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SOME RESULTS ON CONTROLLABILITY AND OPTIMAL CONTROLS FOR A FRACTIONAL NON-AUTONOMOUS SYSTEM WITH A DEVIATED ARGUMENT

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Abstract. In this paper, we are concerned with a non integer order non autonomous system with a deviated argument. Existence and uniqueness of mild solution and controllability of the problem are proved by using the concept of measure of non compactness and Mönch's fixed theorem. Further, we studied the optimal controllability of the problem. In the last section, results are validated by an example.

Keywords: controllability, optimal control, fractional non autonomous system, deviated argument, measure of non compactness, Mönch's condition.

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1 Introduction

Controllability is an important concept in the study of control problems. Different types of control systems in abstract spaces have been investigated by many authors. In the last decade, controllability of fractional order system have drawn the attention of many researchers [9, 11, 16]. In addition, some authors obtained optimal control for non integer order system in abstract spaces [12, 13].

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Mahto and Abbas obtained the optimal control for a class of non integer order delay impulsive differential equations with non local conditions [12]. Qin *et al.* [13], have studied the approximate controllability and the existence of optimal control for a fractional non autonomous system in a Hilbert space assuming that corresponding linear system is approximately controllable.

Mathematical models where coefficients of derivatives depend on time t, can be written as non autonomous systems in abstract spaces. Initially many authors have proved the existence and uniqueness results for non autonomous systems by using the theoretical tools of abstract differential equations [1,5]. Recently, authors have shown great interest in the study of controllability for non autonomous systems [3,5–7,10]. Leiva used the Rothe's type fixed point theorem to prove the controllability for a non autonomous system of impulsive differential equations [3]. Vijayakumar and Murugesu [6] have proved the existence and controllability for a second order non autonomous system without using compactness. For other work on controllability of second order non-autonomous systems, we refer [7]. There are only few papers dealing with the controllability for fractional non autonomous systems. For basic theories of non autonomous systems, we refer [4,14].

Motivated by all above works, we extended the study of controllability to non autonomous systems of order $\gamma \in (0,1)$. Using the idea of probability density functions defined in [2], we defined the mild solution and then by using the fixed point theorem we proved the controllability of the problem. Further we obtained an optimal control. To the best of our knowledge, there are only few papers dealing with the controllability for fractional non autonomous systems.

We consider the following non integer order non-autonomous system with a deviated argument in a Banach space $(Z, \|\cdot\|)$:

$$\begin{cases} {}_CD^\gamma z(t) = -A(t)z(t) + Ew(t) + g(t, z(t), z(b(z(t),t))), & t \in (0, b_0] \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where $0 < \gamma \leq 1$, ${}_CD^\gamma$ denotes the Caputo fractional derivative, $z_0 \in Z$. $\{A(t)\}_{t\in[0,b_0]}$ is a family of closed linear operators in Z satisfying the following properties:

(i) The domain $D(A(t))$ is independent of t and is dense in Z *i.e.* $\overline{D(A(t))} = Z$.

(ii) For any p with $\mathrm{Re}(p) \geq 0$, $(A(t)+p)^{-1}$ exists and satisfies

$$\|(A(t)+p)^{-1}\| \leq \frac{M}{|p|+1}, \quad t \in [0, b_0].$$

(iii) There exist constants L_A and $0 \leq \alpha \leq 1$ such that

$$\|(A(t) - A(\nu))A(s)^{-1}\| \leq L_A \|t - \nu\|^\alpha \quad \text{for } t, \nu, s \in [0, b_0].$$

Linear operator E is bounded and defined on a Banach space W, $w \in L^2([0, b_0], W)$, denotes the control variable, g and b satisfy some suitable conditions to be specified later.

The main purpose of this paper is to prove the controllability and to find the optimal control for the problem (1.1) by using the concept of measure of non compactness and the fixed point theorem.

2 Preliminaries and Assumptions

Under the conditions (i) and (ii), operator $-A(\nu)$ generates an analytic semigroup $e^{-tA(\nu)}$, $\nu \in [0, b_0]$. The fractional power $A^\mu(t)$ of $A(t)$ is well defined for all $0 \leq \mu \leq 1$ (cf., [14]). The space

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$Z_{\mu,t} = \big(D(A^{\mu}(t)), \|\cdot\|_{\mu}\big)$ is a Banach space, where

$$\|\psi\|_{\mu} = \sup_{t\in[0,b_0]} \|A^{\mu}(t)\psi\|, \quad \psi \in D(A^{\mu}(t)).$$

Let $I_0 = [0, b_0]$ and

$$C(I_0, Z_{\mu,t}) = \{z : I_0 \to Z_{\mu,t} \mid z \text{ is continuous on } I_0\},$$

$$C_L(I_0, Z_{\mu,t}) = \{z \in C(I_0, Z_{\mu,t}) \mid \|z(t) - z(s)\|_{\mu} \le L|t-s|\}.$$

Obviously $(C_L(I_0, Z_{\mu,t}), \|\cdot\|_{\mu,b_0})$ is a Banach space, where

$$\|z\|_{\mu,b_0} = \sup_{s\in I_0} \|z(s)\|_{\mu}.$$

Consider a non-autonomous fractional linear system:

$$\begin{cases} {}_C D_t^{\gamma} z(t) = A(t)z(t), \\ z(0) = \xi, \end{cases} \tag{2.1}$$

where $\gamma \in (0,1)$, and closed operator $A(t) : D(A(t)) \subset Z \to Z$ is densely defined in a Banach space Z.

Definition 2.1 *[15] A measure of non compactness defined on a Banach space Z is a function defined from Z to a positive cone of an ordered Banach space $(F, \le)$ such that $\phi(\bar{ch}B) = \phi(B)$ for all bounded subset B of Z, where $\bar{ch}$ denotes the closure of convex hull of B.*

Lemma 2.2 *[15] Let ϕ be a measure of non compactness defined on a Banach space Z, if*

(i) ϕ is monotone, then for every bounded subsets B_1 and B_2 of Z, we have

$$B_1 \subset B_2 \Rightarrow \phi(B_1) \le \phi(B_2).$$

(ii) ϕ is nonsingular $\Leftrightarrow \phi(\{y\} \cup B) = \phi(B)$ for every $y \in Z$, $B \subset Z$.

(iii) $\phi(B) = 0$ if and only if B is precompact in Z.

Lemma 2.3 *[15] Let Ω_0 be a closed convex subset of a Banach space Z and f_0 be a continuous map defined on Ω_0. If f_0 satisfies the (Mönch's) condition: $C_0 \subseteq \Omega_0$ is countable, $C_0 \subseteq \bar{ch}(\{0\} \cup f_0(C_0)) \Rightarrow \bar{C}_0$ is compact, then f_0 has a fixed point in Ω_0.*

In the rest of the paper, we assume the following conditions:

$(H1)$ The map $g : I_0 \times C(I_0, Z_{\mu,t}) \times C_L(I_0, Z_{\mu,t}) \to Z$ satisfies:

(i) $\|g(t_1, z_1(t), \tilde{z}_1(t)) - g(t_2, z_2(t), \tilde{z}_2(t))\|$
$\le L_g[|t_1 - t_2| + \|z_1(t) - z_2(t)\|_{\mu} + \|\tilde{z}_1(t) - \tilde{z}_2(t)\|_{\mu}]$,
for some $L_g > 0$, and for all $t_1, t_2 \in I_0$, $z_1, z_2 \in C\left(I_0, Z_{\mu,t}\right)$, $\tilde{z}_1, \tilde{z}_2 \in C_L\left(I_0, Z_{\mu,t}\right)$.

(ii) For some $r > 0$, there exists $G_r \in L^{\frac{1}{\gamma_1}}(I_0, R^+)$ such that for any $t \in I_0$ and $z, \tilde{z} \in Z$ satisfying $\|z\| \le r$, $\|\tilde{z}\| \le r$,

$$\|g(t, z(t), \tilde{z}(t))\| \le G_r(t),$$

and

$$\liminf_{r\to\infty} \frac{\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}}{r} = \sigma < +\infty,$$

where $\gamma_1 \le \min\{\gamma, \alpha\}$.

$(H2)$ The map $b : C(I_0, Z_{\mu,t}) \times I_0 \to I_0$ satisfies:

$$\|b(z_1(t), t) - b(z_2(t), t)\| \le L_b\|z_1(t) - z_2(t)\|_\mu,$$

for some $L_b > 0$, and for all $t \in I_0$, $z_1, z_2 \in C(I_0, Z_{\mu,t})$.

$(H3)$ The linear operator $E : L^2(I_0, W) \to L^1(I_0, W)$ is bounded. Also the operator $Q : L^2(I_0, W) \to Z$ defined by:

$$Qw = \int_0^{b_0} U_\gamma(b_0 - \nu, \nu)Ew(\nu)\,\mathrm{d}\nu + \int_0^{b_0}\int_0^{\nu} U_\gamma(b_0 - \nu, \nu)V(\nu, s)Ew(s)\,\mathrm{d}s\,\mathrm{d}\nu$$

has bounded inverse *i.e.* $\|E\| \le M_2$ and $\|Q^{-1}\| \le M_3$, for some $M_2, M_3 > 0$.

3 Exact Controllability

Following the idea of the work [2], we define the following functions:

The Laplace transform of the probability density function ρ_γ (defined in [2]) is given by:

$$\int_0^\infty e^{-\nu z}\rho_\gamma(\nu)\,\mathrm{d}\nu = \sum_{j=0}^{\infty} \frac{(-z)^j}{\Gamma(1+\gamma j)}, \quad 0 < \gamma \le 1, \quad z > 0.$$

We define operator families $\{U_\gamma(t, \nu)\}$ associated with the semigroup $e^{-tA(\nu)}$ by

$$U_\gamma(t, \nu) = \gamma \int_0^\infty \theta t^{\gamma-1}\rho_\gamma(\theta)e^{-t^\gamma\theta A(\nu)}\,\mathrm{d}\theta.$$

Further, we define

$$V_1(t, \nu) = [A(t) - A(\nu)]U_\gamma(t - \nu, \nu),$$
$$V_{k+1}(t, \nu) = \int_\nu^t V_k(t, \theta)V_1(\theta, \nu)\,\mathrm{d}\theta, \quad k = 1, 2, \cdots$$

and construct the family $\{V(t, \nu)\}$ by

$$V(t, \nu) = \sum_{k=1}^\infty V_k(t, \nu),$$

and define

$$\psi(t) = -A(t)A^{-1}(0) - \int_0^t V(t, \nu)A(\nu)A^{-1}(0)\,\mathrm{d}\nu.$$

Definition 3.1 *A function $z \in C_L(I_0, Z_{\mu,t})$ is called a mild solution of the problem (1.1) if z satisfies the following integral equation:*

$$\begin{aligned} z(t) = z_0 &+ \int_0^t U_\gamma(t-\nu,\nu)\psi(\nu)A(0)z_0 \, \mathrm{d}\nu \\ &+ \int_0^t U_\gamma(t-\nu,\nu)[Ew(\nu) + g(\nu, z(\nu), z(b(z(\nu),\nu)))] \, \mathrm{d}\nu \\ &+ \int_0^t \int_0^\nu U_\gamma(t-\nu,\nu)V(\nu,s)[Ew(s) + g(s, z(s), z(b(z(s),s)))] \, \mathrm{d}s \, \mathrm{d}\nu. \end{aligned} \tag{3.1}$$

Definition 3.2 *[11] The control problem (1.1) is called controllable on the interval I_0 if for every $z_0, z_{b_0} \in Z$, there is a control function $w \in L^2(I_0, W)$ such that the mild solution $z(t)$ of (1.1) satisfies $z(b_0) = z_{b_0}$.*

For any $z \in C_L(I_0, Z_{\beta,t})$, using $(H3)$, we define the control function

$$\begin{aligned} w_z(t) = Q^{-1}\Big[z_{b_0} - z_0 &- \int_0^{b_0} U_\gamma(b_0-\nu,\nu)\psi(\nu)A(0)z_0 \, \mathrm{d}\nu \\ &- \int_0^{b_0} U_\gamma(b_0-\nu,\nu)g(\nu, z(\nu), z(b(z(\nu),\nu))) \, \mathrm{d}\nu \\ &- \int_0^{b_0} \int_0^\nu U_\gamma(b_0-\nu,\nu)V(\nu,s)g(s, z(s), z(b(z(s),s))) \, \mathrm{d}s \, \mathrm{d}\nu\Big](t). \end{aligned}$$

Lemma 3.3 *[2] The operator-valued functions $U_\gamma(t-\nu,\nu)$ is continuous in the uniform topology in the variables t, ν, where $0 \le \nu \le t-\varepsilon$, $0 \le t \le b_0$, for any $\varepsilon > 0$, and functions U_γ, V and ψ satisfy the following inequalities:*

$$\begin{aligned} \|U_\gamma(t-\nu,\nu)\| &\le C(t-\nu)^{\gamma-1}, \\ \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| &\le C(t''-t')^{\gamma-1}, \\ \|V(t,\nu)\| &\le C(t-\nu)^{\alpha-1}, \\ \|\psi(t)\| &\le C(1+t^\alpha), \end{aligned}$$

where C is a constant.

Lemma 3.4 *There exists $K_w > 0$ such that for all $z \in (C_L(I_0, Z_{\mu,t}), \|\cdot\|_{\mu,b_0})$ satisfying (3.1),*

$$\|w_z(t)\| \le K_1^w + K_2^w \|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}, \quad t \in I_0.$$

Proof. Using $(H1)$ and $(H3)$, we have

$$\begin{aligned} \|w_z(t)\| \le M_3\Big[\|z_{b_0}\| + \|z_0\| &+ C^2\|A(0)z_0\| \int_0^{b_0} (b_0-\nu)^{\gamma-1}(1+\nu^\alpha) \, \mathrm{d}\nu \\ &+ C \int_0^{b_0} (b_0-\nu)^{\gamma-1} \|g(\nu, z(\nu), z(b(z(\nu),\nu)))\| \, \mathrm{d}\nu \end{aligned}$$

$$+ C^2 \int_0^{b_0} \int_0^{\nu} (b_0-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}\|g(s,z(s),z(b(z(s),s)))\|\,\mathrm{d}s\,\mathrm{d}\nu\Big]$$
$$\le M_3\Big[\|z_{b_0}\| + \|z_0\| + C^2\|A(0)z_0\|(b_0)^{\gamma}\Big\{\frac{1}{\gamma} + (b_0)^{\alpha}\beta(\gamma,\alpha+1)\Big\}$$
$$+ C\int_0^{b_0}(b_0-\nu)^{\gamma-1}G_r(\nu)\,\mathrm{d}\nu$$
$$+ C^2\int_0^{b_0}\int_0^{\nu}(b_0-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}G_r(s)\,\mathrm{d}s\,\mathrm{d}\nu\Big],$$

where

$$\beta(\gamma,\alpha) = \int_0^1 t^{\gamma-1}(1-t)^{\alpha-1}\,\mathrm{d}t,$$

denotes the Beta function.

Using Hölder's inequality, we get

$$\|w_z(t)\| \le M_3\Big[\|z_{b_0}\| + \|z_0\| + C^2\|A(0)z_0\|(b_0)^{\gamma}\Big\{\frac{1}{\gamma} + (b_0)^{\alpha}\beta(\gamma,\alpha+1)\Big\}$$
$$+ C\Big(\int_0^{b_0}(b_0-\nu)^{\frac{\gamma-1}{1-\gamma_1}}\,\mathrm{d}\nu\Big)^{1-\gamma_1}\Big(\int_0^{b_0} G_r^{\frac{1}{\gamma_1}}(\nu)\,\mathrm{d}\nu\Big)^{\gamma_1}$$
$$+ C^2\beta(\gamma,\alpha)\Big(\int_0^{b_0}(b_0-\nu)^{\frac{\gamma+\alpha-1}{1-\gamma_1}}\,\mathrm{d}\nu\Big)^{1-\gamma_1}\Big(\int_0^{b_0} G_r^{\frac{1}{\gamma_1}}(\nu)\,\mathrm{d}\nu\Big)^{\gamma_1}\Big]$$
$$\le M_3\Big[\|z_{b_0}\| + \|z_0\| + C^2\|A(0)z_0\|(b_0)^{\gamma}\Big\{\frac{1}{\gamma} + (b_0)^{\alpha}\beta(\gamma,\alpha+1)\Big\}$$
$$+ Cb_0^{\gamma-\gamma_1}\Big(\frac{1-\gamma_1}{\gamma-\gamma_1}\Big)^{1-\gamma_1}\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}$$
$$+ C^2\beta(\gamma,\alpha)b_0^{\gamma+\alpha-\gamma_1}\Big(\frac{1-\gamma_1}{\gamma+\alpha-\gamma_1}\Big)^{1-\gamma_1}\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big]$$
$$= K_1^w + K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]},$$

where

$$K_1^w = M_3\Big[\|z_{b_0}\| + \|z_0\| + C^2\|A(0)z_0\|(b_0)^{\gamma}\Big\{\frac{1}{\gamma} + (b_0)^{\alpha}\beta(\gamma,\alpha+1)\Big\}\Big],$$
$$K_2^w = M_3 C b_0^{\gamma-\gamma_1}\Big[C\beta(\gamma,\alpha)b_0^{\alpha}\Big(\frac{1-\gamma_1}{\gamma+\alpha-\gamma_1}\Big)^{1-\gamma_1} + \Big(\frac{1-\gamma_1}{\gamma-\gamma_1}\Big)^{1-\gamma_1}\Big].$$

□

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Theorem 3.5 *If* $(H1)$-$(H3)$ *hold, then control system (1.1) is controllable provided that*

$$\sigma C\|A^{\mu}\|b_0^{\gamma}\Big[\frac{M_2K_2^w}{\gamma} + \frac{1}{b_0^{\gamma_1}}\Big(\frac{1-\gamma_1}{\gamma-\gamma_1}\Big)^{1-\gamma} + \beta(\gamma,\alpha)\Big(\frac{1-\gamma_1}{\gamma+\alpha-\gamma_1}\Big)^{1-\gamma_1}b_0^{\alpha-\gamma_1} + \frac{b_0^{\alpha}}{\gamma+\alpha}\beta(\gamma,\alpha)M_2CK_2^w\Big] < 1.$$

Proof. Using the control w_z, we show that the operator $F : C_L(I_0, Z_{\mu,t}) \to C_L(I_0, Z_{\mu,t})$ defined by

$$(Fz)(t) = z_0 + \int_0^t U_{\gamma}(t-\nu,\nu)\psi(\nu)A(0)z_0\,\mathrm{d}\nu$$

$$+\int_0^t U_\gamma(t-\nu,\nu)[Ew_z(\nu)+g(\nu,z(\nu),z(b(z(\nu),\nu)))]\,\mathrm{d}\nu$$
$$+\int_0^t\int_0^\nu U_\gamma(t-\nu,\nu)V(\nu,s)[Ew_z(s)+g(s,z(s),z(b(z(s),s)))]\,\mathrm{d}s\,\mathrm{d}\nu$$

has a fixed point. This fixed point is then a solution of given system, clearly $(Fz)(b_0)=z_{b_0}$, which show that given system is controllable on I_0.

We apply Lemma 2.3 to show that F has a fixed point.

Let
$$\mathcal{C}_r(C_L(I_0,Z_{\mu,t}),z_0)=\{z\in C_L(I_0,Z_{\mu,t})\mid \|z-z_0\|_{\mu,b_0}\le r\}.$$
We prove this result in four steps:

Step 1: We show that
$$F(\mathcal{C}_r(C_L(I_0,Z_{\mu,t}),z_0))\subseteq \mathcal{C}_r(C_L(I_0,Z_{\mu,t}),z_0)\quad\text{for some } r>0.$$
If this is not true, then there exist $t_r\in(0,b_0]$ and $z_r\in\mathcal{C}_r(C_L(I_0,Z_{\mu,t}),z_0)$ for all $r>0$ such that
$$r<\|(Fz_r)(t_r)-z_0\|_\mu\le\|A^\mu\|\int_0^{t_r}\|U_\gamma(t_r-\nu,\nu)\|\|\psi(\nu)\|\|A(0)z_0\|\,\mathrm{d}\nu$$
$$+\|A^\mu\|\int_0^{t_r}\|U_\gamma(t_r-\nu,\nu)\|[\|E\|\|w_z(\nu)\|+\|g(\nu,z(\nu),z(b(z(\nu),\nu)))\|]\,\mathrm{d}\nu$$
$$+\|A^\mu\|\int_0^{t_r}\int_0^\nu\|U_\gamma(t_r-\nu,\nu)\|\|V(\nu,s)\|[\|E\|\|w_z(s)\|+\|g(s,z(s),z(b(z(s),s)))\|]\,\mathrm{d}s\,\mathrm{d}\nu.$$
Using Lemma 3.3 and $(H3)$, we get
$$r<C^2\|A(0)z_0\|\|A^\mu\|\int_0^{t_r}(t_r-\nu)^{\gamma-1}(1+\nu^\alpha)\,\mathrm{d}\nu$$
$$+M_2C\|A^\mu\|\Big(K_1^w+K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big)\int_0^{t_r}(t_r-\nu)^{\gamma-1}\,\mathrm{d}\nu$$
$$+C\|A^\mu\|\int_0^{t_r}(t_r-\nu)^{\gamma-1}G_r(\nu)\,\mathrm{d}\nu$$
$$+M_2C^2\|A^\mu\|\Big(K_1^w+K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big)\int_0^{t_r}\int_0^\nu(t_r-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}\,\mathrm{d}s\,\mathrm{d}\nu$$
$$+C^2\|A^\mu\|\int_0^{t_r}\int_0^\nu(t_r-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}G_r(s)\,\mathrm{d}s\,\mathrm{d}\nu.$$
Using Hölder's inequality, we get
$$r<C^2\|A(0)z_0\|\|A^\mu\|\Big[\frac{t_r^\gamma}{\gamma}+t_r^{\alpha+\gamma}\beta(\gamma,\alpha+1)\Big]$$
$$+M_2C\|A^\mu\|\Big(K_1^w+K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big)\frac{t_r^\gamma}{\gamma}$$
$$+C\|A^\mu\|\Big(\int_0^{t_r}(t_r-\nu)^{\frac{\gamma-1}{1-\gamma_1}}\,\mathrm{d}\nu\Big)^{1-\gamma_1}\Big(\int_0^{t_r}G_r^{\frac{1}{\gamma_1}}(\nu)\,\mathrm{d}\nu\Big)^{\gamma_1}$$

$$
\begin{aligned}
&\quad + M_2C^2\|A^\mu\|\Big(K_1^w + K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big)\beta(\gamma,\alpha)\int_0^{t_r}(t_r-\nu)^{\gamma+\alpha-1}\,\mathrm{d}\nu\\
&\quad + C^2\|A^\mu\|\beta(\gamma,\alpha)\Big(\int_0^{t_r}(t_r-\nu)^{\frac{\gamma+\alpha-1}{1-\gamma_1}}\,\mathrm{d}\nu\Big)^{1-\gamma_1}\Big(\int_0^{t_r}G_r^{\frac{1}{\gamma_1}}(\nu)\,\mathrm{d}\nu\Big)^{\gamma_1}\\
&\le C^2\|A(0)z_0\|\|A^\mu\|\Big[\frac{b_0^{\gamma}}{\gamma}+b_0^{\alpha+\gamma}\beta(\gamma,\alpha+1)\Big]\\
&\quad + M_2C\|A^\mu\|\Big(K_1^w + K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big)\frac{b_0^{\gamma}}{\gamma}\\
&\quad + C\|A^\mu\|\Big(\frac{1-\gamma_1}{\gamma-\gamma_1}\Big)^{1-\gamma_1}b_0^{\gamma-\gamma_1}\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\\
&\quad + M_2C^2\|A^\mu\|\Big(K_1^w + K_2^w\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big)\beta(\gamma,\alpha)\cdot\frac{b_0^{\gamma+\alpha}}{\gamma+\alpha}\\
&\quad + C^2\|A^\mu\|\beta(\gamma,\alpha)\Big(\frac{1-\gamma_1}{\gamma+\alpha-\gamma_1}\Big)^{1-\gamma_1}b_0^{\gamma+\alpha-\gamma_1}\|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}.
\end{aligned}
$$

Dividing by r and taking $\liminf$ as $r\to\infty$, we get

$$
1\le\sigma C\|A^\mu\|b_0^{\gamma}\Big[\frac{M_2K_2^w}{\gamma}+\frac{1}{b_0^{\gamma_1}}\Big(\frac{1-\gamma_1}{\gamma-\gamma_1}\Big)^{1-\gamma}+\frac{b_0^{\alpha}}{\gamma+\alpha}\beta(\gamma,\alpha)M_2CK_2^w+\beta(\gamma,\alpha)\Big(\frac{1-\gamma_1}{\gamma+\alpha-\gamma_1}\Big)^{1-\gamma_1}b_0^{\alpha-\gamma_1}\Big],
$$

which is a contradiction.

Step 2: F is continuous on $\mathcal{C}_r(C_L(I_0,Z_{\mu,t}),z_0)$. We consider a sequence $\{z_n\}$ in $\mathcal{C}_r(C_L(I_0,Z_{\mu,t}),z_0)$ s.t. $z_n\to z\in\mathcal{C}_r(C_{L,0}(I_0,Z_{\mu,t}),z_0)$. Then using $(H1)$ and $(H3)$, we get

$$
\begin{aligned}
\|(Fz_n)(t)-(Fz)(t)\|_\mu &\le \|A^\mu\|\int_0^t\|U_\gamma(t-\nu,\nu)\|[\|E\|\|w_{z_n}(\nu)-w_z(\nu)\|\\
&\quad + \|g(\nu,z_n(\nu),z_n(b(z_n(\nu),\nu)))-g(\nu,z(\nu),z(b(z(\nu),\nu)))\|]\,\mathrm{d}\nu\\
&\quad + \|A^\mu\|\int_0^t\int_0^\nu\|U_\gamma(t-\nu,\nu)\|\|V(\nu,s)\|[\|E\|\|w_{z_n}(s)-w_z(s)\|\\
&\quad + \|g(s,z_n(s),z_n(b(z_n(s),s)))-g(s,z(s),z(b(z(s),s)))\|]\,\mathrm{d}s\,\mathrm{d}\nu. \qquad (3.2)
\end{aligned}
$$

Using $(H1)$ and $(H2)$, we get

$$
\begin{aligned}
&\|g(s,z_n(s),z_n(b(z_n(s),s)))-g(s,z(s),z(b(z(s),s)))\|\\
&\le L_g[\|z_n(s)-z(s)\|_\mu+\|z_n(b(z_n(s),s))-z(b(z(s),s))\|_\mu]\\
&\le L_g[\|z_n(s)-z(s)\|_\mu+\|z_n(b(z_n(s),s))-z(b(z_n(s),s))\|_\mu+\|z(b(z_n(s),s))-z(b(z(s),s))\|_\mu]\\
&\le L_g[\|z_n(s)-z(s)\|_\mu+\|z_n(b(z_n(s),s))-z(b(z_n(s),s))\|_\mu+LL_b\|z_n(s)-z(s)\|_\mu]\\
&\le L_g(2+LL_b)\|z_n-z\|_{\mu,b_0}.
\end{aligned}
$$

Using Lemma 3.3 and $(H3)$, we get

$$
\begin{aligned}
&\|w_{z_n}(s)-w_z(s)\|\\
&\le M_3\int_0^{b_0}\int_0^\nu\|U_\gamma(b_0-\nu,\nu)\|\|V(\nu,s)\|\|g(s,z_n(s),z_n(b(z_n(s),s)))-g(s,z(s),z(b(z(s),s)))\|\,\mathrm{d}s\,\mathrm{d}\nu\\
&\le M_3C^2L_g(1+LL_b)\int_0^{b_0}\int_0^\nu(b_0-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}\,\mathrm{d}s\,\mathrm{d}\nu\cdot\|z_n-z\|_{\mu,b_0}
\end{aligned}
$$

$$\leq \frac{b_0^{\gamma+\alpha}}{\gamma+\alpha} M_3 C^2 L_g (1 + LL_b) \beta(\gamma, \alpha) \|z_n - z\|_{\mu, b_0}.$$

Using above inequality in (3.2), we can find $M_0 > 0$ such that

$$\|(Fz_n)(t) - (Fz)(t)\|_\mu \leq M_0 \|z_n - z\|_{\mu, b_0}.$$

Taking supremum over $[0, b_0]$ and limit as $n \to \infty$, we get

$$\|Fz_n - Fz\|_{\mu, b_0} \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that F is continuous on $\mathcal{C}_r(B_L(I_0, Z_{\mu,t}), z_0)$.

Step 3: $F(\mathcal{C}_r(B_L(I_0, Z_{\mu,t}), z_0))$ is equicontinuous on I_0. For this, we assume $y \in F(\mathcal{C}_r(B_L(I_0, Z_{\beta,t}), z_0))$ and $0 \leq t' < t'' \leq b_0$. Then there is a $z \in \mathcal{C}_r(B_L(I_0, Z_{\mu,t}), z_0)$ such that

$$\begin{aligned}
&\|y(t'') - y(t')\| \\
&\leq \int_0^{t'} \|U_\gamma(t'' - \nu, \nu) - U_\gamma(t' - \nu, \nu)\| \|\psi(\nu)\| \|A(0) z_0\| \,\mathrm{d}\nu \\
&+ \int_{t'}^{t''} \|U_\gamma(t'' - \nu, \nu)\| \|\psi(\nu)\| \|A(0) z_0\| \,\mathrm{d}\nu \\
&+ \int_0^{t'} \|U_\gamma(t'' - \nu, \nu) - U_\gamma(t' - \nu, \nu)\| [\|E\| \|w_z(\nu)\| + \|g(\nu, z(\nu), z(b(z(\nu), \nu)))\|] \,\mathrm{d}\nu \\
&+ \int_{t'}^{t''} \|U_\gamma(t'' - \nu, \nu)\| [\|E\| \|w_z(\nu)\| + \|g(\nu, z(\nu), z(b(z(\nu), \nu)))\|] \,\mathrm{d}\nu \\
&+ \int_0^{t'}\int_0^{\nu} \|U_\gamma(t'' - \nu, \nu) - U_\gamma(t' - \nu, \nu)\| \|V(\nu, s)\| [\|E\| \|w_z(s)\| + \|g(s, z(s), z(b(z(s), s)))\|] \,\mathrm{d}s\,\mathrm{d}\nu \\
&+ \int_{t'}^{t''} \int_0^{\nu} \|U_\gamma(t'' - \nu, \nu)\| \|V(\nu, s)\| [\|E\| \|w_z(s)\| + \|g(s, z(s), z(b(z(s), s)))\|] \,\mathrm{d}s\,\mathrm{d}\nu \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{t'} \|U_\gamma(t'' - \nu, \nu) - U_\gamma(t' - \nu, \nu)\| \|\psi(\nu)\| \|A(0) z_0\| \,\mathrm{d}\nu, \\
I_2 &= \int_{t'}^{t''} \|U_\gamma(t'' - \nu, \nu)\| \|\psi(\nu)\| \|A(0) z_0\| \,\mathrm{d}\nu, \\
I_3 &= \int_0^{t'} \|U_\gamma(t'' - \nu, \nu) - U_\gamma(t' - \nu, \nu)\| [\|E\| \|w_z(\nu)\| + \|g(\nu, z(\nu), z(b(z(\nu), \nu)))\|] \,\mathrm{d}\nu, \\
I_4 &= \int_{t'}^{t''} \|U_\gamma(t'' - \nu, \nu)\| [\|E\| \|w_z(\nu)\| + \|g(\nu, z(\nu), z(b(z(\nu), \nu)))\|] \,\mathrm{d}\nu, \\
I_5 &= \int_0^{t'}\int_0^{\nu} \|U_\gamma(t'' - \nu, \nu) - U_\gamma(t' - \nu, \nu)\| \|V(\nu, s)\| [\|E\| \|w_z(s)\| + \|g(s, z(s), z(b(z(s), s)))\|] \,\mathrm{d}s\,\mathrm{d}\nu, \\
I_6 &= \int_{t'}^{t''} \int_0^{\nu} \|U_\gamma(t'' - \nu, \nu)\| \|V(\nu, s)\| [\|E\| \|w_z(s)\| + \|g(s, z(s), z(b(z(s), s)))\|] \,\mathrm{d}s\,\mathrm{d}\nu.
\end{aligned}$$

For $t' > 0$ and $\varepsilon > 0$ small enough, by Lemma 3.3 and the fact that the operator-valued function $U_\gamma(t-\nu,\nu)$ is continuous in uniform topology about the variables t and ν for $0 \le t \le b_0$ and $0 \le \nu \le t'-\varepsilon$, we have

$$\begin{aligned} I_1 &\le \int_0^{t'-\varepsilon} \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| \, \|\psi(\nu)\| \|A(0)z_0\| \, \mathrm{d}\nu \\ &\quad + \int_{t'-\varepsilon}^{t'} \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| \, \|\psi(\nu)\| \|A(0)z_0\| \, \mathrm{d}\nu \\ &\le C \sup_{\nu\in[0,t'-\varepsilon]} \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| \, \|A(0)z_0\| \int_0^{t'-\varepsilon} (1+\nu^\alpha)\, \mathrm{d}\nu \\ &\quad + C^2 \|A(0)z_0\| \int_{t'-\varepsilon}^{t'} |(t''-\nu)^{\gamma-1} + (t'-\nu)^{\gamma-1}| \, (1+\nu^\alpha)\, \mathrm{d}\nu \\ &\to 0, \quad t'' \to t' \text{ and } \varepsilon \to 0. \end{aligned}$$

$$I_2 = \int_{t'}^{t''} \|U_\gamma(t''-\nu,\nu)\| \, \|\psi(\nu)\| \, \|A(0)z_0\| \, \mathrm{d}\nu \le C^2 \|A(0)z_0\| \int_{t'}^{t''} (t''-\nu)^{\gamma-1}(1+\nu^\alpha)\, \mathrm{d}\nu.$$

Obviously $I_2 \to 0$ as $t'' \to t'$.

$$\begin{aligned} I_3 &\le C \int_0^{t'-\varepsilon} \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| \left[M_2\Big(K_1^w + K_2^w \|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big) + G_r(\nu) \right] \mathrm{d}\nu \\ &\quad + C \int_{t'-\varepsilon}^{t'} \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| \left[M_2\Big(K_1^w + K_2^w \|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big) + G_r(\nu) \right] \mathrm{d}\nu \\ &\le C \sup_{\nu\in[0,t'-\varepsilon]} \|U_\gamma(t''-\nu,\nu) - U_\gamma(t'-\nu,\nu)\| \int_0^{t'-\varepsilon} \left[M_2\Big(K_1^w + K_2^w \|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big) + G_r(\nu) \right] \mathrm{d}\nu \\ &\quad + C \int_{t'-\varepsilon}^{t'} |(t''-\nu)^{\gamma-1} + (t'-\nu)^{\gamma-1}| \, (1+\nu^\alpha) \left[M_2\Big(K_1^w + K_2^w \|G_r\|_{L^{\frac{1}{\gamma_1}}[0,b_0]}\Big) + G_r(\nu) \right] \mathrm{d}\nu \\ &\to 0, \quad t'' \to t' \text{ and } \varepsilon \to 0. \end{aligned}$$

Similarly, we can show that I_4, I_5, I_6 are tending to 0 as $t'' \to t'$.

From above inequality, it is clear that $\|y(t'') - y(t')\| \to 0$ as $t'' \to t'$. Therefore $F(C_r(B_L(I_0, Z_{\mu,t}), z_0))$ is equicontinuous on I_0.

Step 4: Next, we show that Mönch's condition is satisfied, *i.e.*if $V \subseteq \mathcal{C}_r(B_L(I_0,Z),z_0)$ is countable and $V \subseteq \bar{ch}\left(\{0\} \cup F(V)\right)$, then $\bar{V}$ is compact. By using the idea used in [16], we can show that $F(V)$ is relatively compact *i.e.*if ϕ is monotone, nonsingular measure of non compactness, then $\phi\big(F(V)\big) = 0$.

Since $V \subseteq \bar{ch}(\{0\} \cup F(V))$, therefore by using the definition of ϕ, we have

$$\phi(V) \le \phi\left(\bar{ch}(\{0\} \cup F(V))\right) = \phi\big(F(V)\big) = 0.$$

This implies that V is relatively compact *i.e.*$\bar{V}$ is compact. Thus the Mönch's condition is satisfied. Therefore by applying Lemma 2.3, F has a fixed point. □

4 Optimal Controllability

In this section, we define the operator

$$\Omega_0^{b_0} = \int_0^{b_0} U_\gamma(b_0 - s, s) E E^* U_\gamma^*(b_0 - s, s)\, \mathrm{d}s,$$

where E^* and U_γ^* are the adjoint operators of E and U_γ respectively.

For $\delta > 0$, let

$$R(\delta, \Omega_0^{b_0}) = (\delta I + \Omega_0^{b_0})^{-1}$$

and

$$\begin{aligned} q(z(t)) = z_{b_0} - z_0 - &\int_0^{b_0} U_\gamma(b_0 - \nu, \nu)\psi(\nu) A(0) z_0\, \mathrm{d}\nu \\ &- \int_0^{b_0} U_\gamma(b_0 - \nu, \nu) g(\nu, z(\nu), b(z(\nu), \nu)))\, \mathrm{d}\nu, \\ &- \int_0^{b_0} \int_0^{\nu} U_\gamma(b_0 - \nu, \nu) V(\nu, s) g(s, z(s), b(z(s), s)))\ \mathrm{d}s\mathrm{d}\nu, \end{aligned}$$

then we have

$$w_z(t) = E^* U_\gamma^*(b_0 - t, t) R(\delta, \Omega_0^{b_0}) q(z(t)).$$

In order to discuss the optimal controllability, we define the performance index

$$\tilde{J}(w) = \int_0^{b_0} \tilde{G}(t, z(t), z(b(z(t), t)), w(t))\, \mathrm{d}t, \tag{4.1}$$

where $\tilde{G}$ is a functional defined on $I_0 \times C(I_0, Z_{\mu,t}) \times C_L(I_0, Z_{\mu,t}) \times W_{ad}$, and W_{ad} denotes the set of all admissible control and consequently is closed and convex in $L^2(I_0, W)$.

Theorem 4.1 *If all conditions of Theorem 3.5 hold, then there exists an optimal control of the problem (1.1) provided that*

$$CL_g(2 + LL_g)\|A^\mu\| b_0^\gamma \left[\frac{1}{\gamma} + C\beta(\gamma, \alpha)\frac{b_0^\alpha}{\alpha + \gamma}\right] < 1.$$

Proof. It is sufficient to prove that there exists $w^0 \in L^2(I_0, W)$ which minimizes $\tilde{J}(w)$.

If $\inf_{w \in W_{ad}} \tilde{J}(w) = \infty$, then the result trivially holds.

If $\inf_{w \in W_{ad}} \tilde{J}(w) = \epsilon_0 < \infty$, then we can find a sequence $\{w^n\}$ in W_{ad} such that $\tilde{J}(w^n) \to \epsilon_0$. Since W_{ad} is a closed and convex subset of $L_2(I_0, W)$, the sequence $\{w^n\}$ has a weakly convergent subsequence $\{w^m\}$ converging to $w^0 \in W_{ad}$. Using Theorem 3.5, for each $w^m \in W_{ad}$, there exists a mild solution z^m of (1.1) satisfying:

$$\begin{aligned} z^m(t) = z_0 + &\int_0^t U_\gamma(t - \nu, \nu)\psi(\nu) A(0) z_0\, \mathrm{d}\nu \\ &+ \int_0^t U_\gamma(t - \nu, \nu)[E w^m(\nu) + g(\nu, z^m(\nu), z^m(b(z^m(\nu), \nu)))]\, \mathrm{d}\nu \end{aligned}$$

$$+\int_0^t\int_0^\nu U_\gamma(t-\nu,\nu)V(\nu,s)[Ew^m(s)+g(s,z^m(s),z^m(b(z^m(s),s)))]\,\mathrm{d}s\mathrm{d}\nu.$$

Similarly corresponding to w^0, there exists a mild solution z^0 of (1.1) satisfying:

$$\begin{aligned}z^0(t)&=z_0+\int_0^t U_\gamma(t-\nu,\nu)\psi(\nu)A(0)z_0\,\mathrm{d}\nu\\&\quad+\int_0^t U_\gamma(t-\nu,\nu)[Ew^0(\nu)+g(\nu,z^0(\nu),z^0(b(z^0(\nu),\nu)))]\,\mathrm{d}\nu\\&\quad+\int_0^t\int_0^\nu U_\gamma(t-\nu,\nu)V(\nu,s)[Ew^0(s)+g(s,z^0(s),z^0(b(z^0(s),s)))]\,\mathrm{d}s\mathrm{d}\nu.\end{aligned}$$

We have

$$\begin{aligned}\|z^m(t)-z^0(t)\|_\mu\le\|A^\mu\|\Big[&\int_0^t\|U_\gamma(t-\nu,\nu)\|\{\|Ew^n(\nu)-Ew^0(\nu)\|\\&+\|g(\nu,z^m(\nu),z^m(b(z^m(\nu),\nu)))-g(\nu,z^0(\nu),z^0(b(z^0(\nu),\nu)))\|\}\,\mathrm{d}\nu\\&+\int_0^t\int_0^\nu\|U_\gamma(t-\nu,\nu)\|\|V(\nu,s)\|\{\|Ew^n(s)-Ew^0(s)\|\\&+\|g(s,z^m(s),z^m(b(z^m(s),s)))-g(s,z^0(s),z^0(b(z^0(s),s)))\|\}\,\mathrm{d}s\mathrm{d}\nu\Big].\end{aligned}\tag{4.2}$$

Using $(H1)$, $(H2)$, we get

$$\begin{aligned}&\|g(t,z^m,z^m(b(z^m(t),t)))-g(t,z^0,z^0(b(z^0(t),t)))\|\\&\le L_g[\|z^m(t)-z^0(t)\|_\mu+\|z^m(b(z^m(s),s))-z^0(b(z^0(s),s))\|_\mu]\\&\le L_g[\|z^m(t)-z^0(t)\|_\mu+\|z^m(b(z^m(s),s))-z^m(b(z^0(s),s))\|_\mu\\&+\|z^m(b(z^0(s),s))-z^0(b(z^0(s),s))\|_\mu]\\&\le L_g(2+LL_b)\|z^m-z^0\|_{\mu,b_0}.\end{aligned}$$

From (4.2), we have

$$\begin{aligned}\|z^m(t)-z^0(t)\|_\mu\le\|A^\mu\|\Big[&\int_0^t C(t-\nu)^{\gamma-1}\{\|Ew^n(\nu)-Ew^0(\nu)\|\\&+L_g(2+LL_b)\|z^m-z^0\|_{\mu,b_0}\}\,\mathrm{d}\nu\\&+\int_0^t\int_0^\nu C^2(t-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}\{\|Ew^n(s)-Ew^0(s)\|\\&+L_g(2+LL_b)\|z^m-z^0\|_{\mu,b_0}\}\Big]\,\mathrm{d}s\mathrm{d}\nu.\\\le\|A^\mu\|\Big[&\int_0^t C(t-\nu)^{\gamma-1}\|Ew^n(\nu)-Ew^0(\nu)\|d\nu\\&+\int_0^t\int_0^\nu C^2(t-\nu)^{\gamma-1}(\nu-s)^{\alpha-1}\|Ew^n(s)-Ew^0(s)\|\,\mathrm{d}s\mathrm{d}\nu\Big]\\&+CL_g(2+LL_g)\|A^\mu\|b_0^\gamma\Big[\frac{1}{\gamma}+C\beta(\gamma,\alpha)\frac{b_0^\alpha}{\alpha+\gamma}\Big]\|z^m-z^0\|_{\mu,b_0}.\end{aligned}$$

Since $CL_g(2+LL_g)\|A^\mu\|b_0^\gamma\left[\frac{1}{\gamma}+C\beta(\gamma,\alpha)\frac{b_0^\alpha}{\alpha+\gamma}\right]<1$ and $\|Ew^m(t)-Ew^0(t)\|\to 0$, we conclude that $z^m\to z^0$.

Applying Balder's theorem, we get

$$\begin{aligned}\epsilon_0&=\lim_{m\to\infty}\int_0^{b_0}\tilde{G}(t,z^m(t),z^m(b(z^m(t),t)),w^m(t))\,\mathrm{d}t\\&\le\int_0^{b_0}\tilde{G}(t,z^0(t),z^0(b(z^0(t),0)),w^0(t))\,\mathrm{d}t\\&=\tilde{J}(w^0)\ge\epsilon_0.\end{aligned}$$

This shows that $\tilde{J}(w^0)=\epsilon_0$, *i.e.* $\tilde{J}$ attains its minimum value at $w^0\in L^2(I_0,W)$. □

5 Application

Consider the following example:

$$\begin{cases}{}_CD^{\frac{3}{4}}y(x,t)-c(x,t)\dfrac{\partial^2}{\partial x^2}y(x,t) &= Ew_0(x,t)+h_1(x,y(x,b(y(x,t),t)))\\ &+\;h_2(t,x,y(x,t)),\quad x\in[0,1],\quad t\in(0,b_0],\\ y(x,0)=y_0(x),\end{cases}\tag{5.1}$$

where

$$h_1(x,y(x,t))=\int_0^x K(x,z)y(x,c_1d(t)|y(z,t)|)\,\mathrm{d}z,$$

and the function $h_2:R_+\times[0,1]\times R\to R$ is locally Hölder continuous in t, locally Lipschitz continuous in y, uniformly in x and measurable in x. $c(t,x)$ is uniformly Hölder continuous *i.e.*there exist $K>0$ and $\bar{\alpha}\in(0,1)$ such that

$$\|c(t_1,x)-c(t_2,x)\|\le K|t_1-t_2|^{\bar{\alpha}}.$$

Let $I_0=[0,b_0]$ and $Y=C([0,1],R)$. It is well know that, if we define the operator $A(\nu)$ by

$$A(\nu)y(x,\nu)=-c(x,\nu)\frac{\partial^2}{\partial x^2}y(x,\nu)$$

with the domain

$$D(A(\nu))=H^2(0,1)\cap H_0^1(0,1),$$

then $-A(\nu)$ generates an analytic semigroup $e^{-tA(\nu)}$. If we take $\mu=\frac{1}{3}$, then fractional power $A^{\frac{1}{3}}(\nu)$ is well defined (see [14]). $\left(D(A^{\frac{1}{3}}(\nu)),\|\cdot\|_{\frac{1}{3}}\right)$ is a Banach space, where for $w\in D(A^{\frac{1}{3}}(\nu))$

$$\|w\|_{\frac{1}{3}}=\|A^{\frac{1}{3}}(\nu)w\|.$$

We denote this Banach space by $Y_{\frac{1}{3}}$.

Let $C\left(I_0,Y\right)$ denote the set of all continuous functions on I_0, and

$$C_L\left(I_0,Y\right)=\left\{y\in C\left(I_0,Y\right)\mid\|y(s_1)-y(s_2)\|\le L|s_1-s_2|\right\}.$$

We define $h : I_0 \times C\left(I_0, Y\right) \times C_L\left(I_0, Y\right) \to Y$ by

$$h(t, y, \phi)(x) = h_1(x, \phi) + h_2(t, x, y),$$

and $(Ew)(t)(x) = w_0(x, t)$, $y(t)(x) = y(x, t)$, then the abstract formulation of the problem (5.1) is:

$$\begin{aligned} {}_{C}D^{\frac{3}{2}} y(t) &= Ay(t) + Ew(t) + h(t, y(t), y(b(y(t), t))), \\ y(0) &= y_0. \end{aligned}$$

We assume that b and E satisfy the required assumptions. It can be easily proved that all other assumptions of Theorem 3.5 are satisfied, therefore by using Theorem 3.5, we conclude that system (5.1) is controllable on I_0.

Define performance index

$$\tilde{J}(w) = \int_0^{b_0} \left(\|y(t)\|^2 + \|w(t)\|^2\right) \mathrm{d}t.$$

It can be easily checked that all the assumptions of Theorem 4.1 are satisfied, therefore by using Theorem 4.1 we find an admissible control w_0 that minimizes $\tilde{J}(w)$.

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