Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2021, Number 6, pp. 119–135 (December 2021) http://www.jneea.com

# LONG TIME DECAY OF THE FRACTIONAL NAVIER–STOKES EQUATIONS IN SOBOLEV–GEVERY SPACES

**XIAOCHUN SUN**[*], **JIA LIU**[†]
College of Mathematics and Statistics, Northwest Normal University,
Lanzhou, 730070 P. R. China

Received on June 3, 2020, revised version on May 10, 2021

Accepted on May 11, 2021

Communicated by Martin Bohner

---

**Abstract.** In this paper, we prove that if $u \in C\big([0,+\infty), \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$ is a global solution of a 3D fractional Navier–Stokes equation, where $\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}$ is the Sobolev–Gevery space with parameters $a > 0$ and $\alpha \in (\frac{2}{3}, 1]$, then $\|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)}$ decays to zero as time approaches infinity. Our technique is based on Fourier analysis.

**Keywords:** fractional Navier–Stokes equation, long time decay, Sobolev–Gevery space.

**2010 Mathematics Subject Classification:** 35Q30, 76B03, 42B37.

---

## 1 Introduction

The Cauchy problem for the 3D generalized heat equation is given by

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u = Q(u,u), & x \in \mathbb{R}^3,\, t \in (0,\infty), \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3,\, t \in (0,\infty), \\ u(0,x) = u^0(x), & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

[*]e-mail address: sunxc@mail.bnu.edu.cn, sunxiaochun@nwnu.edu.cn
[†]e-mail address: 2552645827@qq.com

© 2021 Journal of Nonlinear Evolution Equations and Applications, JNEEA.com

with the bilinear operator $Q$ defined as

$$Q^j(u,v)=\sum_{k,l,m} q_{k,l}^{j,m}\partial_m(u^k v^l),\quad j=1,2,3, \tag{1.2}$$

where

$$q_{k,l}^{j,m}=\sum_{n,p=1}^{3} a_{k,l}^{j,m,p,n}\mathcal{F}^{-1}\left(\frac{\xi_n\xi_p}{|\xi|^2}\hat{u}(\xi)\right)$$

and $a_{k,l}^{j,m,p,n}$ are real numbers.

The particular case of the above system is the fractional Navier–Stokes system for incompressible fluid

$$\begin{cases}\partial_t u+\nu(-\Delta)^\alpha u+u\cdot\nabla u=-\nabla p, & x\in\mathbb{R}^3,\ t\in(0,\infty),\\ \operatorname{div} u=0, & x\in\mathbb{R}^3,\ t\in(0,\infty),\\ u(0,x)=u^0(x), & x\in\mathbb{R}^3,\end{cases} \tag{1.3}$$

where $\nu>0$ is the viscosity coefficient of the fluid, $\alpha>0$ represents the 'strength of dissipation', $u=u(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))$ denotes the velocity vector field of the fluid and $p=p(t,x)$ denotes the scalar pressure at the point $(t,\ x)\in(0,\infty)\times\mathbb{R}^3$. Moreover, $u^0=(u_1^0(x),u_2^0(x),u_3^0(x))$ is a given vector field of initial velocity which is assumed to be divergence-free. We define $(-\Delta)^\alpha u$ implicitly via

$$\mathcal{F}((-\Delta)^\alpha u)(t,\xi))=|\xi|^{2\alpha}\mathcal{F}(u)(t,\xi).$$

For simplicity, we will denote the Laplacian $(-\Delta)^{\frac12}$ by $\Lambda$ and obviously $\widehat{\Lambda^\alpha u}(\xi)=|\xi|^\alpha\hat{u}(\xi)$.

Lions [10] proved the global existence of the classical solutions to equation (1.3) with $\alpha\ge\frac54$. There are many weak-strong uniqueness results for (1.3) (see [7, 13, 15]). Zhai [14] proved the global existence and uniqueness of regular solutions in spatial variables with $L^r(\mathbb{R}^n)$ data and also studied the well-posedness for (1.3) in critical spaces close to $\dot{B}_{\infty,\infty}^{-(2\alpha-1)}$ for $\frac12<\alpha<1$. We [8] established the global existence and uniqueness of regular solutions in spatial variables for the higher order elliptic Navier–Stokes system. We [12] also proved the time-local existence and uniqueness of the mild solution to the fractional Navier–Stokes–Coriolis system in homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^3)^3$ with $\frac14<\alpha\le\frac32$ and $\frac32-\alpha<s<\frac54$.

When $\alpha=1$, (1.3) becomes the classical Navier–Stokes system. There are several authors who have studied the behavior of the norm of the solutions at infinity in various Banach spaces. Benameur [3] proved that if $u$ is a Lei–Lin solution of a 3D Navier–Stokes equation, then $\lim_{t\to\infty}\sup\|u(t)\|_{\chi^{-1}}=0$. In [4], Benameur and Jlali proved that long-time decay of the global solution of 3D Navier–Stokes equations in Lei–Lin–Gevery spaces. Gallagher, Iftimie and Planchon [9] showed that if $u$ is a global solution of a 3D Navier–Stokes equation, then $\lim_{t\to\infty}\|u(t)\|_{\dot{H}^{\frac12}}=0$. Benameur and Jlali [5] proved that $\|u(t)\|_{H^1_{a,\sigma}}$, where $\sigma>1$, approaches zero at infinity. Orf [11] proved that if $u\in C\big([0,+\infty),\dot{H}_{a,1}^{\frac12}(\mathbb{R}^2)\big)$ is a global solution of a 3D incompressible Navier–Stokes equation, then $\|u\|_{\dot{H}^{\frac12}_{a,1}}$ decays to zero as time approaches infinity. In this paper, greatly inspired by the work of Orf, we are devoted to studying the non blow-up result of the global solution of the fractional Navier–Stokes equation and obtain the result $\lim_{t\to\infty}\|u(t)\|_{\dot{H}^{\frac52-2\alpha}_{a,\frac1\alpha}}=0$ with $\frac23<\alpha\le1$.

For simplicity, we take $\nu = 1$ for the rest of the paper.

**Theorem 1.1** *Let $a > 0$ and $\frac{2}{3} < \alpha \leq 1$. If $u^0 \in \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)$ is such that* $\operatorname{div} u^0 = 0$, *then there exists a positive time $T^*$ such that* (1.3) *has a unique solution $u \in C\big([0,T^*), \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big) \cap L^2\big([0,T^*), \dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$.*

**Remark 1.1** *If $u$ is a solution of* (1.3) *in $C\big([0,T^*), \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$, then $u \in L^2_{\text{loc}}\big([0,T^*), \dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$.*

In the second theorem, we give blow up criteria if the maximal time is finite.

**Theorem 1.2** *Let $a > 0$ and $\frac{2}{3} < \alpha \leq 1$ and let $u \in C\big([0,T^*), \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big) \cap L^2\big([0,T^*), \dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$ be a maximal solution of* (1.3) *given by Theorem* 1.1.

(i) *If $\|u(0)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)} < \frac{1}{C}$, then $T^* = +\infty$.*

(ii) *If $T^*$ is finite, then $\int_0^{T^*} \|u(t)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)} \,\mathrm{d}t = +\infty$.*

In the next theorem, we show that the norm of the global solution in $\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)$ goes to zero at infinity.

**Theorem 1.3** *Let $a > 0$ and $\frac{2}{3} < \alpha \leq 1$. If $u \in C\big(\mathbb{R}^+, \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$ is a global solution of* (1.3), *then we have*

$$\lim_{t \to +\infty} \|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)} = 0.$$

In the last theorem, we prove the stability of global solutions of (1.3).

**Theorem 1.4** *Let $u \in C\big(\mathbb{R}^+, \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$ be a global solution of* (1.3) *and let $v^0 \in C\big(\mathbb{R}^+, \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$ be such that*

$$\|v^0 - u^0\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)} \leq \frac{1}{16} e^{-\dfrac{c}{2}\displaystyle\int_0^\infty \|u(z)\|^4_{\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)} \mathrm{d}z}.$$

*Then, equation* (1.3) *with the initial data* $v^0$ *has a global solution. Moreover, if* $v$ *is the corresponding global solution, then for all* $t \geq 0$ *we have*

$$\|v(t)-u(t)\|^2_{\dot H^{\frac52-2\alpha}_{a,\frac1\alpha}(\mathbb{R}^3)} + \frac14\int_0^t \|v(s)-u(s)\|^2_{\dot H^{\frac52-\alpha}_{a,\frac1\alpha}(\mathbb{R}^3)}\,\mathrm{d}s \leq \|v^0-u^0\|^2_{\dot H^{\frac52-2\alpha}_{a,\frac1\alpha}(\mathbb{R}^3)} e^{\frac{c}{2}\int_0^\infty \|u(s)\|^4_{\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha}(\mathbb{R}^3)}\,\mathrm{d}s}.$$

The remainder of this paper is organized in the following way. In Section 2, we give some notations and important preliminary results. Section 3 is devoted to proving the existence of solutions in the critical Sobolev–Gevery spaces $\dot H^{\frac52-2\alpha}_{a,\frac1\alpha}(\mathbb{R}^3)$. In Section 4, we show the blow-up result of maximal solution in $L^2\big([0,T^*),\dot H^{\frac52-\alpha}_{a,\frac1\alpha}(\mathbb{R}^3)\big)$. In Section 5, we prove the non blow-up result in $\dot H^{\frac52-2\alpha}_{a,\frac1\alpha}(\mathbb{R}^3)$. Finally we give the proof of the stability result for global solutions in Section 6.

## 2 Notations and preliminary results

**2.1 Notations.** In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as

$$\mathcal{F}(f)(\xi)=\hat f(\xi)=\int_{\mathbb{R}^3}\exp(-ix\cdot\xi)f(x)\,\mathrm{d}x,\quad \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x)=(2\pi)^{-3}\int_{\mathbb{R}^3}\exp(-i\xi\cdot x)g(\xi)\,\mathrm{d}\xi,\quad x=(x_1,x_2,x_3)\in\mathbb{R}^3.$$

- The convolution product of a suitable pair of functions $f$ and $g$ on $\mathbb{R}^3$ is given by

$$(f*g)(x):=\int_{\mathbb{R}^3}f(y)g(x-y)\,\mathrm{d}y.$$

- If $f=(f_1,f_2,f_3)$ and $g=(g_1,g_2,g_3)$ are two vector fields, we set $f\otimes g:=(g_1f,g_2f,g_3f)$ and $\operatorname{div}(f\otimes g):=(\operatorname{div}(g_1f),\operatorname{div}(g_2f),\operatorname{div}(g_3f))$.

- The homogeneous Sobolev–Gevery spaces are defined as follows: for $a$, $s\geq 0$, $\sigma\geq 1$ and $|D|=(-\Delta)^{\frac12}$ we set

$$\dot H^s_{a,\sigma}(\mathbb{R}^3)=\big\{f\in L^2(\mathbb{R}^3): e^{a|D|^{\frac1\sigma}}f\in \dot H^s(\mathbb{R}^3)\big\}.$$

  Moreover, we equip $\dot H^s_{a,\sigma}(\mathbb{R}^3)$ with the norm

$$\|f\|_{\dot H^s_{a,\sigma}}=\|e^{a|D|^{\frac1\sigma}}f\|_{\dot H^s}$$

  and the associated inner product

$$\langle f,g\rangle_{\dot H^s_{a,\sigma}}=\langle e^{a|D|^{\frac1\sigma}}f,e^{a|D|^{\frac1\sigma}}g\rangle_{\dot H^s}.$$

- We define also the following spaces:

$$\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha}) = \left\{ f \in S'(\mathbb{R}^+ \times \mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{5-4\alpha} \Big( \sup_{0\le t<\infty} |\hat{f}(t,\xi)| \Big)^2 \mathrm{d}\xi < \infty \right\}$$

with the norm

$$\|f\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} = \left( \int_{\mathbb{R}^3} |\xi|^{5-4\alpha} \Big( \sup_{0\le t<\infty} |\hat{f}(t,\xi)| \Big)^2 \mathrm{d}\xi \right)^{\frac{1}{2}}$$

and

$$L^2(\dot{H}^{\frac{5}{2}-\alpha}) = \left\{ f \in S'(\mathbb{R}^+ \times \mathbb{R}^3) : \int_0^\infty \int_{\mathbb{R}^3} |\xi|^{5-2\alpha} |\hat{f}(t,\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}t < \infty \right\}$$

with the norm

$$\|f\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} = \left( \int_0^\infty \int_{\mathbb{R}^3} |\xi|^{5-2\alpha} |\hat{f}(t,\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}t \right)^{\frac{1}{2}}.$$

- By $\mathcal{P}$ we denote the Leray projection operator defined by

$$\mathcal{P} = I_d - \nabla(-\Delta)^{-1} \operatorname{div},$$

$$\mathcal{F}(\mathcal{P}f)(\xi) = \hat{f}(\xi) - \frac{\xi \cdot \hat{f}(\xi)}{|\xi|^2}\xi.$$

**2.2 Preliminary results.** In this section, we recall some classical results and we give new technical lemmas.

**Lemma 2.1 ([1])** *Let $E$ be a Banach space, $B$ a continuous bilinear map from $E \times E$ into $E$, and let $\alpha$ be a positive real number such that $\alpha < \frac{1}{4\|B\|}$, where*

$$\|B\| = \sup_{\|u\|\le 1,\ \|v\|\le 1} \|B(u,v)\|.$$

*Then, for any a in the ball $B(0,\alpha)$ in $E$, there exists a unique $x$ in $B(0,2\alpha)$ such that $x = a + B(x,x)$.*

**Lemma 2.2 ([6])** *Let $a > 0$, $\sigma \ge 1$ and $(s_1, s_2) \in \mathbb{R}^2$ be such that $s_1 < \frac{3}{2}$, $s_2 < \frac{3}{2}$ and $s_1 + s_2 > 0$. Then, there exists a constant $C = C(s_1, s_2)$ such that for all $u \in \dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)$ and $v \in \dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)$ we have*

$$\|uv\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} \le C\|u\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)} \|v\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)}. \tag{2.1}$$

**Lemma 2.3** *Let $a > 0$ and $\frac{2}{3} < \alpha \le 1$. Let $Q$ be the bilinear form defined in* (1.2). *Then, there exists a constant $C > 0$ such that for all $u, v \in \dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)$ we have*

$$\|Q(u,v)\|_{\dot{H}^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}}} \le C\|u\|_{\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}} \|v\|_{\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}. \tag{2.2}$$

*Proof.* Thanks to the inequality (2.1), we get

$$
\begin{aligned}
\|Q(u,v)\|_{\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}}} &\le C\sup_{k,l}\Big(\|u^k\partial v^l\|_{\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}}}+\|v^l\partial u^k\|_{\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}}}\Big)\\
&\le C\Big(\|u\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|\nabla v\|_{\dot H^{\frac{3}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}+\|v\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|\nabla u\|_{\dot H^{\frac{3}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\Big)\\
&\le 2C\|u\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|v\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}.
\end{aligned}
$$

This ends the proof. □

**Lemma 2.4** *Let $u$ be the solution in $C([0,T],S')$ of the Cauchy problem*

$$
\begin{cases}\partial_t u+(-\Delta)^\alpha u=f,\\ u(0,x)=u^0(x),\end{cases}
$$

*with $f\in L^2\big([0,T],\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}}\big)$ and $u^0\in \dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}$. Then,*

$$
u\in\bigcap_{p=2}^{\infty}L^p\big([0,T],\dot H^{\frac{5}{2}-2\alpha+\frac{2\alpha}{p}}_{a,\frac{1}{\alpha}}\big)\cap C\big([0,T],\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}\big).
$$

*Moreover, we have the following estimates*:

$$
\|u\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\int_0^t\|\Lambda^\alpha u\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}s\le\|u^0\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\int_0^t\|f(s)\|^2_{\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}s, \tag{2.3}
$$

$$
\Big[\int_{\mathbb{R}^3}|\xi|^{5-4\alpha}e^{2a|\xi|^\alpha}\Big(\sup_{0\le t'\le t}|\hat u(t',\xi)|\Big)^2\mathrm{d}\xi\Big]^{\frac{1}{2}}\le\|u^0\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\|f\|_{L^2([0,t),\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}})}, \tag{2.4}
$$

$$
\|u\|_{L^p_T(\dot H^{\frac{5}{2}-2\alpha+\frac{2\alpha}{p}}_{a,\frac{1}{\alpha}})}\le\|u^0\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\|f\|_{L^2([0,t),\dot H^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}})}. \tag{2.5}
$$

*Proof.* The first estimate is just the energy estimate. To prove the second one we write Duhamel's formula in Fourier space. Namely,

$$
\hat u(t,\xi)=e^{-t|\xi|^{2\alpha}}\widehat{u^0}(\xi)+\int_0^t e^{-(t-s)|\xi|^{2\alpha}}\hat f(s,\xi)\,\mathrm{d}s.
$$

By the Cauchy–Schwartz inequality we get

$$
\sup_{0\le t'\le t}|\hat u(t',\xi)|\le|\widehat{u^0}(\xi)|+\frac{1}{\sqrt2|\xi|^\alpha}\|\hat f(\xi,\cdot)\|_{L^2([0,t))}
$$

for any $0<t<T$. Multiplying the obtained inequality by $|\xi|^{\frac{5}{2}-2\alpha}e^{a|\xi|^\alpha}$, we obtain

$$
|\xi|^{\frac{5}{2}-2\alpha}e^{a|\xi|^\alpha}\sup_{0\le t'\le t}|\hat u(t',\xi)|\le|\xi|^{\frac{5}{2}-2\alpha}e^{a|\xi|^\alpha}|\widehat{u^0}(\xi)|+\frac{|\xi|^{\frac{5}{2}-2\alpha}e^{a|\xi|^\alpha}}{\sqrt2|\xi|^\alpha}\|\hat f(\xi,\cdot)\|_{L^2([0,t))}.
$$

Taking the $L^2$-norm with respect to the frequency variable $\xi$, we conclude that

$$\left[\int_{\mathbb{R}^3} |\xi|^{5-4\alpha} e^{2a|\xi|^\alpha} \left(\sup_{0\le t'\le t} |\hat{u}(t',\xi)|\right)^2 \mathrm{d}\xi\right]^{\frac{1}{2}} \le \|u^0(\xi)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} + \|f\|_{L^2([0,t),\dot{H}^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}})}.$$

Since the map $t \mapsto \hat{u}(t,\xi)$ is continuous over $[0,T]$ for almost all fixed $\xi \in \mathbb{R}^3$, the Lebesgue dominated convergence theorem ensures that $u \in C\big([0,T], \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$.

Similarly, we have

$$|\xi|^{\frac{5}{2}-\alpha} e^{a|\xi|^\alpha} |\hat{u}| \le |\xi|^{\frac{5}{2}-\alpha} e^{-t|\xi|^{2\alpha}} e^{a|\xi|^\alpha} |\widehat{u^0}| + \int_0^t |\xi|^{\frac{5}{2}-\alpha} e^{-(t-s)|\xi|^{2\alpha}} e^{a|\xi|^\alpha} |\hat{f}(s,\xi)|\,\mathrm{d}s.$$

Taking the $L^2$-norm with respect to time and using Young's inequality, we obtain

$$\begin{aligned}\left[\int_0^t |\xi|^{5-2\alpha} e^{2a|\xi|^\alpha} |\hat{u}(\xi,s)|^2\,\mathrm{d}s\right]^{\frac{1}{2}} &\le \left(\int_0^t |\xi|^{2\alpha} e^{-2s|\xi|^{2\alpha}}\,\mathrm{d}s\right)^{\frac{1}{2}} |\xi|^{\frac{5}{2}-2\alpha} e^{a|\xi|^\alpha} |\widehat{u^0}| \\ &\quad + \int_0^t |\xi|^{2\alpha} e^{-s|\xi|^{2\alpha}}\,\mathrm{d}s \left(\int_0^t |\xi|^{5-6\alpha} e^{2a|\xi|^\alpha} |\hat{f}(s,\xi)|^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\ &\le |\xi|^{\frac{5}{2}-\alpha} e^{a|\xi|^\alpha} |\widehat{u^0}| + \left(\int_0^t |\xi|^{5-6\alpha} e^{2a|\xi|^\alpha} |\hat{f}(s,\xi)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the $L^2$-norm with respect to the frequency variable $\xi$, we obtain

$$\|u\|_{L^2_T(\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}})} \le \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} + \|f\|_{L^2_T(\dot{H}^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}})}. \tag{2.6}$$

Finally, the last inequality follows by interpolation

$$\|u\|_{\dot{H}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{p}}_{a,\frac{1}{\alpha}}} \le \|u\|^{1-\frac{2}{p}}_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} \|u\|^{\frac{2}{p}}_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}$$

and

$$\|u\|^p_{\dot{H}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{p}}_{a,\frac{1}{\alpha}}} \le \|u\|^{p-2}_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} \|u\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}.$$

Taking the $L^1$-norm with respect to time and using the inequalities (2.4) and (2.6), we get (2.5). $\square$

**Lemma 2.5** *There exists a positive constant* $\varepsilon_0 > 0$ *such that for any initial data* $u^0$ *in* $\dot{H}^{\frac{5}{2}-2\alpha}$ *with* $\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} < \varepsilon_0$ *equation* (1.3) *has a unique global in time solution* $u \in \tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha}) \cap L^2(\dot{H}^{\frac{5}{2}-\alpha})$ *which is analytic in the sense that*

$$\|e^{\sqrt{t}|D|^\alpha} u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|e^{\sqrt{t}|D|^\alpha} u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} \le c_0 \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}}, \tag{2.7}$$

*where* $e^{\sqrt{t}|D|^\alpha}$ *is a Fourier multiplier whose symbol is given by* $e^{\sqrt{t}|\xi|^\alpha}$ *and* $c_0$ *is a universal constant.*

*Proof.* The proof of Lemma 2.5 is inspired by the work of Bae (see [2]). This proof is done in three steps. We first apply the Fourier transform to the integral form of the fractional Navier–Stokes equation

$$\hat{u}(t,\xi) = e^{-t|\xi|^{2\alpha}}\widehat{u^0} - \int_0^t e^{-(t-s)|\xi|^{2\alpha}}\mathcal{F}(\mathcal{P}(\operatorname{div}(u\otimes u)))\,\mathrm{d}s. \tag{2.8}$$

**Step 1.** First, we estimate $u$ in $\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})$. Multiplying (2.8) by $|\xi|^{\frac{5}{2}-2\alpha}$, we get

$$\begin{aligned}
|\xi|^{\frac{5}{2}-2\alpha}|\hat{u}(t,\xi)| &\le |\xi|^{\frac{5}{2}-2\alpha}|\widehat{u^0}(\xi)| + \int_0^t e^{-(t-s)|\xi|^{2\alpha}}|\xi|^{\frac{7}{2}-2\alpha}|\widehat{u\otimes u}|\,\mathrm{d}s\\
&\le |\xi|^{\frac{5}{2}-2\alpha}|\widehat{u^0}(\xi)| + \sup_{0\le t<\infty}\int_0^t |\xi|^{2\alpha}e^{-(t-s)|\xi|^{2\alpha}}|\xi|^{\frac{7}{2}-4\alpha}|\widehat{u\otimes u}|\,\mathrm{d}s\\
&\le |\xi|^{\frac{5}{2}-2\alpha}|\widehat{u^0}(\xi)| + \int_0^t |\xi|^{2\alpha}e^{-(t-s)|\xi|^{2\alpha}}\,\mathrm{d}s \sup_{0\le t<\infty}|\xi|^{\frac{7}{2}-4\alpha}|\widehat{u\otimes u}(t,\xi)|\\
&\le |\xi|^{\frac{5}{2}-2\alpha}|\widehat{u^0}(\xi)| + |\xi|^{\frac{7}{2}-4\alpha}\sup_{0\le t<\infty}|\widehat{u\otimes u}(t,\xi)|.
\end{aligned}$$

Taking the $L^2$-norm with respect to the frequency variable $\xi$, we get

$$\|u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} \le \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + C_{\frac{5}{2}-2\alpha,\frac{5}{2}-2\alpha}\|u\|^2_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})}. \tag{2.9}$$

Now, we estimate $u$ in $L^2(\dot{H}^{\frac{5}{2}-2\alpha})$. Multiplying (2.8) by $|\xi|^{\frac{5}{2}-\alpha}$, we obtain

$$|\xi|^{\frac{5}{2}-\alpha}|\hat{u}(t,\xi)| \le |\xi|^{\frac{5}{2}-\alpha}e^{-t|\xi|^{2\alpha}}|\widehat{u^0}(\xi)| + \int_0^t e^{-(t-s)|\xi|^{2\alpha}}|\xi|^{\frac{7}{2}-\alpha}|\widehat{u\otimes u}(s,\xi)|\,\mathrm{d}s.$$

Taking the $L^2$-norm with respect to time and using Young's inequality, we deduce that

$$\begin{aligned}
&\left(\int_0^\infty \left[|\xi|^{\frac{5}{2}-\alpha}|\hat{u}(t,\xi)|\right]^2\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\le \left(\int_0^\infty |\xi|^{2\alpha}e^{-2t|\xi|^{2\alpha}}|\xi|^{5-4\alpha}|\widehat{u^0}(\xi)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\qquad + \left(\int_0^\infty\left[\int_0^t |\xi|^{\frac{7}{2}-\alpha}e^{-(t-s)|\xi|^{2\alpha}}|\widehat{u\otimes u}(s,\xi)|\,\mathrm{d}s\right]^2\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\le |\xi|^{\frac{5}{2}-2\alpha}|\widehat{u^0}(\xi)| + \int_0^\infty |\xi|^{2\alpha}e^{-s|\xi|^{2\alpha}}\,\mathrm{d}s\left(\int_0^\infty |\xi|^{7-6\alpha}|\widehat{u\otimes u}(s,\xi)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad\le |\xi|^{\frac{5}{2}-2\alpha}|\widehat{u^0}(\xi)| + \left(\int_0^\infty |\xi|^{7-6\alpha}|\widehat{u\otimes u}(s,\xi)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Taking the $L^2$-norm in $\xi$ and using Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned}
\|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} &\le \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + \left(\int_0^\infty\int_{\mathbb{R}^3}|\xi|^{7-6\alpha}|\widehat{u\otimes u}(s,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\le \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + C_{\frac{5}{2}-\alpha,\frac{5}{2}-2\alpha}\left(\int_0^\infty \|u\|^2_{\dot{H}^{\frac{5}{2}-\alpha}}\|u\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}}\,\mathrm{d}s\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + C_{\frac{5}{2}-\alpha,\frac{5}{2}-2\alpha}\Big(\int_0^\infty \|u\|^2_{\dot{H}^{\frac{5}{2}-\alpha}}\,\mathrm{d}s \sup_{0\leq t<\infty}\|u\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}}\Big)^{\frac{1}{2}},$$

which yields

$$\|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} \leq \|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + C_{\frac{5}{2}-\alpha,\frac{5}{2}-2\alpha}\|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\|u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})}. \tag{2.10}$$

**Step 2.** Combining (2.9) and (2.10), we get

$$\|u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} \leq 2\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + C\big(\|u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\big)^2 \tag{2.11}$$

with $C = C_{\frac{5}{2}-2\alpha,\frac{5}{2}-2\alpha} + C_{\frac{5}{2}-\alpha,\frac{5}{2}-2\alpha}$. Let $0 < \varepsilon_0 < C_0$ be such that $\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} < \varepsilon_0$, where $C_0 = \frac{3}{16C}\min(\frac{1}{4C},\frac{1}{4})$. Moreover, let $\frac{16C\varepsilon_0}{3} < r < \min(\frac{1}{4C},\frac{1}{4})$, and take

$$B_r = \Big\{u \in \tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha}) \cap L^2(\dot{H}^{\frac{5}{2}-\alpha}) : \|u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} \leq r\Big\}.$$

We also use $\psi$ definied by

$$\psi(u) = e^{-t(-\Delta)^\alpha}u^0 - \int_0^t e^{-(t-s)(-\Delta)^\alpha}\mathcal{P}(\mathrm{div}(u\otimes u)(s))\,\mathrm{d}s.$$

Then, we have

$$\|\psi(u)\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|\psi(u)\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} \leq 2\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + C\big(\|u\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|u\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\big)^2.$$

This yields

$$\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} < \varepsilon_0 < \frac{3r}{16C}.$$

Finally, we get

$$\|\psi(u)\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|\psi(u)\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} \leq r.$$

So,

$$\psi(B_r) \subset B_r.$$

Moreover, for all $u_1, u_2 \in B_r$, we have

$$\begin{aligned}\|\psi(u_1)-\psi(u_2)\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} &\leq \|B(u_1-u_2,u_1) + B(u_2,u_1-u_2)\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})}\\ &\leq C\big(\|u_1\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})} + \|u_2\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})}\big)\|u_1-u_2\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})}\\ &\leq 2Cr\|u_1-u_2\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})}\\ &\leq \frac{1}{2}\|u_1-u_2\|_{\tilde{L}^\infty(\dot{H}^{\frac{5}{2}-2\alpha})},\end{aligned}$$

where $B(u,v) = -\int_0^t e^{-(t-s)(-\Delta)^\alpha}\mathcal{P}(\mathrm{div}(u\otimes v)(s))\,\mathrm{d}s$. Similarly, we have

$$\begin{aligned}\|\psi(u_1)-\psi(u_2)\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} &\leq \|B(u_1-u_2,u_1) + B(u_2,u_1-u_2)\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\\ &\leq C\big(\|u_1\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})} + \|u_2\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\big)\|u_1-u_2\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\\ &\leq 2Cr\|u_1-u_2\|_{L^2(\dot{H}^{\frac{5}{2}-\alpha})}\end{aligned}$$

$$\le \frac{1}{2}\|u_1 - u_2\|_{L^2(\dot H^{\frac{5}{2}-\alpha})}.$$

This implies the existence of a global solution in $\tilde L^\infty(\dot H^{\frac{5}{2}-2\alpha}) \cap L^2(\dot H^{\frac{5}{2}-\alpha})$ for small initial data in $\dot H^{\frac{5}{2}-2\alpha}(\mathbb{R}^3)$. Moreover, we have the estimate

$$\|u\|_{\tilde L^\infty(\dot H^{\frac{5}{2}-2\alpha})} + \|u\|_{L^2(\dot H^{\frac{5}{2}-\alpha})} \le \|u^0\|_{\dot H^{\frac{5}{2}-2\alpha}}.$$

**Step 3.** Multiplying (2.8) by $e^{\sqrt{t}|\xi|^\alpha}$, we obtain

$$\begin{aligned} & e^{\sqrt{t}|\xi|^\alpha}|\hat u(t,\xi)| \\ & \le e^{\sqrt{t}|\xi|^\alpha - t|\xi|^{2\alpha}}|\widehat{u^0}| + \int_0^t e^{-(t-s)|\xi|^{2\alpha}+\sqrt{t}|\xi|^\alpha}|\xi||\widehat{u\otimes u}|\,\mathrm{d}s \\ & \le e^{\sqrt{t}|\xi|^\alpha - \frac{1}{2}t|\xi|^{2\alpha}}e^{-\frac{1}{2}t|\xi|^{2\alpha}}|\widehat{u^0}| + \int_0^t e^{\sqrt{t}|\xi|^\alpha - \sqrt{s}|\xi|^\alpha - \frac{1}{2}(t-s)|\xi|^{2\alpha}}e^{-\frac{1}{2}(t-s)|\xi|^{2\alpha}}e^{\sqrt{s}|\xi|^\alpha}|\xi||\widehat{u\otimes u}|\,\mathrm{d}s. \end{aligned}$$

Since $e^{\sqrt{t}|\xi|^\alpha - \frac{1}{2}t|\xi|^{2\alpha}}$ is uniformly bounded in time and $\xi$, we have

$$\begin{aligned} & e^{\sqrt{t}|\xi|^\alpha}|\hat u(t,\xi)| \\ & \le c_0\Big(e^{-\frac{1}{2}t|\xi|^{2\alpha}}|\widehat{u^0}| + \int_0^t e^{-\frac{1}{2}(t-s)|\xi|^{2\alpha}}|\xi|\int e^{\sqrt{s}|\xi-\eta|^\alpha}|\hat u(\xi-\eta)|e^{\sqrt{s}|\eta|^\alpha}|\hat u(\eta)|\,\mathrm{d}\eta\,\mathrm{d}s\Big) \\ & \le c_0\Big(e^{-\frac{1}{2}t|\xi|^{2\alpha}}|\widehat{u^0}| + \int_0^t |\xi|e^{-\frac{1}{2}(t-s)|\xi|^{2\alpha}}|\widehat{V\otimes V}|\,\mathrm{d}s\Big) \end{aligned}$$

with $V(t,\cdot) = e^{\sqrt{t}|D|^\alpha}u(t,\cdot)$ and $c_0 = \sqrt{e} = e^{\frac{1}{2}}$. Then, going through the previous steps, we get

$$\|V\|_{\tilde L^\infty(\dot H^{\frac{5}{2}-2\alpha})} + \|V\|_{L^2(\dot H^{\frac{5}{2}-\alpha})} \le c_0\|u^0\|_{\dot H^{\frac{5}{2}-2\alpha}},$$

which implies that

$$\|e^{\sqrt{t}|D|^\alpha}u\|_{\tilde L^\infty(\dot H^{\frac{5}{2}-2\alpha})} \le c_0\|u^0\|_{\dot H^{\frac{5}{2}-2\alpha}}. \tag{2.12}$$

This completes the proof. □

## 3 Proof of Theorem 1.1

Let $B(u,u)$ be the solution to the fractional heat equation

$$\begin{cases} \partial_t B(u,u) + (-\Delta)^\alpha B(u,u) = Q(u,u), \\ \operatorname{div} B(u,u) = 0, \\ B(u,u)(0) = 0, \end{cases} \tag{3.1}$$

with the bilinear operator $Q$ defined in (1.2) and

$$B(u,u) = -\int_0^t e^{-(t-s)(-\Delta)^\alpha}\mathcal{P}(\operatorname{div}(u\otimes u))\,\mathrm{d}s.$$

Thanks to Lemma 2.3, we have

$$\int_0^T \|Q(u,v)\|^2_{\dot H^{\frac52-3\alpha}_{a,\frac1\alpha}}\,\mathrm{d}s \le C\|u\|^2_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}\|v\|^2_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}.$$

Thus, combining Duhamel's formula and the inequality (2.5), we obtain

$$\|B(u,u)\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})} \le C\|u\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}\|u\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}.$$

This implies that

$$\|B\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})} \le C < C_0.$$

Thanks to Minkowski's inequality, we have

$$\|e^{-t(-\Delta)^\alpha}u^0\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})} \le \|u^0\|_{\dot H^{\frac52-2\alpha}_{a,\frac1\alpha}}. \tag{3.2}$$

Thus, if $\|u^0\|_{\dot H^{\frac52-2\alpha}_{a,\frac1\alpha}} \le \frac{1}{4C_0}$, we get

$$\|e^{-t(-\Delta)^\alpha}u^0\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})} \le \frac{1}{4C_0} < \frac{1}{4\|B\|}.$$

According to Lemma 2.1, there exists a unique solution of (1.1) in the ball with center $0$ and radius $\frac{1}{2C_0}$ in the space $L^4([0,T];\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})$ such that $u(t,x) = e^{-t(-\Delta)^\alpha}u^0 + B(u,u)$.

We now consider the case of a large initial data $u^0 \in \dot H^{\frac52-2\alpha}_{a,\frac1\alpha}$. Let $\rho_{u_0} > 0$ be such that

$$\left(\int_{|\xi|\ge\rho_{u_0}} e^{2a|\xi|^\alpha}|\xi|^{5-4\alpha}|\widehat{u^0}|^2\,\mathrm{d}\xi\right)^{\frac12} < \frac{1}{8C_0}.$$

Using the inequality (3.2) and defining $v_0 = \mathcal{F}^{-1}(\chi_{|\xi|<\rho_{u_0}}\widehat{u^0})$, we get

$$\begin{aligned}
&\|e^{-t(-\Delta)^\alpha}u^0\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}\\
&\quad\le \|e^{-t(-\Delta)^\alpha}\mathcal{F}^{-1}(\chi_{|\xi|\ge\rho_{u_0}}\widehat{u^0})\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})} + \|e^{-t(-\Delta)^\alpha}v_0\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}\\
&\quad\le \frac{1}{8C_0} + \|e^{-t(-\Delta)^\alpha}v_0\|_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})}.
\end{aligned}$$

Moreover, we can deduce that

$$\begin{aligned}
\|e^{-t(-\Delta)^\alpha}v_0\|^4_{L^4_T(\dot H^{\frac52-\frac{3\alpha}{2}}_{a,\frac1\alpha})} &= \int_0^T\left[\int_{|\xi|<\rho_{u_0}}|\xi|^{5-3\alpha}e^{2a|\xi|^\alpha}|\widehat{u^0}|^2\,\mathrm{d}\xi\right]^2\mathrm{d}t\\
&\le \rho_{u_0}^{2\alpha}\int_0^T\left[\int_{|\xi|<\rho_{u_0}}|\xi|^{5-4\alpha}e^{2a|\xi|^\alpha}|\widehat{u^0}|^2\,\mathrm{d}\xi\right]^2\mathrm{d}t
\end{aligned}$$

$$\le T\rho_{u_0}^{2\alpha}\|u^0\|^4_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}.$$

This yields

$$\|e^{-t(-\Delta)^\alpha}v_0\|_{L^4_T(\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})} \le (T\rho_{u_0}^{2\alpha})^{\frac{1}{4}}\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}.$$

Thus, if

$$8C_0\rho_{u_0}^{\frac{\alpha}{2}}\|u^0\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} \le \frac{1}{T^{\frac{1}{4}}}, \tag{3.3}$$

then we obtain the existence of a unique solution of (1.1) in the ball with center 0 and radius $\frac{1}{2C_0}$ in the space $L^4([0,T];\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})$. And we observe that if $u$ is a solution of (1.1) in $L^4([0,T];\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})$, then $Q(u,u)$ belongs to $L^2_T(\dot{H}^{\frac{5}{2}-3\alpha}_{a,\frac{1}{\alpha}})$ by Lemma 2.3. Hence, Lemma 2.4 implies that the solution $u$ belongs to $C\big([0,T];\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}\big)\cap L^2\big([0,T];\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}\big)$.

## 4 Proof of Theorem 1.2

We begin by proving the blow-up result (ii). Suppose that

$$\int_0^{T^*}\|u(t)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}t<\infty.$$

Let a time $T\in(0,T^*)$ be such that $\int_T^{T^*}\|u(t)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}t<\frac{1}{4C}$. Lemma 2.2 implies that for all $t\in[T,T^*)$ and $z\in[T,t]$ we have

$$\begin{aligned}\|u(z)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+2\int_T^z\|u(s)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}s &\le \|u(T)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+2C\int_T^z\|u(s)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\|u(s)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}s\\ &\le \|u(T)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\frac{1}{2}\sup_{T\le s\le t}\|u(s)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}.\end{aligned}$$

Hence, we can deduce that

$$\sup_{T\le z\le t}\|u(z)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\le\|u(T)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\frac{1}{2}\sup_{T\le s\le t}\|u(s)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}},$$

which implies that

$$\sup_{T\le s\le t}\|u(s)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\le C_T$$

with $C_T=\frac{1}{4}+\big(\frac{1}{16}+\|u(T)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\big)^{\frac{1}{2}}$.

Let $M=\max\big(\sup_{0\le t\le T}\|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}},C_T\big)$. Then, for all $t\in[0,T^*)$ we get

$$\|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\le M.$$

By the interpolation formula and Hölder's inequality, we deduce that

$$u \in L^4\big([0,T^*), \dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}\big).$$

Let $0 < t_0 < T^*$ be such that

$$\|u\|_{L^4([t_0,T^*),\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})} \le \frac{1}{4C_0}.$$

Now, we consider the Navier–Stokes system starting at $t = t_0$

$$\begin{cases} \partial_t v + (-\Delta)^\alpha v + v\cdot\nabla v = -\nabla q, \\ \operatorname{div} v = 0, \\ v(0) = u(t_0). \end{cases}$$

Then, we obtain

$$\begin{aligned} \|v(t)\|_{L^4([0,T^*-t_0),\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})} &= \|u(t+t_0)\|_{L^4([0,T^*-t_0),\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})} \\ &= \|u(t)\|_{L^4([t_0,T^*),\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}})} \\ &\le \frac{1}{4C_0}, \end{aligned}$$

which implies the existence of a unique solution to the above system on $[0, T^*-t_0)$ that can be extended to the interval $[0,T^*)$. This is absurd.

Next, we prove the first claim of Theorem 1.2. We have

$$\partial_t u + (-\Delta)^\alpha u + u\cdot\nabla u = -\nabla p.$$

Taking the inner product in $\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)$ with $u$ and using Lemma 2.2, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} + \|\Lambda^\alpha u\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} &\le |\langle u\cdot\nabla u, u\rangle_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}| \\ &\le \|u\otimes u\|_{\dot H^{\frac{7}{2}-3\alpha}_{a,\frac{1}{\alpha}}}\|u\|_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}} \\ &\le C\|u\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\|u\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}. \end{aligned}$$

Let

$$T = \sup\big\{t \ge 0 : \sup_{0\le z\le t}\|u(z)\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} < \tfrac{1}{C}\big\}.$$

For all $0 < t \le T$ we have

$$\|u(t)\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} + \int_0^T \|u(z)\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z \le \|u^0\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} < \big(\tfrac{1}{C}\big)^2.$$

Then, $T = T^*$ and $\int_0^{T^*}\|u(z)\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z < \infty$. Therefore, $T^* = \infty$ and for every $t \ge 0$ we get

$$\|u(t)\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} + \int_0^T \|u(z)\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z \le \|u^0\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}.$$

# 5 Proof of Theorem 1.3

In this section we prove that

$$\lim_{t\to\infty} \|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} = 0.$$

For $0 < \varepsilon < \frac{\varepsilon_0}{C}$ let $u \in C\big(\mathbb{R}^+, \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$. As $\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{\frac{5}{2}-2\alpha}(\mathbb{R}^3)$, we infer that $u \in C\big(\mathbb{R}^+, \dot{H}^{\frac{5}{2}-2\alpha}(\mathbb{R}^3)\big)$. We can conclude that there exists a time $t_0 > 0$ (see Remark 1.5 in [11]). For every $t \geq t_0$ we have

$$\|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} < \varepsilon.$$

Applying Lemma 2.5, for all $t \geq t_0$ we get

$$\|e^{\sqrt{t-t_0}|D|^\alpha} u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \leq \|u(t_0)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} < \varepsilon.$$

Consider the following system

$$\begin{cases} \partial_t v + (-\Delta)^\alpha v + v \cdot \nabla v = -\nabla p, \\ \operatorname{div} v = 0, \\ v(0) = u(t_0). \end{cases}$$

By the uniqueness of solutions to (1.1) in $\tilde{L}^\infty\big(\dot{H}^{\frac{5}{2}-2\alpha}(\mathbb{R}^3)\big)$, for all $t \geq 0$ we obtain

$$\begin{aligned} \|e^{\sqrt{t}|D|^\alpha} v(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} &= \|e^{\sqrt{t}|D|^\alpha} u(t+t_0)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \\ &= \|e^{\sqrt{t+t_0-t_0}|D|^\alpha} u(t+t_0)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \\ &< \varepsilon. \end{aligned}$$

Let a time $t_1 > t_0 > 0$ be such that $\sqrt{t_1 - t_0} > a$. For all $t \geq t_1 - t_0$, we get

$$\begin{aligned} \|e^{a|D|^\alpha} v(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} &= \|e^{a|D|^\alpha - \sqrt{t}|D|^\alpha} e^{\sqrt{t}|D|^\alpha} v(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \\ &\leq \|e^{\sqrt{t}|D|^\alpha} v(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \\ &< \varepsilon. \end{aligned}$$

Now, we consider the following system

$$\begin{cases} \partial_t w + (-\Delta)^\alpha w + w \cdot \nabla w = -\nabla p, \\ \operatorname{div} w = 0, \\ w(0) = v(t_1). \end{cases}$$

We obtain

$$\begin{aligned} \|e^{a|D|^\alpha} w(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} &= \|e^{a|D|^\alpha} v(t+t_1)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \\ &\leq \|e^{\sqrt{t+t_1}|D|^\alpha} v(t+t_1)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \\ &< \varepsilon, \end{aligned}$$

which yields the result

$$\lim_{t\to\infty} \|e^{a|D|^\alpha} w(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}} = 0.$$

## 6 Proof of Theorem 1.4

The proof of Theorem 1.4 is identical to the proofs presented in [3, 9]. Let $v \in C\big([0,T^*),\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)\big)$ be the maximal solution of (1.1) corresponding to the initial conditional $v^0$. We want to prove that $T^*=\infty$, if $\|u(0)-v^0\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}<\varepsilon$.

Put $w=v-u$ and $w^0=v^0-u(0)$. We have

$$\partial_t w+(-\Delta)^\alpha w+w\cdot\nabla w+u\cdot\nabla w+w\cdot\nabla u=-\nabla p.$$

Then, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+2\|\Lambda^\alpha w\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\le I_1+I_2$$

with

$$I_1=2|\langle w\cdot\nabla w,w\rangle_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}|$$

and

$$I_2=2|\langle u\cdot\nabla w,w\rangle_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}|+2|\langle w\cdot\nabla u,w\rangle_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}|.$$

By using Cauchy–Schwartz inequality, Lemma 2.2 and Young's inequality, we obtain

$$I_1\le C\|w\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\|w\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}$$

and

$$\begin{aligned}
I_2&\le 2\big(\|u\otimes w\|_{\dot H^{\frac{7}{2}-3\alpha}_{a,\frac{1}{\alpha}}}+\|w\otimes u\|_{\dot H^{\frac{7}{2}-3\alpha}_{a,\frac{1}{\alpha}}}\big)\|w\|_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\\
&\le 2C\|u\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|w\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|w\|_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\\
&\le 2C\|u\|_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|w\|^{\frac{1}{2}}_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\|w\|^{\frac{3}{2}}_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\\
&\le \frac{C}{2}\|u\|^4_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|w\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\frac{3}{2}\|w\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}.
\end{aligned}$$

Then, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+2\|w\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\le\|w\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\|w\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}+\frac{C}{2}\|u\|^4_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|w\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\frac{3}{2}\|w\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}.$$

Put

$$T=\sup\big\{t\in[0,T^*):\sup_{0\le z\le t}\|w(z)\|_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}<\tfrac{1}{4}\big\}.$$

For all $t\in[0,T)$ we have

$$\|w(t)\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}+\frac{1}{4}\int_0^t\|w(z)\|^2_{\dot H^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z\le\frac{C}{2}\int_0^t\|u(z)\|^4_{\dot H^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\|w(z)\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z+\|w(0)\|^2_{\dot H^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}}.$$

Gronwall's lemma yields

$$\|w(t)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} + \frac{1}{4}\int_0^t \|w(z)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z \le \|w(0)\|^2_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}} e^{\frac{c}{2}\int_0^\infty \|u(z)\|^4_{\dot{H}^{\frac{5}{2}-\frac{3\alpha}{2}}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z} < \frac{1}{16}.$$

So, $T = T^*$ and $\int_0^{T^*} \|w(z)\|^2_{\dot{H}^{\frac{5}{2}-\alpha}_{a,\frac{1}{\alpha}}}\,\mathrm{d}z < \infty$. Therefore, $T^* = \infty$ and the proof is finished.

**Acknowledgements** The authors would like to express their deep gratitude to the editor and the referee for their careful reading, valuable comments and suggestions which made this article more readable.

This work is supported in part by National Natural Science Foundation of China (Grant No. 11601434) and SRPNWNU (Grant No. NWNU-LKQW-14-2).

## References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, A Series of Comprehensive Studies in Mathematics, vol. 343, Springer, Heidelberg, 2011.

[2] H. Bae, *Existence and analyticity of Lei-Lin solution to the Navier–Stokes equations*, Proceedings of the American Mathematical Society **143** (2015), no. 7, 2887–2892.

[3] J. Benameur, *Long time decay to the Lei–Lin solution of 3D Navier–Stokes equations*, Journal of Mathematical Analysis and Applications **422** (2015), no. 1, 424–434.

[4] J. Benameur, L. Jlali, *Long time decay of 3D-NSE in Lei–Lin–Gevrey spaces*, Mathematica Slovaca **70** (2020), no. 4, 877–892.

[5] J. Benameur, L. Jlali, *Long time decay for 3D Navier–Stokes equations in Sobolev–Gevery spaces*, Electronic Journal of Differential Equations **2016** (2016), no. 104, 13 pages.

[6] J. Benameur, L. Jlali, *On the blow-up criterion of 3D-NSE in Sobolev–Gevrey spaces*, Journal of Mathematical Fluid Mechanics **18** (2016), no. 4, 805–822.

[7] Y. Ding, X. Sun, *Uniqueness of weak solutions for fractional Navier–Stokes equations*, Frontiers of Mathematics in China **10** (2015), no. 1, 33–51.

[8] Y. Ding, X. Sun, *Strichartz estimates for parabolic equations with higher order differential operators*, Science China Mathematics **58** (2015), no. 5, 1047–1062.

[9] I. Gallagher, D. Iftimie, F. Planchon, *Non-blowup at large times and stability for global solutions to the Navier–Stokes equations*, Comptes Rendus Mathématique. Académie des Sciences. Paris **334** (2002), no. 4, 289–292.

[10] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.

[11] H. Orf, *Long time decay for global solutions to the Navier–Stokes equations in Sobolev–Gevery spaces*, arXiv:1903.03034v1 [math.AP].

[12] X. Sun, Y. Ding, *Dispersive effect of the Coriolis force and the local well-posedness for the fractional Navier–Stokes–Coriolis system*, Journal of Evolution Equations **20** (2020), no. 2, 335–354.

[13] H. Wu and J. Fan, *Weak-strong uniqueness for the generalized Navier–Stokes equations*, Applied Mathematics Letters **25** (2012), no. 3, 423–428.

[14] Z. Zhai, *Strichartz type estimates for fractional heat equations*, Journal of Mathematical Analysis and Applications **356** (2009), no. 2, 642–658.

[15] Y. Zhou, *Regularity criteria for the generalized viscous MHD equations*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **24** (2007), no. 3, 491–505.