what precedes, there exists a unique u characterized by the solution $\{\zeta, z, \beta, \phi\}$ of the system (1.4). □

3 Conclusion

In this work, we have examined an ill-posed problem with incomplete data using the regularization method. This method allowed us to generate the missing information on Σ_1 without which the control of the system was delicate. We obtained the characterization of the control problem (1.4) by the regularized problem (2.1). By using the low-regret method and by passing to the limit with $\gamma \longrightarrow 0$, we obtained the no-regret control.

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