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STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF INFINITE CLASS UNDER THE LIGHT OF MEASURE THEORY

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Abstract. The aim of this work is to study weighted Stepanov-like pseudo almost automorphic functions with infinite delay using the measure theory. We present a new concept of weighted ergodic functions which is more general than the classical one. Then, we establish many interesting results on the space of such functions. We also study the existence and uniqueness of (μ, ν)-weighted Stepanov-like pseudo almost automorphic solutions of infinite class for some neutral partial functional differential equations in a Banach space when the delay is distributed using the spectral decomposition of the phase space developed by Adimy and his co-authors. Here, we assume that the undelayed part is not necessarily densely defined and satisfies the well-known Hille–Yosida condition. The delayed part is assumed to be pseudo almost automorphic with respect to the first argument and Lipschitz continuous with respect to the second argument.

Keywords: Ergodicity, evolution equations, measure theory, partial functional differential equations, weighted Stepanov-like pseudo almost automorphic function.

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1 Introduction

In this work, we study the existence and uniqueness of Stepanov-like pseudo almost automorphic solutions of infinite class for the following neutral partial functional differential equation

$$u'(t) = Au_t + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where A is a linear operator on a Banach space X satisfying the Hille–Yosida condition. The phase space $\mathscr{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X satisfying some axioms which will be described in the sequel; for every $t \geq 0$ the history $u_t \in \mathscr{B}$ is defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \in (-\infty, 0],$$

L is a bounded linear operator from $\mathscr{B}$ to X and $f\colon \mathbb{R}^+ \to X$ is a continuous function.

Almost automorphic functions are more general than almost periodic ones and they were introduced by Bochner [5, 6]. (For more details on almost automorphy and related topics we refer to the recent books [12, 14], where the author discusses not only the theory of almost automorphic functions, but also shows applications of such functions to differential equations.) In [13] the authors introduced and studied a new class of Stepanov-like almost automorphic functions with values in a Banach space. Almost automorphic solutions in the context of differential equations have been studied by many authors. In [17], the authors presented a new approach dealing with weighted pseudo almost periodic functions and their applications in evolution equations and partial functional differential equations by using the measure theory.

The aim of this work is to generalise the results established in [17] by proving the existence of Stepanov-like pseudo almost automorphic solutions of equation (1.1) when the delay is distributed on $(-\infty, 0]$. Our approach is based on the variation of constants formula and the spectral decomposition of the phase space developed in [3].

This work is organised as follow. In Section 2, we recall some preliminary results on variation of constants formula and spectral decomposition. In Section 3 and Section 4, we recall some preliminary results on Stepanov-like pseudo almost automorphic functions that will be used in this work. In Section 5, we prove some properties of S^p-pseudo almost automorphic function of infinite class. In Section 6, we discuss the main result of this paper. Using the strict contraction principle we show the existence and uniqueness of Stepanov-like pseudo almost automorphic solution of infinite class for equation (1.1). Finally, for illustration, we study the existence and uniqueness of S^p-pseudo almost automorphic solution for some model arising in population dynamics.

2 Variation of constants formula and spectral decomposition

In this work, we assume that the state space $(\mathscr{B}, |.|_{\mathscr{B}})$ is a normed linear space of functions mapping $(-\infty, 0]$ into X and satisfying the following fundamental axioms.

($\mathbf{A}_1$) There exist a positive constant H and functions $K(.), M(.)\colon \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$ if $u\colon (-\infty, a] \to X$, $u_\sigma \in \mathscr{B}$, and $u(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold

(i) $u_t \in \mathscr{B}$,

(ii) $|u(t)| \leq H|u_t|_{\mathscr{B}}$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_{\mathscr{B}}$ for every $\varphi \in \mathscr{B}$,

(iii) $|u_t|_{\mathscr{B}} \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}|u(s)| + M(t-\sigma)|u_\sigma|_{\mathscr{B}}$.

(A$_2$) For the function $u(.)$ in **(A$_1$)**, $t \mapsto u_t$ is a $\mathscr{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

(B) The space $\mathscr{B}$ is a Banach space.

We assume that

(C$_1$) if $(\varphi_n)_{n\geq 0}$ is a sequence in $\mathscr{B}$ such that $\varphi_n \to 0$ in $\mathscr{B}$ as $n \to +\infty$, then $(\varphi_n(\theta))_{n\geq 0}$ converges to 0 in X.

Let $C((-\infty, 0], X)$ be the space of continuous functions from $(-\infty, 0]$ into X. We suppose the following assumptions hold:

(C$_2$) $\mathscr{B} \subset C((-\infty, 0], X)$,

(C$_3$) there exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$ and $x \in X$ we have $e^{\lambda \cdot}x \in \mathscr{B}$ and

$$K_0 = \sup_{\substack{\operatorname{Re}\lambda > \lambda_0,\, x \in X, \\ x \neq 0}} \frac{|e^{\lambda \cdot}x|_{\mathscr{B}}}{|x|} < \infty,$$

where $(e^{\lambda \cdot}x)(\theta) = e^{\lambda\theta}x$ for $\theta \in (-\infty, 0]$ and $x \in X$.

To equation (1.1) we associate the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u_t = Au_t + L(u_t) + f(t) \quad \text{for } t \geq 0, \\ u_0 = \varphi \in \mathscr{B}, \end{cases} \tag{2.1}$$

where $f\colon \mathbb{R}^+ \to X$ is a continuous function. Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which is defined by

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\}, \\ A_0 x = Ax \text{ for } x \in D(A_0). \end{cases}$$

We make the following assumption.

(H$_0$) A satisfies the Hille–Yosida condition.

Lemma 2.1 ([2]) *A_0 generates a strongly continuous semi-group $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

The phase space $\mathscr{B}_A$ of equation (2.1) is defined by $\mathscr{B}_A = \left\{\varphi \in \mathscr{B} : \varphi(0) \in \overline{D(A)}\right\}$. For each $t \geq 0$ we define the linear operator $\mathcal{U}(t)$ on $\mathscr{B}_A$ by $\mathcal{U}(t)\varphi = v_t(., \varphi)$, where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}v_t = Av_t + L(v_t) \quad \text{for } t \geq 0, \\ v_0 = \varphi \in \mathscr{B}. \end{cases}$$

Theorem 2.2 ([3]) *Assume that $\mathscr{B}$ satisfies* $(\mathbf{A}_1)$*,* $(\mathbf{A}_2)$*,* $(\mathbf{B})$*,* $(\mathbf{C}_1)$ *and* $(\mathbf{C}_2)$*. Then, $\mathcal{A}_\mathcal{U}$ defined on $\mathscr{B}_A$ by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) = \big\{\varphi \in C^1\big((-\infty,0];X\big) \cap \mathscr{B}_A : \varphi' \in \mathscr{B}_A \textit{ and } \varphi'(0) = A\varphi(0) + L(\varphi)\big\}, \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \textit{ for } \varphi \in D(\mathcal{A}_\mathcal{U}) \end{cases}$$

is the infinitesimal generator of the semi-group $(\mathcal{U}(t))_{t\geq 0}$ on $\mathscr{B}_A$.

Let $\langle X_0\rangle$ be the space defined by $\langle X_0\rangle = \{X_0x : x \in X\}$, where the function X_0x is defined by

$$(X_0x)(\theta) = \begin{cases} 0, & \text{if } \theta \in (-\infty,0), \\ x, & \text{if } \theta = 0. \end{cases}$$

The space $\mathscr{B}_A \oplus \langle X_0\rangle$ equipped with the norm $|\phi + X_0c|_\mathscr{B} = |\phi|_\mathscr{B} + |c|$ for $(\phi,c) \in \mathscr{B}_A \times X$ is a Banach space. Consider the extension $\widetilde{\mathcal{A}_\mathcal{U}}$ of $\mathcal{A}_\mathcal{U}$ defined on $\mathscr{B}_A \oplus \langle X_0\rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \big\{\varphi \in C^1\big((-\infty,0];X\big) : \varphi \in D(A) \text{ and } \varphi' \in \overline{D(A)}\big\}, \\ \widetilde{\mathcal{A}_\mathcal{U}}\varphi = \varphi' + X_0(A\varphi + L(\varphi) - \varphi'). \end{cases}$$

Lemma 2.3 ([3]) *Assume that $\mathscr{B}$ satisfies* $(\mathbf{A}_1)$*,* $(\mathbf{A}_2)$*,* $(\mathbf{B})$*,* $(\mathbf{C}_1)$*,* $(\mathbf{C}_2)$ *and* $(\mathbf{C}_3)$*. Then, $\widetilde{\mathcal{A}_\mathcal{U}}$ satisfies the Hille–Yosida condition on $\mathscr{B}_A \oplus \langle X_0\rangle$,* i.e.*, there exist $\widetilde{M} \geq 0,\ \widetilde{\omega} \in \mathbb{R}$ such that $(\widetilde{\omega},+\infty) \subset \rho(\widetilde{\mathcal{A}_\mathcal{U}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \quad \textit{for } n \in \mathbb{N} \textit{ and } \lambda > \widetilde{\omega}.$$

Now, we can state the variation of constants formula associated to equation (2.1). Let C_{00} be the space of X-valued continuous function on $(-\infty,0]$ with compact support. We assume that

- **(D)** if $(\varphi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathscr{B}$ and converges compactly to φ on $(-\infty,0]$, then $\varphi \in \mathscr{B}$ and $|\varphi_n - \varphi| \to 0$.

Theorem 2.4 ([3]) *Assume that* $(\mathbf{C}_1)$*,* $(\mathbf{C}_2)$ *and* $(\mathbf{C}_3)$ *hold. Then, the integral solution x of equation* (2.1) *is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0f(s))\,\mathrm{d}s \ \ \textit{for} \ \ t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_\mathcal{U})^{-1}$.

Definition 2.5 *We say that a semi-group $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if $\sigma(\mathcal{A}_\mathcal{U}) \cap i\mathbb{R} = \emptyset$.*

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semi-group defined on the subspace $\mathscr{B}_0 = \{\varphi \in \mathscr{B} : \varphi(0) = 0\}$ of $\mathscr{B}$ by

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta), & \text{if } t+\theta \leq 0, \\ 0, & \text{if } t+\theta \geq 0. \end{cases}$$

Definition 2.6 *Assume that $\mathscr{B}$ satisfies axioms* (**B**) *and* (***D***)*. The space $\mathscr{B}$ is said to be a fading memory space, if for all $\varphi \in \mathscr{B}_0$ we have $|S_0(t)\varphi| \to 0$ as $t \to +\infty$. Moreover, $\mathscr{B}$ is said to be a uniform fading memory space, if $|S_0(t)| \to 0$ as $t \to +\infty$ with respect to the operator norm.*

Lemma 2.7 ([11]) *If $\mathscr{B}$ is a uniform fading memory space, then we can choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$.*

Proposition 2.8 ([11]) *If the phase space $\mathscr{B}$ is a fading memory space, then the space $BC((-\infty, 0], X)$ of bounded continuous X-valued functions on $(-\infty, 0]$ endowed with the uniform norm topology is continuously embedded in $\mathscr{B}$. In particular, $\mathscr{B}$ satisfies* (**C**$_3$) *for $\lambda_0 > 0$.*

For the sequel, we make the following assumption.

(**H**$_1$) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

(**H**$_2$) $\mathscr{B}$ is a uniform fading memory space.

We get the following result on the spectral decomposition of the phase space $\mathscr{B}_A$.

Theorem 2.9 ([3]) *Assume that $\mathscr{B}$ satisfies* (**A**$_1$), (**A**$_2$), (**B**), (**C**$_1$) *and that* (**H**$_0$), (**H**$_1$), (**H**$_2$) *hold. Then, the space $\mathscr{B}_A$ is decomposed as a direct sum $\mathscr{B}_A = S \oplus U$ of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semi-group on U is a group and there exist positive constants $\overline{M}$ and ω such that*

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \quad \text{for } t \geq 0 \text{ and } \varphi \in S,$$
$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{\omega t}|\varphi| \quad \text{for } t \leq 0 \text{ and } \varphi \in U.$$

The spaces S and U are called, respectively, the stable and unstable space. By Π^s and Π^u we denote, respectively, the projection operator on S and U.

3 Almost automorphic functions and (μ, ν) ergodic functions

In this section, we recall some properties about (μ, ν)-pseudo almost automorphic functions. Let $BC(\mathbb{R}; X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm. We denote by $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$ for all $a, b \in \mathbb{R}$ ($a \leq b$).

Definition 3.1 *A bounded continuous function $\phi\colon \mathbb{R} \to X$ is called almost automorphic if for each real sequence (s_m) there exists a subsequence (s_n) such that*

$$g(t) = \lim_{n \to +\infty} \phi(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to +\infty} g(t - s_n) = \phi(t)$$

for each $t \in \mathbb{R}$. We denote by $AA(\mathbb{R}; X)$ the space of all such functions.

Proposition 3.2 ([15]) *$AA(\mathbb{R};X)$ equipped with the supremum norm is a Banach space.*

Definition 3.3 *Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi\colon \mathbb{R}\times X_1 \to X_2$ is called almost automorphic in $t\in\mathbb{R}$ uniformly for each x in X_1 if for every real sequence (s_m) there exists a subsequence (s_n) such that*

$$g(t,x) = \lim_{n\to+\infty} \phi(t+s_n,x) \ \text{ in } \ X_2$$

is well defined for each $t\in\mathbb{R}$ and each $x\in X_1$, and

$$\lim_{n\to+\infty} g(t-s_n,x) = \phi(t,x) \ \text{ in } \ X_2$$

for each $t\in\mathbb{R}$ and for every $x\in X_1$. We denote by $AA(\mathbb{R}\times X_1;X_2)$ the space of all such functions.

Definition 3.4 *A bounded continuous function $\phi\colon\mathbb{R}\to X$ is called compact almost automorphic if for each real sequence (s_m) there exists a subsequence (s_n) such that*

$$g(t) = \lim_{n\to+\infty}\phi(t+s_n) \quad \text{and} \quad \lim_{n\to+\infty} g(t-s_n) = \phi(t)$$

uniformly on compact subsets of $\mathbb{R}$. We denote by $AA_c(\mathbb{R};X)$ the space of all such functions.

It is well-known that $AA_c(\mathbb{R};X)$ is a closed subset of $(BC(\mathbb{R};X),|.|_\infty)$, and so we immediately get the following result.

Lemma 3.5 ([15]) *$AA_c(\mathbb{R};X)$ equipped with the supremum norm is a Banach space.*

Definition 3.6 *Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi\colon\mathbb{R}\times X_1\to X_2$ is called compact almost automorphic in $t\in\mathbb{R}$ if for every real sequence (s_m) there exists a subsequence (s_n) such that*

$$g(t,x) = \lim_{n\to+\infty}\phi(t+s_n,x) \quad \text{and} \quad \lim_{n\to+\infty} g(t-s_n,x) = \phi(t,x) \quad \text{in } X_2,$$

where the limits are uniform on compact subsets of $\mathbb{R}$ for each $x\in X_1$. We denote by $AA_c(\mathbb{R}\times X_1;X_2)$ the space of all such functions.

In the sequel, we recall some preliminary results concerning the (μ,ν)-pseudo almost automorphic functions. The symbol $\mathcal{E}(\mathbb{R};X,\mu,\nu)$ stands for the space of functions

$$\mathcal{E}(\mathbb{R};X,\mu,\nu) = \left\{ u\in BC(\mathbb{R};X) : \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|u(t)|\,\mathrm{d}\mu(t) = 0\right\}.$$

To study delayed differential equations for which the history belong to $\mathscr{B}$, we need to introduce the space

$$\begin{aligned}&\mathcal{E}(\mathbb{R};X,\mu,\nu,\infty)\\ &\quad = \left\{ u\in BC(\mathbb{R};X) : \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\left(\sup_{\theta\in(-\infty,t]}|u(\theta)|\right)\mathrm{d}\mu(t) = 0\right\}.\end{aligned}$$

In addition to the above-mentioned spaces, we consider the following spaces

$$\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu) = \Big\{u\in BC(\mathbb{R}\times X_1;X_2) : \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|u(t,x)|_{X_2}\,\mathrm{d}\mu(t)=0\Big\},$$

$$\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,\infty) = \Big\{u\in BC(\mathbb{R}\times X_1;X_2) : \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in(-\infty,t]}|u(\theta,x)|_{X_2}\Big)\,\mathrm{d}\mu(t)=0\Big\},$$

where in both cases the limit (as $\tau\to+\infty$) is uniform in compact subset of X_1. In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R};X,\mu,\nu,\infty)$ and $\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,\infty)$ are continuously embedded in $\mathcal{E}(\mathbb{R};X,\mu,\nu)$ and $\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu)$, respectively.

Definition 3.7 *We say that a continuous function f is ρ-weighted pseudo almost automorphic if $f=g+\phi$, where g is almost automorphic and ϕ is ergodic with respect to some weighted function ρ in the sense that*

$$\lim_{\tau\to+\infty}\frac{1}{m(\rho,\tau)}\int_{-\tau}^{+\tau}|\phi(t)|\rho(t)\,\mathrm{d}t=0,$$

where $m(\rho,\tau)=\int_{-\tau}^{+\tau}\rho(t)\,\mathrm{d}t$ and ρ is assumed to be positive and locally integrable.

On the other hand, one can observe that a ρ-weighted pseudo almost automorphic functions is μ-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is ρ:

$$\mathrm{d}\mu(t)=\rho(t)\mathrm{d}t.$$

Example 3.8 ([4]) *Let ρ be a non-negative $\mathcal{N}$-measurable function. Denote by μ the positive measure defined by*

$$\mu(A)=\int_A\rho(t)\,\mathrm{d}t \;\; \textit{for} \;\; A\in\mathcal{N}, \tag{3.1}$$

where $\mathrm{d}t$ denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3.1) is called the Radon–Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

4 (μ,ν)-Stepanov-like pseudo almost automorphic functions

Definition 4.1 *The Bochner transform $f^b(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by $f^b(t,s)=f(t+s)$.*

Remark 4.2 *If $f=h+\varphi$, then $f^b=h^b+\varphi^b$. Moreover, $(\lambda f)^b=\lambda f^b$ for each scalar λ.*

Definition 4.3 *The Bochner transform $F^b(t,s,u)$, $t\in\mathbb{R}$, $s\in[0,1]$, $u\in X$, of a function $F(t,u)$ on $\mathbb{R}\times X$, with values in X, is defined by*

$$F^b(t,s,u)=F(t+s,u)\;\textit{for each}\; u\in X.$$

Definition 4.4 *Let $p \in [1, +\infty)$. The space $BS^p(\mathbb{R}; X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in X such that $f^b \in L^\infty(\mathbb{R}; L^p([0,1], X))$. This is a Banach space with the norm*

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} |f(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}}.$$

Definition 4.5 *A bounded continuous function $\phi \colon \mathbb{R} \to X$ is called (μ, ν)-pseudo almost automorphic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}; X)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}; X, \mu, \nu)$. We denote by $PAA(\mathbb{R}; X, \mu, \nu)$ the space of all such functions.*

Definition 4.6 *Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi \colon \mathbb{R} \times X_1 \to X_2$ is called uniformly (μ, ν)-pseudo almost automorphic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathscr{E}(\mathbb{R} \times X_1; X_2, \mu, \nu)$. We denote by $PAA(\mathbb{R} \times X_1; X_2, \mu, \nu)$ the space of all such functions.*

We now introduce some new spaces used in the sequel.

Definition 4.7 *A function $f \in BS^p(\mathbb{R}; X)$ is called (μ, ν)-S^p-pseudo-almost automorphic* (*or Stepanov-like pseudo-almost automorphic*) *if it can be expressed as $f = h + \varphi$, where $h^b \in AA(\mathbb{R}; L^p((0,1); X))$ and $\varphi^b \in \mathscr{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu)$. The collection of all such functions will be denoted by $PAAS^p(\mathbb{R}; X, \mu, \nu)$.*

In other words, a function $f \in L^p(\mathbb{R}; X)$ is said to be S^p-pseudo-almost automorphic if its Bochner transform $f^b \colon \mathbb{R} \to L^p((0,1); X)$ is pseudo-almost automorphic in the sense that there exist two functions $h, \varphi \colon \mathbb{R} \to X$ such that $f = h + \varphi$, where $h^b \in AA(\mathbb{R}; L^p((0,1); X))$ and $\varphi^b \in \mathscr{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu)$, i.e., according to [13, Definition 2.5, p. 2660] for each real sequence (s_m) there exists a subsequence (s_n) and a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}; X)$ such that

$$\lim_{n\to+\infty} \left(\int_t^{t+1} |g(s) - h(s+s_n)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to+\infty} \left(\int_t^{t+1} |g(s-s_n) - h(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} = 0$$

pointwise on $\mathbb{R}$ and

$$\lim_{\tau\to+\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{\tau} \left(\int_t^{t+1} |\varphi(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0 \ \text{ for } t \in \mathbb{R}.$$

Definition 4.8 *A function $f \in BS^p(\mathbb{R}; X)$ is called (μ, ν)-S^p-pseudo-almost automorphic of infinite class* (*or Stepanov-like pseudo-almost automorphic of infinite class*) *if it can be expressed as $f = h + \varphi$, where $h^b \in AA(\mathbb{R}; L^p((0,1); X))$ and $\varphi^b \in \mathscr{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$,* i.e.,

$$\lim_{\tau\to+\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{\tau} \sup_{\theta\in(-\infty,t]} \left(\int_\theta^{\theta+1} |\varphi(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0.$$

The collection of all such functions will be denoted by $PAAS^p(\mathbb{R}; X, \mu, \nu, \infty)$.

Definition 4.9 *A function $f\colon \mathbb{R}\times X_1 \to X_2$, $(t,x)\mapsto f(t,x)$, with $f(.,x)\in L^p(\mathbb{R};X_2)$ is called (μ,ν)-S^p-pseudo-almost automorphic of infinite class (or Stepanov-like pseudo-almost automorphic of infinite class) if it can be expressed as $f = h+\varphi$, where $h^b \in AA(\mathbb{R}\times X_1; L^p((0,1);X_2))$ and $\varphi^b \in \mathscr{E}[(\mathbb{R}\times X_1; L^p((0,1);X_2),\mu,\nu,\infty]$,* i.e.,

$$\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{\tau}\sup_{\theta\in(-\infty,t]}\left(\int_{\theta}^{\theta+1}|\varphi(s,x)|_{X_2}^{p}\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)=0.$$

The collection of all such functions will be denoted by $PAAS^p[\mathbb{R}\times X_1; L^p((0,1);X_2),\mu,\nu,\infty]$.

Definition 4.10 *A bounded continuous function $\phi:\mathbb{R}\to X$ is called (μ,ν)-S^p-pseudo compact almost automorphic of infinite class if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA_c(\mathbb{R};L^p((0,1);X))$ and $\phi_2\in\mathscr{E}(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$. We denote by $PAA_cS^p(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$ the space of all such functions.*

Definition 4.11 *Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi\colon\mathbb{R}\times X_1\to X_2$ is called uniformly (μ,ν)-S^p-pseudo almost automorphic of infinite class (respectively, uniformly pseudo compact almost automorphic of infinite class) if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA(\mathbb{R}\times X_1, L^p((0,1);X_2))$ and $\phi_2\in\mathscr{E}(\mathbb{R}\times X_1;L^p((0,1);X_2),\mu,\nu,\infty)$ (respectively, if $\phi=\phi_1+\phi_2$, where $\phi_1\in AA_c(\mathbb{R}\times X_1,L^p((0,1);X_2))$ and $\phi_2\in\mathscr{E}(\mathbb{R}\times X_1;L^p((0,1);X_2),\mu,\nu,\infty)$). We denote by $PAAS^p(\mathbb{R}\times X_1;L^p((0,1);X_2),\mu,\nu,\infty)$ (respectively, $PAA_cS^p(\mathbb{R}\times X_1;L^p((0,1);X_2),\mu,\nu,\infty)$) the space of all such functions.*

5 Properties of μ-Stepanov-like pseudo almost automorphic functions of infinite class

For $\mu,\nu\in\mathcal{M}$, we formulate the following hypothesis.

(H$_3$) Let $\mu,\nu\in\mathcal{M}$ be such that

$$\limsup_{\tau\to+\infty}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}=\alpha<\infty.$$

Proposition 5.1 *$PAAS^p(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$ is a closed subspace of $BS^p(\mathbb{R};X)$.*

Proof. Let $(x_n)_n$ be a sequence in $PAAS^p(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$ such that $\lim_{n\to\infty}x_n = x$ in $BS^p(\mathbb{R};X)$. For each n let $x_n=y_n+z_n$ with $y_n^b\in AA(\mathbb{R};L^p((0,1);X))$ and $z_n^b\in\mathscr{E}(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$. Then, $(y_n)_n$ converges to some $y\in BS^p(\mathbb{R};X)$ and $(z_n)_n$ also converges to some $z\in BS^p(\mathbb{R};X)$. In view of [13, Theorem 2.3], we infer that $y\in AA(\mathbb{R};L^p((0,1);X))$. It remains to show that $z\in\mathscr{E}(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$. By the Minkowski inequality, we have

$$
\begin{aligned}
&\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\bigg(\int_{\theta}^{\theta+1}|z(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad=\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\bigg(\int_{\theta}^{\theta+1}|z(s)-z_n(s)+z_n(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad\le\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\bigg[\bigg(\int_{\theta}^{\theta+1}|z(s)-z_n(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}+\bigg(\int_{\theta}^{\theta+1}|z_n(s)|^p)\,\mathrm{d}s\bigg)^{\frac{1}{p}}\bigg]\mathrm{d}\mu(t)\\
&\quad\le\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{t\in\mathbb{R}}\bigg(\int_{t}^{t+1}|z(s)-z_n(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\qquad+\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\bigg(\int_{\theta}^{\theta+1}|z_n(s)|^p)\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad\le\|z-z_n\|_{S^p}\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\bigg(\int_{\theta}^{\theta+1}|z_n(s)|^p)\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t).
\end{aligned}
$$

So, $z\in\mathcal{E}(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$, and hence $x\in PAAS^p(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$. □

Proposition 5.2 *The space $PAAS^p(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$ endowed with the $\|\,.\,\|_{S^p}$ norm is a Banach space.*

The next result is a characterization of (μ,ν)-ergodic functions of infinite class.

Theorem 5.3 *Assume that* ($\mathbf{H}_3$) *holds, let $\mu,\nu\in\mathcal{M}$ and let I be a bounded interval* (*we do not exclude the case $I=\emptyset$*)*. Assume that $f\in BS^p(\mathbb{R};X)$. Then, the following assertions are equivalent*:

(i) $f\in\mathcal{E}(\mathbb{R};L^p((0,1);X),\mu,\nu,\infty)$,

(ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in(-\infty,t]}\bigg(\int_{\theta}^{\theta+1}|f(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t)=0,$

(iii) *for any $\varepsilon>0$ we have*

$$\lim_{\tau\to+\infty}\frac{\mu\Bigg(\Bigg\{t\in[-\tau,\tau]\setminus I:\ \sup\limits_{\theta\in(-\infty,t]}\bigg(\displaystyle\int_{\theta}^{\theta+1}|f(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}>\varepsilon\Bigg\}\Bigg)}{\nu([-\tau,\tau]\setminus I)}=0.$$

Proof. The proof runs along the same lines as the proof of Theorem 2.13 in [4].

(i) $\Leftrightarrow$ (ii) Denote by $A=\nu(I)$,

$$B=\int_I\sup_{\theta\in(-\infty,t]}\bigg(\int_{\theta}^{\theta+1}|f(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\mathrm{d}\mu(t).$$

We have A and $B\in\mathbb{R}$, since the interval I is bounded and the function f is bounded and continuous. For $\tau>0$ such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$ we have

$$
\begin{aligned}
&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad=\frac{1}{\nu([-\tau,\tau])-A}\left[\int_{[-\tau,\tau]}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)-B\right]\\
&\quad=\frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\left[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)-\frac{B}{\nu([-\tau,\tau])}\right].
\end{aligned}
$$

From above equalities and the fact that $\nu(\mathbb{R})=+\infty$, we deduce that (ii) is equivalent to

$$
\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)=0,
$$

that is (i).

(iii) $\Rightarrow$ (ii) Denote by A_τ^ε and B_τ^ε the following sets

$$
A_\tau^\varepsilon=\left\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}>\varepsilon\right\}
$$

and

$$
B_\tau^\varepsilon=\left\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\leq\varepsilon\right\}.
$$

Assume that (iii) holds, that is,

$$
\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}=0. \tag{5.1}
$$

From the following equality

$$
\begin{aligned}
&\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad=\int_{A_\tau^\varepsilon}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)+\int_{B_\tau^\varepsilon}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}d\mu(t)\\
&\qquad\leq\|f\|_{S^p}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}
\end{aligned}
$$

for τ sufficiently large. By ($\mathbf{H}_3$) it follows that for all $\varepsilon>0$ we have

$$
\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\leq\alpha\varepsilon.
$$

Consequently, (ii) holds.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. From the following inequalities

$$\int_{[-\tau,\tau]\setminus I} \sup_{\theta\in(-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \geq \int_{A^\varepsilon_\tau} \sup_{\theta\in(-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t),$$

$$\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta\in(-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \geq \varepsilon \frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)},$$

$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta\in(-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \geq \frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)},$$

which hold for τ sufficiently large, we obtain equation (5.1). So, (iii) holds. $\square$

For $\mu \in \mathcal{M}$, we formulate the following hypotheses.

(H$_4$) For all a, b and $c \in \mathbb{R}$ such that $0 \leq a < b \leq c$, there exist δ_0 and $\alpha_0 > 0$ such that $|\delta| \geq \delta_0 \Rightarrow \mu(a+\delta, b+\delta) \geq \alpha_0\mu(\delta, c+\delta)$.

(H$_5$) For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\}) \leq \beta\mu(A), \text{ when } A \in \mathcal{N} \text{ satisfies } A \cap I = \emptyset.$$

We have the following result due to [4].

Lemma 5.4 ([4]) *Hypothesis* (**H**$_5$) *implies* (**H**$_4$).

Proposition 5.5 ([7]) *Let $\mu, \nu \in \mathcal{M}$ satisfy* (**H**$_4$) *and let $f \in PAA(\mathbb{R}; X, \mu, \nu)$ be such that $f = g + h$, where $g \in AA(\mathbb{R}; X)$ and $h \in \mathscr{E}(\mathbb{R}; X, \mu, \nu)$. Then, $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$ (the closure of the range of f).*

Corollary 5.6 ([7]) *Assume that* (**H**$_4$) *holds. Then, the decomposition of a (μ, ν)-pseudo almost automorphic functions in the form $f = g + \phi$, where $g \in AA(\mathbb{R}; X)$ and $\phi \in \mathscr{E}(\mathbb{R}; X, \mu, \nu)$, is unique.*

Definition 5.7 *Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoting this as $\mu_1 \sim \mu_2$, if there exist constants $\alpha > 0$ and $\beta > 0$ and a bounded interval I (we allow also the situation when $I = \emptyset$) such that $\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A)$, when $A \in \mathcal{N}$ satisfies $A \cap I = \emptyset$.*

From [4] we know that $\sim$ is a binary equivalence relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by $\mathrm{cl}(\mu) = \{\varpi \in \mathcal{M} : \mu \sim \varpi\}$.

Theorem 5.8 *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then the spaces $PAAS^p(\mathbb{R}; L^p((0,1); X), \mu_1, \nu_1, \infty)$ and $PAAS^p(\mathbb{R}; L^p((0,1); X), \mu_2, \nu_2, \infty)$ coincide, that is, $PAAS^p(\mathbb{R}; L^p((0,1); X), \mu_1, \nu_1, \infty) = PAAS^p(\mathbb{R}; L^p((0,1); X), \mu_2, \nu_2, \infty)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (we allow also the situation when $I = \emptyset$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{N}$ satisfying $A \cap I = \emptyset$, *i.e.*,

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field, we for τ sufficiently large we obtain

$$\begin{aligned}
&\frac{\alpha_1\mu_1\left(\left\{t \in [-\tau,\tau] \setminus I : \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} > \varepsilon\right\}\right)}{\beta_2\nu_1([-\tau,\tau] \setminus I)} \\
&\quad \leq \frac{\mu_2\left(\left\{t \in [-\tau,\tau] \setminus I : \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} > \varepsilon\right\}\right)}{\nu_2([-\tau,\tau] \setminus I)} \\
&\quad \leq \frac{\beta_1\mu_1\left(\left\{t \in [-\tau,\tau] \setminus I : \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} > \varepsilon\right\}\right)}{\alpha_2\nu_1([-\tau,\tau] \setminus I)}.
\end{aligned}$$

By Theorem 5.3, we deduce that $\mathscr{E}(\mathbb{R}; X, \mu_1, \nu_1, \infty) = \mathscr{E}(\mathbb{R}; X, \mu_2, \nu_2, \infty)$. According to Definition 4.8, we deduce that $PAAS^p(\mathbb{R}; X, \mu_1, \nu_1, \infty) = PAAS^p(\mathbb{R}; X, \mu_2, \nu_2, \infty)$. □

For $\mu, \nu \in \mathcal{M}$ we set $\mathrm{cl}(\mu,\nu) = \{(\overline{\omega}_1, \overline{\omega}_2) \in \mathcal{M}^2 : \mu \sim \overline{\omega}_1 \text{ and } \nu \sim \overline{\omega}_2\}$.

Proposition 5.9 ([7]) *Let $\mu, \nu \in \mathcal{M}$ satisfy* ($\mathbf{H}_5$). *Then, $PAA(\mathbb{R}; X, \mu, \nu)$ is translation invariant, that is, $f \in PAA(\mathbb{R}; X, \mu, \nu)$ implies $f_\alpha \in PAP(\mathbb{R}; X, \mu, \nu)$ for all $\alpha \in \mathbb{R}$.*

Corollary 5.10 *Let $\mu \in \mathcal{M}$ satisfy* ($\mathbf{H}_4$). *Then, $PAAS^p(\mathbb{R}; X, \mu, \nu, \infty)$ is translation invariant, that is, $f \in PAAS^p(\mathbb{R}; X, \mu, \nu, \infty)$ implies $f_\alpha \in PAAS^p(\mathbb{R}; X, \mu, \nu, \infty)$ for all $\alpha \in \mathbb{R}$.*

Proof. It suffices to prove that $\mathcal{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$ is translation invariant. Let $f \in \mathcal{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$ and

$$F^t(\theta) = \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}}.$$

Then, $F^t \in \mathcal{E}(\mathbb{R}; \mathbb{R}, \mu, \nu)$. But since $\mathcal{E}(\mathbb{R}; \mathbb{R}, \mu, \nu)$ is translation invariant, it follows that

$$\begin{aligned}
&\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} F^t(\theta+\alpha) \,\mathrm{d}\mu(t) \\
&\quad = \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s+\alpha)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0.
\end{aligned}$$

This implies that $f(.+\alpha) \in PAAS^p(\mathbb{R}; X, \mu, \nu, \infty)$ and ends the proof. □

We have the following result.

Theorem 5.11 *Let $u \in PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. Then, the function $t \mapsto u_t$ belongs to $PAA_cS^p(\mathscr{B}; L^p((0,1); X)), \mu, \nu, \infty)$.*

Proof. Assume that $u = g + h$, where $g^p \in AA(\mathbb{R}; L^p((0,1); X))$ and $h^p \in \mathcal{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. We can see that $u_t = g_t + h_t$. We want to show that $g_t^p \in AA_c(\mathbb{R}; L^p((0,1); X))$ and $\varphi_t^p \in \mathcal{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$.

Firstly, for a given sequence $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v \in BS^p(\mathbb{R}; X)$ such that $g(s + s_n) \to v(s)$ uniformly on compact subsets of $\mathbb{R}$. Let Ω be an arbitrary compact subset of $\mathbb{R}$ and let $L > 0$ be such that $\Omega \subset [-L, L]$. For $\varepsilon > 0$ fix $N_{\varepsilon,L} \in \mathbb{N}$ such that $\|g(s + s_n) - v(s)\|_{S^p} \leq \varepsilon$ for $s \in [-L, L]$, whenever $n \geq N_{\varepsilon,L}$. For $t \in \Omega$ and $n \geq N_{\varepsilon,L}$ we have

$$\|g_{t+s_n} - v_t\|_{S^p} \leq \sup_{\theta\in[-L,L]} \|g(\theta + s_n) - v(\theta)\|_{S^p} \leq \varepsilon.$$

In view of above, g_{t+s_n} converges to v_t uniformly on Ω. Similarly, one can prove that v_{t-s_n} converges to g_t uniformly on Ω. Thus, the function $s \mapsto g_s$ belongs to $AA_c(L^p((0,1); \mathscr{B}))$.

On the other hand, let $\xi \in (-\infty, 0]$. Then, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]} \left(\int_{\theta}^{\theta+1} \big(|h_s(\xi)|\big)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad = \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]} \left(\int_{\theta}^{\theta+1} \big(|h(s+\xi)|\big)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad \leq \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]} \left(\int_{\theta+\xi}^{\theta+1+\xi} \big(|h(s)|\big)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t).
\end{aligned}$$

Since $\theta \in (-\infty, t]$ and $\xi \in (-\infty, 0]$, we obtain $\theta' = (\theta + \xi) \in (-\infty, t]$. It follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]} \left(\int_{\theta}^{\theta+1} \big(|h_s(\xi)|\big)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad \leq \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]} \left(\int_{\theta}^{\theta+1} \big(|h(s)|\big)^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t),
\end{aligned}$$

which shows that u_t belongs to $PAAS^p(\mathscr{B}; L^p((0,1); X)), \mu, \nu, \infty)$. Thus, we obtain the desired result. $\square$

6 Weighted Stepanov-like pseudo almost automorphic solutions of infinite class

Theorem 6.1 *Assume that $\mathscr{B}$ satisfies* ($\mathbf{A}_1$), ($\mathbf{A}_2$), (**B**), ($\mathbf{C}_1$), ($\mathbf{C}_2$) *and* ($\mathbf{H}_1$) *and that the semi-group $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If $f \in BS^p(\mathbb{R};X)$, then there exists a unique bounded solution u of equation* (1.1) *on $\mathbb{R}$, given by*

$$\begin{aligned} u_t = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))\,\mathrm{d}s \\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))\,\mathrm{d}s \ \ for\ \ t\in\mathbb{R}, \end{aligned} \tag{6.1}$$

where Π^s and Π^u are the projections of $\mathscr{B}_A$ onto the stable and unstable subspaces, respectively.

Proof. Let us first prove that the limits in equation (6.1) exist. For $t\in\mathbb{R}$ we have

$$\begin{aligned} \int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|\,\mathrm{d}s &\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}|f(s)|\,\mathrm{d}s \\ &\leq \overline{M}\widetilde{M}|\Pi^s|\sum_{n=1}^{\infty}\left(\int_{t-n}^{t-n+1}e^{-\omega(t-s)}|f(s)|\,\mathrm{d}s\right). \end{aligned}$$

Let q be such that $\frac{1}{q}+\frac{1}{p}=1$. Using the Hölder inequality, we obtain

$$\begin{aligned} &\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|\,\mathrm{d}s \\ &\quad\leq \overline{M}\widetilde{M}|\Pi^s|\sum_{n=1}^{\infty}\left[\left(\int_{t-n}^{t-n+1}e^{-q\omega(t-s)}\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right] \\ &\quad\leq \frac{\overline{M}\widetilde{M}|\Pi^s|}{\sqrt[q]{q\omega}}\sum_{n=1}^{\infty}\left[(e^{-q\omega(n-1)}-e^{-q\omega n})^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right] \\ &\quad\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}-1)^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{-\omega n} \\ &\quad\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{-\omega n}. \end{aligned}$$

Since the series $\sum_{n=1}^{+\infty}e^{-q\omega n}$ is convergent, it follows that

$$\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|\,\mathrm{d}s<\gamma \tag{6.2}$$

with

$$\gamma=\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{-\omega n}.$$

Set $F(n,s,t)=\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))$ for $n\in\mathbb{N}$ and $s\leq t$. For n sufficiently large and $\sigma\leq t$ we have

$$
\begin{aligned}
&\left|\int_{-\infty}^{\sigma} F(n,s,t)\,\mathrm{d}s\right| \\
&\quad \le \overline{M}\widetilde{M}|\Pi^s|\sum_{n=1}^{\infty}\left[\left(\int_{\sigma-n}^{\sigma-n+1} e^{-q\omega(t-s)}\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_{\sigma-n}^{\sigma-n+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right] \\
&\quad \le \frac{\overline{M}\widetilde{M}|\Pi^s|}{\sqrt[q]{q\omega}}\sum_{n=1}^{\infty}\left[(e^{-q\omega(t-\sigma+n-1)}-e^{-q\omega(t-\sigma+n)})^{\frac{1}{q}}\left(\int_{\sigma-n}^{\sigma-n+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right] \\
&\quad \le \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\times e^{-\omega(t-\sigma)}\sum_{n=1}^{\infty}e^{-\omega n} \\
&\quad \le \gamma e^{-\omega(t-\sigma)}.
\end{aligned}
$$

It follows that for n and m sufficiently large and $\sigma \le t$ we have

$$
\begin{aligned}
&\left|\int_{-\infty}^{t} F(n,s,t)\,\mathrm{d}s-\int_{-\infty}^{t} F(m,s,t)\,\mathrm{d}s\right| \\
&\quad \le \left|\int_{-\infty}^{\sigma} F(n,s,t)\,\mathrm{d}s\right|+\left|\int_{-\infty}^{\sigma} F(m,s,t)\,\mathrm{d}s\right|+\left|\int_{\sigma}^{t} F(n,s,t)\,\mathrm{d}s-\int_{\sigma}^{t} F(m,s,t)\,\mathrm{d}s\right| \\
&\quad \le 2\gamma e^{-\omega(t-\sigma)}+\left|\int_{\sigma}^{t} F(n,s,t)\,\mathrm{d}s-\int_{\sigma}^{t} F(m,s,t)\,\mathrm{d}s\right|.
\end{aligned}
$$

Since $\lim_{n\to+\infty}\int_{\sigma}^{t} F(n,s,t)\,\mathrm{d}s$ exists, we infer that

$$
\limsup_{n,m\to+\infty}\left|\int_{-\infty}^{t} F(n,s,t)\,\mathrm{d}s-\int_{-\infty}^{t} F(m,s,t)\,\mathrm{d}s\right| \le 2\gamma e^{-\omega(t-\sigma)}.
$$

If $\sigma \to -\infty$, then

$$
\limsup_{n,m\to+\infty}\left|\int_{-\infty}^{t} F(n,s,t)\,\mathrm{d}s-\int_{-\infty}^{t} F(m,s,t)\,\mathrm{d}s\right| = 0.
$$

Thus, by the completeness of the phase space $\mathscr{B}$, we deduce that the limit

$$
\lim_{n\to+\infty}\int_{-\infty}^{t} F(n,s,t)\,\mathrm{d}s=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0f(s))\,\mathrm{d}s
$$

exists. In addition, one can see from equation (6.2) that the function

$$
\eta_1 : t \mapsto \lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0f(s))\,\mathrm{d}s
$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$
\eta_2 : t \mapsto \lim_{n\to+\infty}\int_{t}^{\infty}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_nX_0f(s))\,\mathrm{d}s
$$

is well defined and bounded on $\mathbb{R}$. Using the same argument as in the proof of [1, Theorem 5.9], it can be shown that the integral solution u given by the formula

$$
u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))\,\mathrm{d}s+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s))\,\mathrm{d}s
$$

for $t\in\mathbb{R}$ is the only bounded integral solution of equation (1.1) on $\mathbb{R}$. $\square$

Theorem 6.2 *Let $g \in \mathcal{E}(\mathbb{R}; L^p((0,1);X), \mu, \nu, \infty)$ and let Ψ be the mapping defined by*

$$\Psi g(t) = \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\,\mathrm{d}s + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\,\mathrm{d}s$$

for $t \in \mathbb{R}$. Then, $\Psi g \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$.

Proof. For each $n = 1, 2, 3, \dots$ set

$$X_n(t) = \lim_{\lambda\to+\infty} \int_{t-n}^{t-n+1} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\,\mathrm{d}s + \lim_{\lambda\to+\infty} \int_{t+n-1}^{t+n} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\,\mathrm{d}s$$

for $t \in \mathbb{R}$. We have

$$|X_n(t)| \le \overline{M}\widetilde{M}|\Pi^s| \int_{t-n}^{t-n+1} e^{-\omega(t-s)}|g(s)|\,\mathrm{d}s + \overline{M}\widetilde{M}|\Pi^u| \int_{t+n-1}^{t+n} e^{-\omega(t-s)}|g(s)|\,\mathrm{d}s.$$

Set $\Delta = \max\big(\overline{M}\widetilde{M}|\Pi^s|, \overline{M}\widetilde{M}|\Pi^u|\big)$. Let q be such that $\frac{1}{q} + \frac{1}{p} = 1$. Using the Hölder inequality, we obtain

$$\begin{aligned}
|X_n(t)| &\le \Delta\bigg(\int_{t-n}^{t-n+1} e^{-q\omega(t-s)}\,\mathrm{d}s\bigg)^{\frac{1}{q}}\bigg(\int_{t-n}^{t-n+1} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} \\
&\quad + \Delta\bigg(\int_{t+n-1}^{t+n} e^{q\omega(t-s)}\,\mathrm{d}s\bigg)^{\frac{1}{q}}\bigg(\int_{t+n-1}^{t+n} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} \\
&\le \frac{\Delta}{\sqrt[q]{q\omega}}(e^{-q\omega(n-1)} - e^{-q\omega n})^{\frac{1}{q}}\bigg(\int_{t-n}^{t-n+1} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} \\
&\quad + \frac{\Delta}{\sqrt[q]{q\omega}}(e^{q\omega(1-n)} - e^{-q\omega n})^{\frac{1}{q}}\bigg(\int_{t+n-1}^{t+n} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} \\
&\le \frac{\Delta e^{-\omega n}}{\sqrt[q]{q\omega}}(e^{q\omega} - 1)^{\frac{1}{q}}\bigg[\bigg(\int_{t-n}^{t-n+1} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} + \bigg(\int_{t+n-1}^{t+n} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\bigg] \\
&\le \frac{\Delta e^{-\omega n}}{\sqrt[q]{q\omega}}(e^{q\omega} + 1)^{\frac{1}{q}}\bigg[\bigg(\int_{t-n}^{t-n+1} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} + \bigg(\int_{t+n-1}^{t+n} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\bigg].
\end{aligned}$$

Since the series

$$\frac{\Delta}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}} \times \sum_{n=1}^{+\infty} e^{-q\omega n} = \frac{\Delta}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}} \times \frac{e^{-q\omega}}{1-e^{-q\omega}}$$

is convergent, it follows from the Weierstrass M-test that the sequence of functions $\sum_{n=1}^{N} X_n(t)$ is uniformly convergent on $\mathbb{R}$. Since $g \in \mathcal{E}(\mathbb{R}; L^p((0,1);X), \mu, \nu, \infty)$ and

$$|X_n(t)| \le C_q(\Delta,\omega)\bigg[\bigg(\int_{t-n}^{t-n+1} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}} + \bigg(\int_{t+n-1}^{t+n} |g(s)|^p\,\mathrm{d}s\bigg)^{\frac{1}{p}}\bigg],$$

where

$$C_q(\Delta,\omega) = \frac{\Delta}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{p}} \times \sum_{n=1}^{+\infty} e^{-q\omega n},$$

we conclude that $X_n \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$. Thus, $\sum_{n=1}^{N} X_n(t) \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ and its uniform limit belongs $\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ by Theorem 5.1. Observing that $\Psi g(t) = \sum_{n=1}^{+\infty} X_n(t)$, we deduce that $\Psi g \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$. $\square$

Theorem 6.3 *Let $g \in \mathcal{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$. Then, $\Psi g \in \mathcal{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$.*

Proof. For each $n = 1,2,3,\dots$ let X_n be defined as in the proof of Theorem 6.2. We have

$$|X_n(t)|^p \le C_q^p(\Delta,\omega)\left[\left(\int_{t-n}^{t-n+1}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} + \left(\int_{t+n-1}^{t+n}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right]^p.$$

Using the Minkowski inequality, we obtain

$$\begin{aligned}
\left(\int_\theta^{\theta+1}|X_n(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} &\le C_q(\Delta,\omega)\left[\left(\int_\theta^{\theta+1}\left[\left(\int_{s-n}^{s-n+1}|g(\xi)|^p\,\mathrm{d}\xi\right)^{\frac{1}{p}}\right]^p\mathrm{d}s\right)^{\frac{1}{p}}\right.\\
&\qquad\left.+\left(\int_\theta^{\theta+1}\left[\left(\int_{s+n-1}^{s+n}|g(\xi)|^p\,\mathrm{d}\xi\right)^{\frac{1}{p}}\right]^p\mathrm{d}s\right)^{\frac{1}{p}}\right]\\
&\le C_q(\Delta,\omega)\left[\left(\sup_{s\in[\theta,\theta+1]}\int_{s-n}^{s-n+1}|g(\xi)|^p\,\mathrm{d}\xi\right)^{\frac{1}{p}}\right.\\
&\qquad\left.+\left(\sup_{s\in[\theta,\theta+1]}\int_{s+n-1}^{s+n}|g(\xi)|^p\,\mathrm{d}\xi\right)^{\frac{1}{p}}\right]\\
&\le C_q(\Delta,\omega)\left[\left(\int_{\theta-n}^{\theta-n+2}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}+\left(\int_{\theta+n-1}^{\theta+n+1}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right]\\
&\le C_q(\Delta,\omega)\left[\left(\int_{\theta-n}^{\theta-n+1}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}+\left(\int_{\theta-n+1}^{\theta-n+2}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right.\\
&\qquad\left.+\left(\int_{\theta+n-1}^{\theta+n}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}+\left(\int_{\theta+n}^{\theta+n+1}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1}|X_n(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\le C_q(\Delta,\omega)\left[\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_{\theta-n}^{\theta-n+1}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\right.\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_{\theta-n+1}^{\theta-n+2}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_{\theta+n-1}^{\theta+n}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad\left.+\frac{1}{\mu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,t]}\left(\int_{\theta+n}^{\theta+n+1}|g(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\right].
\end{aligned}$$

We conclude that $X_n \in \mathcal{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$. Thus, $\sum_{n=1}^N X_n(t) \in \mathcal{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$ and its uniform limit belongs $\mathcal{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$ by Proposition 5.1. Observing that

$$\Psi g(t) = \sum_{n=1}^{+\infty} X_n(t),$$

we deduce that $\Psi g \in \mathcal{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$. □

Theorem 6.4 *Let* $h \in AA_c(\mathbb{R}; L^p((0,1); X))$. *Then,* $\Psi h \in AA_c(\mathbb{R}; L^p((0,1); X))$.

Proof. For a given sequence $(s_m)_{m\in\mathbb{N}}$ of real numbers fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v \in BS^p(\mathbb{R}; X)$ such that $h(t+s_n)$ converges to $v(t)$ and $v(t-s_n)$ converges to $h(t)$ uniformly on compact subsets of $\mathbb{R}$. From [8], if

$$w(t+s_n)=\lim_{\lambda\to\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s+s_n))\,\mathrm{d}s+\lim_{\lambda\to\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s+s_n))\,\mathrm{d}s$$

for $t \in \mathbb{R}$, $n \in \mathbb{N}$, then $w(t+s_n)$ converges to

$$z(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))\,\mathrm{d}s + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 v(s))\,\mathrm{d}s.$$

It remains to prove that the convergence is uniform on all compact subset of $\mathbb{R}$. We get the following estimates

$$\begin{aligned} w(s+s_n)-z(s) = {} & \lim_{\lambda\to+\infty}\int_{-\infty}^{s}\mathcal{U}^s(s-\theta)\Pi^s(\widetilde{B}_\lambda X_0[h(\theta+s_n)-v(\theta)])\,\mathrm{d}\theta \\ & + \lim_{\lambda\to+\infty}\int_{+\infty}^{s}\mathcal{U}^u(s-\theta)\Pi^u(\widetilde{B}_\lambda X_0[h(\theta+s_n)-v(\theta)])\,\mathrm{d}\theta \end{aligned}$$

and

$$\begin{aligned} |w(s+s_n)-z(s)| \le {} & \Delta\int_{-\infty}^{s} e^{-\omega(s-\theta)}|v(\theta)-h(\theta+s_n)|\,\mathrm{d}\theta \\ & + \Delta\int_{s}^{+\infty} e^{\omega(s-\theta)}|v(\theta)-h(\theta+s_n)|\,\mathrm{d}\theta, \end{aligned}$$

where $\Delta = \max\big(\overline{M}\widetilde{M}|\Pi^s|, \overline{M}\widetilde{M}|\Pi^u|\big)$. For each $k = 1,2,3,\dots$ set

$$X_k(s) = \Delta\int_{s-k}^{s-k+1} e^{-\omega(s-\theta)}|v(\theta)-h(\theta+s_n)|\,\mathrm{d}\theta + \Delta\int_{s+k-1}^{s+k} e^{\omega(s-\theta)}|v(\theta)-h(\theta+s_n)|\,\mathrm{d}\theta.$$

Using the Hölder inequality, we obtain

$$\begin{aligned} X_k(s) \le {} & \frac{\Delta e^{-q\omega k}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{p}}\Bigg[\bigg(\int_{s-k}^{s-k+1}|v(\theta)-h(\theta+s_n)|^p\,\mathrm{d}\theta\bigg)^{\frac{1}{p}} \\ & + \bigg(\int_{s+k-1}^{s+k}|v(\theta)-h(\theta+s_n)|^p\,\mathrm{d}\theta\bigg)^{\frac{1}{p}}\Bigg], \end{aligned}$$

and by the Minkowski inequality we have

$$\begin{aligned}
&\left(\int_t^{t+1} |X_k(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\leq \frac{\Delta e^{-q\omega k}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{p}}\left[\left(\int_t^{t+1}\left[\left(\int_{s-k}^{s-k+1} |v(\theta)-h(\theta+s_n)|^p \,\mathrm{d}\theta\right)^{\frac{1}{p}}\right]^p \mathrm{d}s\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(\int_t^{t+1}\left[\left(\int_{s+k-1}^{s+k} |v(\theta)-h(\theta+s_n)|^p \,\mathrm{d}\theta\right)^{\frac{1}{p}}\right]^p \mathrm{d}s\right)^{\frac{1}{p}}\right] \\
&\leq \frac{\Delta e^{-q\omega k}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{p}}\left[\left(\sup_{s\in[t,t+1]}\int_{s-k}^{s-k+1} |v(\theta)-h(\theta+s_n)|^p \,\mathrm{d}\theta\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(\sup_{s\in[t,t+1]}\int_{s+k-1}^{s+k} |v(\theta)-h(\theta+s_n)|^p \,\mathrm{d}\theta\right)^{\frac{1}{p}}\right] \\
&\leq \frac{2\Delta e^{-q\omega k}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{p}}\sup_{t\in\mathbb{R}}\left(\int_t^{t+1} |v(\theta)-h(\theta+s_n)|^p \,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\sum_{k=1}^{+\infty} X_k(s) = \Delta\int_{-\infty}^{s} e^{-\omega(s-\theta)}|v(\theta)-h(\theta+s_n)|\,\mathrm{d}\theta + \Delta\int_{+\infty}^{s} e^{\omega(s-\theta)}|v(\theta)-h(\theta+s_n)|\,\mathrm{d}\theta,$$

it follows that

$$\left(\int_t^{t+1} |w(s+s_n)-z(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \leq 2C_q(\Delta,\omega)\sup_{t\in\mathbb{R}}\left(\int_t^{t+1} |v(\theta)-h(\theta+s_n)|^p \,\mathrm{d}s\right)^{\frac{1}{p}}.$$

Fix $L>0$ and $N_\varepsilon\in\mathbb{N}$ such that $\Omega\subset[\frac{-L}{2},\frac{L}{2}]$ with $|h(s+s_n)-v(s)|\leq\varepsilon$ for $n\geq N_\varepsilon$ and $s\in[-L,L]$. Then, for each $t\in\Omega$ one has

$$\left(\int_t^{t+1} |w(s+s_n)-z(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \leq 2C_q(\Delta,\omega)\varepsilon.$$

This proves that the convergence is uniform on Ω, as the last estimate is independent of $t\in\Omega$. Proceeding as previously, one can prove that

$$\lim_{n\to+\infty}\left(\int_t^{t+1} |w(s)-z(s-s_n)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} = 0,$$

which implies that $\Psi h\in AA_c(\mathbb{R};L^p((0,1);X))$. □

For the existence of pseudo almost automorphic solution, we make the following assumption.

(**H**$_6$) $f\colon\mathbb{R}\to X$ is $cl(\mu,\nu)$-S^p-pseudo almost automorphic of infinite class.

Theorem 6.5 *Assume that $\mathscr{B}$ satisfies* (**A**$_1$), (**A**$_2$), (**B**), (**C**$_1$), (**C**$_2$) *and that* (**H**$_0$), (**H**$_1$), (**H**$_2$), (**H**$_3$), (**H**$_4$), (**H**$_6$) *hold. Then, equation* (1.1) *has a unique $cl(\mu,\nu)$-S^p-pseudo almost automorphic solution of infinite class.*

Proof. Since f is a S^p-pseudo almost automorphic function, it has the decomposition $f = f_1 + f_2$, where $f_1^b \in AA(\mathbb{R}; L^p((0,1); X_1))$ and $f_2^b \in \mathcal{E}(\mathbb{R}; L^p((0,1); X_1), \mu, \nu, \infty)$. Using Theorem 6.1, Theorem 6.3 and Theorem 6.4, we get the desired result. □

Our next objective is to show the existence of pseudo almost automorphic solutions of infinite class for the following problem

$$u'(t) = Au(t) + L(u_t) + f(t, u_t) \text{ for } t \in \mathbb{R}, \tag{6.3}$$

where $f : \mathbb{R} \times \mathscr{B} \to X$ is continuous.

In what follows, we prove some preliminary results concerning the composition of μ-pseudo almost automorphic functions of infinite class.

Lemma 6.6 *Assume that* ($\mathbf{H}_3$) *holds and let* $f \in BS^p(\mathbb{R}; X)$. *Then,* f *belongs to* $\mathscr{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$ *if and only if for any* $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])} = 0,$$

where

$$M_{\tau,\varepsilon}(f) = \left\{ t \in [-\tau, \tau] : \sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |f(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \geq \varepsilon \right\}.$$

Proof. Suppose that $f \in \mathscr{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. Then,

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&= \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad + \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(f)} \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\geq \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in (-\infty,t]} \left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\geq \frac{\varepsilon \mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])}.
\end{aligned}$$

Consequently,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])} = 0.$$

Suppose that $f \in BS^p(\mathbb{R}; X)$ is such that for any $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau, \tau])} = 0.$$

We can assume that

$$\left(\int_t^{t+1} |f(s)|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \leq N$$

for all $t \in \mathbb{R}$. Using ($\mathbf{H}_3$), we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad = \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} \sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\qquad + \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)} \sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad \leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} \mathrm{d}\mu(t) \\
&\qquad + \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)} \sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1} |f(s)|^p \ \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\
&\quad \leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)} \mathrm{d}\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \mathrm{d}\mu(t) \\
&\quad \leq \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])} N + \frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.
\end{aligned}$$

This implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]}\left(\int_\theta^{\theta+1} |f(s)|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \leq \alpha\varepsilon \ \text{ for any } \varepsilon > 0.$$

Therefore, $f \in \mathscr{E}(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$. □

Theorem 6.7 *Let* $\mu,\nu \in \mathcal{M}$*,* $\phi = \phi_1 + \phi_2 \in PAAS^p(\mathbb{R}\times X; L^p((0,1);X),\mu,\nu,\infty)$ *with* $\phi_1^b \in AA(\mathbb{R}\times X; L^p((0,1);X))$*,* $\phi_2^b \in \mathcal{E}(\mathbb{R}\times X; L^p((0,1);X),\mu,\nu,\infty)$ *and* $h \in PAAS^p(\mathbb{R}; L^p((0,1);X),\mu,\nu,\infty)$*. Assume that*:

(i) $\phi_1(t,x)$ *is uniformly continuous on any bounded subset uniformly for* $t \in \mathbb{R}$*,*

(ii) *there exists a function* $L_\phi\colon \mathbb{R}\to[0,+\infty)$ *satisfying*

$$\left(\int_t^{t+1} |\phi(s,x_1)-\phi(s,x_2)|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \leq L_\phi(t)\|x_1-x_2\|_{L^p} \tag{6.4}$$

for $t \in \mathbb{R}$ *and for* $x_1, x_2 \in L^p((0,1);X)$.

If

$$\beta = \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in(-\infty,t]} L_\phi(\theta)\,\mathrm{d}\mu(t) < \infty, \tag{6.5}$$

then the function $t\mapsto\phi(t,h(t))$ *belongs to* $PAAS^p(\mathbb{R}\times X; L^p((0,1);X),\mu,\nu,\infty)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$ and $h = h_1 + h_2$, where $\phi_1^b \in AA(\mathbb{R} \times X; L^p((0,1);X))$, $\phi_2^b \in \mathcal{E}(\mathbb{R} \times X; L^p((0,1);X), \mu, \nu, \infty)$ and $h_1^b \in AA(\mathbb{R}; L^p((0,1);X))$, $h_2^b \in \mathcal{E}(\mathbb{R}; L^p((0,1);X), \mu, \nu, \infty)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + [\phi(t, h(t)) - \phi(t, h_1(t))] + \phi_2(t, h_1(t)).$$

From [18] we know that $\phi_1(., h_1(.)) \in AA(\mathbb{R}; L^p((0,1);X))$. It remains to prove that both $\phi(., h(.)) - \phi(., h_1(.))$ and $\phi_2(., h_1(.))$ belong to $\mathcal{E}(\mathbb{R}; L^p((0,1);X), \mu, \nu, \infty)$. Clearly, $\phi(., h(.)) - \phi(., h_1(.))$ is bounded and continuous. We can assume that

$$\left(\int_t^{t+1} |\phi(s, h(s)) - \phi(s, h_1(s))|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \leq N \text{ for all } t \in \mathbb{R}.$$

Since $h(t)$, $h_1(t)$ are bounded, we can choose a bounded subset $B \subset \mathbb{R}$ such that $h(\mathbb{R}) \subset B$ and $h_1(\mathbb{R}) \subset B$. Under assumption (ii), for a given $\varepsilon > 0$ if $\|x_1 - x_2\|_{L^p} \leq \varepsilon$, then

$$\left(\int_t^{t+1} |\phi(t, x_1) - \phi(t, x_2)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \leq \varepsilon L_\phi(t) \text{ for all } t \in \mathbb{R}.$$

Since $\eta \in \mathscr{E}(\mathbb{R} \times X; L^p((0,1);X), \mu, \nu, \infty)$, Lemma 6.6 yields that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \mu(M_{\tau,\varepsilon}(\eta)) = 0.$$

Then, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi(s, h(s)) - \phi(s, h_1(s))|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \right) \mathrm{d}\mu(t) \\
&\quad = \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(\eta)} \left(\sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi(s, h(s)) - \phi(s, h_1(s))|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \right) \mathrm{d}\mu(t) \\
&\qquad + \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(\eta)} \left(\sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \right) \mathrm{d}\mu(t) \\
&\quad \leq \frac{N}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(\eta)} \mathrm{d}\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(\eta)} \left(\sup_{\theta \in (-\infty, t]} |L_\phi(\theta)| \right) \mathrm{d}\mu(t) \\
&\quad \leq \frac{N}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(\eta)} \mathrm{d}\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in (-\infty, t]} |L_\phi(\theta)| \right) \mathrm{d}\mu(t) \\
&\quad \leq \frac{N \mu(M_{\tau,\varepsilon}(\eta))}{\nu([-\tau,\tau])} + \frac{\varepsilon}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in (-\infty, t]} |L_\phi(\theta)| \right) \mathrm{d}\mu(t).
\end{aligned}$$

This implies that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi(s, h(s)) - \phi(s, h_1(s))|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) \leq \varepsilon\beta$$

for any $\varepsilon > 0$, which shows that $t \mapsto \phi(t, h(t)) - \phi(t, h_1(t))$ belongs to $\mathcal{E}(\mathbb{R}; L^p((0,1);X), \mu, \nu, \infty)$.

Since ϕ_2^b is uniformly continuous on the compact set $\Omega = \overline{\{h_1^b(t) : \ t \in \mathbb{R}\}}$ with respect to the second variable x, we deduce that for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\xi_1 - \xi_2\|_{L^p} \leq \delta \Rightarrow |\phi_2^b(t, \xi_1(t)) - \phi_2^b(t, \xi_2(t))| \leq \varepsilon$$

for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in \Omega$. Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset \Omega$ such that

$$\Omega \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i, \delta).$$

Then, by the Minkowski inequality, we have

$$\begin{aligned}
&\left(\int_t^{t+1} |\phi_2(t, h_1(t))|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \\
&\quad \leq \left(\int_t^{t+1} |\phi_2(t, h_1(t)) - \phi_2(t, z_i)|^p \, \mathrm{d}s\right)^{\frac{1}{p}} + \left(\int_t^{t+1} |\phi_2(t, z_i)|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \\
&\quad \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \left(\int_t^{t+1} |\phi_2(t, z_i)|^p \, \mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi_2(\theta, z_i)|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0$$

for every $i \in \{1, ..., n(\varepsilon)\}$, we deduce that for all $\varepsilon > 0$ we have

$$\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi_2(\theta, h_1(\theta))|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \leq \varepsilon.$$

This implies that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in (-\infty, t]} \left(\int_\theta^{\theta+1} |\phi_2(\theta, h_1(\theta))|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0.$$

Consequently, $t \mapsto \phi_2(t, h(t))$ belongs to $\mathcal{E}(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. □

For the sequel, we make the following assertions.

($\mathbf{H}_7$) The unstable space $U \equiv \{0\}$.

($\mathbf{H}_8$) The function $f\colon \mathbb{R} \times \mathscr{B} \to X$ is uniformly pseudo compact almost automorphic such that there exists a function $L_f \in L^p(\mathbb{R}; \mathbb{R}^+)$ such that $|f(t, \varphi_1) - f(t, \varphi_2)| \leq L_f(t)\|\varphi_1 - \varphi_2\|_{S^p}$ for all $t \in \mathbb{R}$, $\varphi_1, \varphi_2 \in BS^p((-\infty, 0]; X))$, where L_f satisfies conditions of Theorem 6.7.

Theorem 6.8 *Assume that $\mathscr{B}$ satisfies* ($\mathbf{A}_1$), ($\mathbf{A}_2$), ($\mathbf{B}$), ($\mathbf{C}_1$), ($\mathbf{C}_2$) *and that* ($\mathbf{H}_0$), ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$), ($\mathbf{H}_4$), ($\mathbf{H}_7$), ($\mathbf{H}_8$) *hold. Then, equation* (6.3) *has a unique $cl(\mu, \nu)$-S^p-pseudo compact almost automorphic mild solution of infinite class.*

Proof. Let x be a function in $PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. From Theorem 5.11 it follows that the function $t \mapsto x_t$ belongs to $PAA_cS^p(\mathscr{B}; L^p((0,1); X), \mu, \nu, \infty)$. Hence, Theorem 6.7 implies that the function $g(.) := f(., x_.)$ is in $PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. Since the unstable space $U \equiv \{0\}$, we obtain $\Pi^u \equiv 0$. Consider now the mapping

$$\mathcal{H} : PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty) \to PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \left[\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s, x_s))\,\mathrm{d}s\right](0).$$

From Theorem 6.1, Theorem 6.3 and Theorem 6.4, we can infer that $\mathcal{H}$ maps $PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$ into $PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. It suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. Since $\mathscr{B}$ is a uniform fading memory space, by the Lemma 2.7, we can choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$. Let $C = \max\left\{\sup_{t\in\mathbb{R}} |M(t)|,\ \sup_{t\in\mathbb{R}} |K(t)|\right\}$ and consider two cases.

Case 1: $L_f \in L^1(\mathbb{R})$. Let $x_1, x_2 \in PAA_cS^p(\mathbb{R}; L^p((0,1); X), \mu, \nu, \infty)$. Then, we have

$$\begin{aligned}
&|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)| \\
&\quad \leq \left|\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s, x_{1s}) - f(s, x_{2s})))\,\mathrm{d}s\right| \\
&\quad \leq \overline{M}\widetilde{M}|\Pi^s| \int_{-\infty}^{t} e^{-\omega(t-s)} L_f(s)|x_{1s} - x_{2s}|_{\mathscr{B}}\,\mathrm{d}s \\
&\quad \leq \overline{M}\widetilde{M}|\Pi^s| \int_{-\infty}^{t} L_f(s)\Big(K(s)\sup_{0\leq\xi\leq s}|x_1(\xi) - x_2(\xi)| + M(s)|x_{1_0} - x_{2_0}|_{\mathscr{B}}\Big)\,\mathrm{d}s \\
&\quad \leq 2\overline{M}\widetilde{M}|\Pi^s|C\left(\int_{-\infty}^{t} L_f(s)\,\mathrm{d}s\right)\|x_1 - x_2\|_{S^p}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&|\mathcal{H}^2x_1(t) - \mathcal{H}^2x_2(t)| \\
&\quad \leq \left|\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s, \mathcal{H}x_{1s}) - f(s, \mathcal{H}x_{2s}))\,\mathrm{d}s\right| \\
&\quad \leq (\overline{M}\widetilde{M}|\Pi^s|C)^2\|x_1 - x_2\|_{S^p} \int_{-\infty}^{t} L_f(s) \int_{-\infty}^{s} L_f(\delta)\ \mathrm{d}\delta\,\mathrm{d}s \\
&\quad \leq \frac{(2\overline{M}\widetilde{M}|\Pi^s|C)^2}{2}\left(\int_{-\infty}^{t} L_f(s)\,\mathrm{d}s\right)^2\|x_1 - x_2\|_{S^p}.
\end{aligned}$$

Induction with respect to n, gives

$$|\mathcal{H}^nx_1(t) - \mathcal{H}^nx_2(t)| \leq \frac{(2\overline{M}\widetilde{M}|\Pi^s|C)^n}{n!}\left(\int_{-\infty}^{t} L_f(s)\,\mathrm{d}s\right)^n\|x_1 - x_2\|_{S^p}.$$

Therefore,

$$\|\mathcal{H}^nx_1 - \mathcal{H}^nx_2\|_{S^p} \leq \frac{(2\overline{M}\widetilde{M}|\Pi^s|C|L_f|_{L^1(\mathbb{R})})^n}{n!}\|x_1 - x_2\|_{S^p}.$$

Since

$$\lim_{n\to+\infty}\frac{(2\overline{M}\widetilde{M}|\Pi^s|C|L_f|_{L^1(\mathbb{R})})^n}{n!}=0,$$

there exists $n_0\in\mathbb{N}$ such that

$$\frac{(2\overline{M}\widetilde{M}|\Pi^s|C|L_f|_{L^1(\mathbb{R})})^{n_0}}{n_0!}<1.$$

Hence, $\mathcal{H}^{n_0}$ is a contraction. By the Banach fixed point theorem $\mathcal{H}$ has a unique point fixed and this fixed point satisfies the integral equation

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,u_s))\,\mathrm{d}s.$$

Case 2: $L_f\in L^p(\mathbb{R})$, where $1<p<\infty$. First, put $\mu(t)=\int_{-\infty}^{t}(L_f(s))^p\,\mathrm{d}s$. Then, we define an equivalent norm over $PAAS^p(\mathbb{R};X)$ by the formula

$$|f|_c=\sup_{t\in\mathbb{R}}e^{-c\mu(t)}\left(\int_t^{t+1}|f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}},$$

where c is a fixed positive number to be specified later. Using the Hölder inequality we have

$$\begin{aligned}
&|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)|\\
&\leq\left|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s})-f(s,x_{2s})))\,\mathrm{d}s\right|\\
&\leq\overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)|x_{1s}-x_{2s}|_{\mathscr{B}}\,\mathrm{d}s\\
&\leq\overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)\left(K(s)\sup_{0\leq\xi\leq s}|x_1(\xi)-x_2(\xi)|+M(s)|x_{1_0}-x_{2_0}|_{\mathscr{B}}\right)\mathrm{d}s\\
&\leq\overline{M}\widetilde{M}|\Pi^s|C\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)\left(\sup_{0\leq\xi\leq s}|x_1(\xi)-x_2(\xi)|+|x_{1_0}-x_{2_0}|_{\mathscr{B}}\right)\mathrm{d}s\\
&\leq 2\overline{M}\widetilde{M}|\Pi^s|C\int_{-\infty}^{t}e^{-\omega(t-s)}e^{-c\mu(s)}e^{c\mu(s)}L_f(s)\|x_{1s}-x_{2s}\|_{S^p}\,\mathrm{d}s\\
&\leq 2\overline{M}\widetilde{M}|\Pi^s|C\int_{-\infty}^{t}\left(e^{-\omega(t-s)}e^{c\mu(s)}L_f(s)\right)\sup_{s\in\mathbb{R}}\left(e^{-c\mu(s)}\|x_{1s}-x_{2s}\|_{S^p}\right)\mathrm{d}s\\
&\leq 2\overline{M}\widetilde{M}|\Pi^s|C\int_{-\infty}^{t}\left(e^{-\omega(t-s)}e^{c\mu(s)}L_f(s)\,\mathrm{d}s\right)|x_1-x_2|_c\\
&\leq 2\overline{M}\widetilde{M}|\Pi^s|C\left(\int_{-\infty}^{t}e^{pc\mu(s)}(L_f(s))^p\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_{-\infty}^{t}e^{-\omega q(t-s)}\,\mathrm{d}s\right)^{\frac{1}{q}}|x_1-x_2|_c\\
&\leq 2\overline{M}\widetilde{M}|\Pi^s|C\left(\int_{-\infty}^{t}e^{pc\mu(s)}\mu'(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_{-\infty}^{t}e^{-\omega q(t-s)}\,\mathrm{d}s\right)^{\frac{1}{q}}|x_1-x_2|_c\\
&\leq 2\overline{M}\widetilde{M}|\Pi^s|C\left(\frac{1}{(pc)^{\frac{1}{p}}}\times\frac{1}{(\omega q)^{\frac{1}{q}}}\right)e^{c\mu(t)}|x_1-x_2|_c.
\end{aligned}$$

Consequently,

$$|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)|_c \leq \frac{2\overline{M}\widetilde{M}|\Pi^s|C}{(pc)^{\frac{1}{p}} \times (\omega q)^{\frac{1}{q}}}|x_1 - x_2|_c.$$

Note that the function $c \mapsto 1/(pc)^{\frac{1}{p}}$ converges to 0 when c tends to $+\infty$. It follows that for $c > 0$ large enough we have

$$\frac{2\overline{M}\widetilde{M}|\Pi^s|C}{(pc)^{\frac{1}{p}} \times (\omega q)^{\frac{1}{q}}} < 1.$$

Thus, $\mathcal{H}$ is a contractive mapping. Using the same argument as in Theorem 3.3 of [16], we conclude that equation (6.3) has one and only one $cl(\mu,\nu)$-S^p-pseudo almost automorphic solution of infinite class. This ends the proof. □

Corollary 6.9 *Assume that $\mathscr{B}$ is a uniform fading memory space and* ($\mathbf{C}_1$), ($\mathbf{C}_2$), ($\mathbf{H}_0$), ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$) *and* ($\mathbf{H}_7$) *hold. Moreover, assume that f is Lipschitz continuous with respect the second argument. If*

$$\mathrm{Lip}(f) < \frac{\omega}{2\overline{M}\widetilde{M}|\Pi^s|},$$

then equation (6.3) *has a unique* $cl(\mu)$*-pseudo almost automorphic solution of infinite class, where* $\mathrm{Lip}(f)$ *is the Lipschitz constant of* f.

Proof. Let us set $k := \mathrm{Lip}(f)$. We have

$$\begin{aligned}|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)| &\leq \left|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s}) - f(s,x_{2s})))\,\mathrm{d}s\right| \\ &\leq 2|\Pi^s|\overline{M}\widetilde{M}k\|x_1 - x_2\|_{S^p}\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\,\mathrm{d}s\right) \\ &\leq \frac{2|\Pi^s|\overline{M}\widetilde{M}k\|x_1 - x_2\|_{S^p}}{\omega}.\end{aligned}$$

Consequently, $\mathcal{H}$ is a strict contraction if

$$k < \frac{\omega}{2\overline{M}\widetilde{M}|\Pi^s|}.$$

□

7 Application

For illustration, we will study the existence of solutions for the following model

$$\begin{cases}\dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-\infty}^{0} G(\theta)z(t+\theta,x)\,\mathrm{d}\theta + \sin\left(\dfrac{1}{2+\cos t+\cos\sqrt{2}t}\right) \\ \qquad\qquad + \arctan t + \displaystyle\int_{-\infty}^{t} e^{\omega(-t+\theta)}h(\theta, z(t+\theta,x))\,\mathrm{d}\theta \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi], \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R},\end{cases} \tag{7.1}$$

where $G\colon (-\infty,0] \to \mathbb{R}$ is a continuous function, $h\colon (-\infty,0]\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies Lipschitz condition with respect to the second argument and ω is a positive real number. For example, we can take

$$G(\theta) = \frac{\theta^2-1}{(\theta^2+1)^2} \text{ for } \theta \in (-\infty,0]$$

and

$$h(\theta,x) = \theta^2 + \sin\left(\frac{x}{2}\right) \text{ for } (\theta,x) \in (-\infty,0]\times\mathbb{R}.$$

To rewrite equation (7.1) in the abstract form, we introduce the space $X = C_0([0,\pi];\mathbb{R})$ of continuous functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the uniform norm topology. Let $A\colon D(A) \to X$ be defined by

$$\begin{cases} D(A) = \{y \in X \cap C^2([0,\pi];\mathbb{R}) : y'' \in X\}, \\ Ay = y''. \end{cases}$$

Then, A satisfies the Hille–Yosida condition in X. Moreover, the part A_0 of A in $\overline{D(A)}$ is the generator of strongly continuous compact semi-group $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. It follows that ($\mathbf{H}_0$) and ($\mathbf{H}_1$) are satisfied.

As the phase space we set $\mathscr{B} = C_\gamma$, $\gamma > 0$, where

$$C_\gamma = \left\{\varphi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\right\}$$

with the following norm

$$|\varphi|_\gamma = \sup_{\theta\leq 0} |e^{\gamma\theta}\varphi(\theta)|.$$

This space is a uniform fading memory space, In other words, ($\mathbf{H}_2$) holds. Note also that it satisfies ($\mathbf{C}_1$), ($\mathbf{C}_2$).

We define $f\colon \mathbb{R}\times\mathscr{B} \to X$ and $L\colon \mathscr{B} \to X$ as follows

$$\begin{aligned} f(t,\varphi)(x) = \sin&\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) + \arctan t \\ &+ \int_{-\infty}^{t} e^{\omega(-t+\theta)} h(\theta,\varphi(\theta)(x))\,\mathrm{d}\theta \text{ for } x\in[0,\pi] \text{ and } t\in\mathbb{R} \end{aligned}$$

and

$$L(\varphi)(x) = \int_{-\infty}^{0} G(\theta)\varphi(\theta)(x)\,\mathrm{d}\theta \text{ for } x \in [0,\pi].$$

Let us set $v(t) = z(t,x)$. Then, equation (7.1) takes the following abstract form

$$v'(t) = Av(t) + L(v_t) + f(t,v_t) \text{ for } t \in \mathbb{R}. \tag{7.2}$$

Consider the measures μ and ν whose Radon–Nikodym derivatives are respectively $\rho_1, \rho_2\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0, \\ e^t & \text{for } t \leq 0, \end{cases}$$

and $\rho_2(t) = |t|$ for $t \in \mathbb{R}$, *i.e.*, $\mathrm{d}\mu(t) = \rho_1(t)\mathrm{d}t$ and $\mathrm{d}\nu(t) = \rho_2(t)\mathrm{d}t$, where $\mathrm{d}t$ denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)\,\mathrm{d}t \;\text{ and }\; \nu(A) = \int_A \rho_2(t)\,\mathrm{d}t \;\text{ for }\; A \in \mathscr{B}.$$

From [4] we know that $\mu, \nu \in \mathcal{M}$ and that μ, ν satisfy hypothesis ($\mathbf{H}_5$). Furthermore, by [9] the function

$$t \mapsto \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)$$

is compact almost automorphic.

We have

$$\limsup_{\tau\to+\infty}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}=\limsup_{\tau\to+\infty}\frac{\displaystyle\int_{-\tau}^{0} e^t\,\mathrm{d}t+\int_0^{\tau}\,\mathrm{d}t}{\displaystyle 2\int_0^{\tau} t\,\mathrm{d}t}=\limsup_{\tau\to+\infty}\frac{1-e^{-\tau}+\tau}{\tau^2}=0<\infty,$$

which implies that ($\mathbf{H}_3$) is satisfied.

Let $p \geq 1$. Since $\frac{-\pi}{2} \leq \arctan\theta \leq \frac{\pi}{2}$ for all $\theta \in \mathbb{R}$, we have

$$\begin{aligned}&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in(-\infty,0]}\Big(\int_{\theta}^{\theta+1}|\arctan s|^p\,\mathrm{d}s\Big)^{\frac{1}{p}}\,\mathrm{d}\mu(t)\\&\quad\leq\frac{\pi}{2}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\mathrm{d}\mu(t)\leq\frac{\pi}{2}\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\to 0\ \text{ as } \tau\to+\infty.\end{aligned}$$

It follows that $t \mapsto \arctan t \in \mathcal{E}(\mathbb{R}; L^p((0,1);X), \mu, \nu, \infty)$. Consequently, f is uniformly (μ,ν)-S^p-pseudo almost automorphic of infinite class. Moreover, L is a bounded linear operator from $\mathscr{B}$ to X.

For every $\varphi_1, \varphi_2 \in BS^p(\mathbb{R}; X)$ and $t \geq 0$ we have

$$|f(t,\varphi_1)-f(t,\varphi_2)|\leq\frac{1}{2}\int_{-\infty}^{t}e^{\omega(-t+\theta)}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|\,\mathrm{d}\theta.$$

For each $n = 1, 2, 3, \ldots$, set

$$X_n(t)=\frac{1}{2}\int_{t-n}^{t-n+1}e^{\omega(-t+\theta)}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|\,\mathrm{d}\theta.$$

Let q be such that $\frac{1}{q}+\frac{1}{p}=1$. Using the Hölder inequality, we obtain

$$\begin{aligned}X_n(t)&\leq\frac{1}{2}\Big(\int_{t-n}^{t-n+1}e^{-q\omega(t-\theta)}\,\mathrm{d}\theta\Big)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|^p\,\mathrm{d}\theta\Big)^{\frac{1}{p}}\\&\leq\frac{1}{2\sqrt[q]{q\omega}}(e^{-q\omega(n-1)}-e^{-q\omega n})^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|^p\,\mathrm{d}\theta\Big)^{\frac{1}{p}}\\&\leq\frac{e^{-\omega n}}{2\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|^p\,\mathrm{d}\theta\Big)^{\frac{1}{p}}.\end{aligned}$$

Since the series

$$\frac{(e^{q\omega}+1)^{\frac{1}{q}}}{2\sqrt[q]{q\omega}} \times \sum_{n=1}^{+\infty} e^{-q\omega n}$$

is convergent, it follows from the Weierstrass M-test that the sequence of functions $\sum_{n=1}^{N} X_n(t)$ is uniformly convergent on $\mathbb{R}$. Thus, we have

$$\begin{aligned}|f(t,\varphi_1)-f(t,\varphi_2)| &\le \frac{C_q(\omega)}{2}\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|^p\,\mathrm{d}\theta\right)^{\frac{1}{p}}\\ &\le \frac{C_q(\omega)}{2}\sup_{0\le x\le\pi}\|\varphi_1(x)-\varphi_2(x)\|_{S^p},\end{aligned}$$

where

$$C_q(\omega)=\frac{(e^{q\omega}+1)^{\frac{1}{p}}}{\sqrt[q]{q\omega}}\times\sum_{n=1}^{+\infty}e^{-q\omega n}=\frac{(e^{q\omega}+1)^{\frac{1}{p}}}{\sqrt[q]{q\omega}}\times\frac{e^{-q\omega}}{1-e^{-q\omega}}.$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-S^p-pseudo almost automorphic of infinite class.

To show that ($\mathbf{H}_6$) holds we need a result established in [10].

Lemma 7.1 ([10]) *If* $\int_{-\infty}^{0}|G(\theta)|\,\mathrm{d}\theta < 1$, *then the semi-group* $(\mathcal{U}(t))_{t\ge0}$ *is hyperbolic and the unstable space* $U\equiv\{0\}$.

We can see that in our case we have

$$\int_{-\infty}^{0}|G(\theta)|\,\mathrm{d}\theta=\lim_{r\to+\infty}\int_{-r}^{0}\left|\frac{\theta^2-1}{(\theta^2+1)^2}\right|\mathrm{d}\theta=\lim_{r\to+\infty}\left[\frac{\theta}{\theta^2+1}\right]_{-r}^{0}=\lim_{r\to+\infty}\frac{r}{r^2+1}<1.$$

Hence, ($\mathbf{H}_6$) holds. Consequently, by Corollary 6.9 we deduce the following result.

Corollary 7.2 *Under the above assumptions, if*

$$\mathrm{Lip}(h)=\frac{1}{2}<\frac{1}{C_q(\omega)},$$

i.e., *if* $C_q(\omega)<2$, *then equation* (7.2) *has a unique* $cl(\mu,\nu)$-S^p*-pseudo almost automorphic solution* v *of infinite class.*

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References

[1] M. Adimy, A. Elazzouzi, K. Ezzinbi, *Reduction principle and dynamic behaviors for a class of partial functional differential equations*, Nonlinear Analysis. Theory, Methods & Applications **71** (2009), no. 5–6, 1709–1727.

[2] M. Adimy, A. Elazzouzi, K. Ezzinbi, *Bohr–Neugebauer type theorem for some partial neutral functional differential equations*, Nonlinear Analysis. Theory, Methods & Applications **66** (2007), no. 5, 1145–1160.

[3] M. Adimy, K. Ezzinbi, M. Laklach, *Spectral decomposition for partial neutral functional differential equations*, Canadian Applied Mathematics Quarterly **9** (2001), no. 1, 1–34.

[4] J. Blot, P. Cieutat, K. Ezzinbi, *New approach for weighted pseudo almost periodic functions under the light of measure theory, basic results and applications*, Applicable Analysis **92** (2013), no. 3, 493–526.

[5] S. Bochner, *Continuous mappings of almost automorphic and almost automorphic functions*, Proceedings of the National Academy of Sciences of the United States of America **52** (1964), no. 4, 907–910.

[6] S. Bochner, *A new approach to almost-periodicity*, Proceedings of the National Academy of Sciences of the United States of America **48** (1962), no. 12, 2039–2043.

[7] T. Diagana, K. Ezzinbi, M. Miraoui, *Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory*, CUBO A Mathematical Journal **16** (2014), no. 2, 1–31.

[8] K. Ezzinbi, H. Touré, I. Zabsonré, *Pseudo almost automorphic solutions of class r for some partial functional differential equations*, Afrika Matematika **25** (2014), 25–41.

[9] K. Ezzinbi, S. Fatajou, G. M. N'Guérékata, *Pseudo almost automorphic solutions for dissipative differential equations in Banach spaces*, Journal of Mathematical Analysis and Applications **351** (2009), no. 2, 765–772.

[10] K. Ezzinbi, S. Fatajou, G. M. N'Guérékata, *C^n-almost automorphic solutions for partial neutral functional differential equations*, Applicable Analysis **86** (2007), no. 9, 1127–1146.

[11] Y. Hino, S. Murakami, T. Naito, *Functional differential equations with infinite delay*, Lectures Notes in Mathematics, vol. 1473, Springer-Verlag Berlin Heidelberg, 1991.

[12] G. M. N'Guérékata, *Spectral theory of bounded functions and applications to evolution equations*, Nova Science Publisher Inc., New York, 2017.

[13] G. M. N'Guérékata, A. Pankov, *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Analysis. Theory, Methods & Applications **68**, (2008), no. 9, 2658–2667.

[14] G. M. N'Guérékata, *Topics in almost automorphy*, Springer-Verlag, New York, 2005.

[15] G. M. N'Guérékata, *Almost automorphic and almost periodic functions in abstract spaces*, Springer Science+Business Media, New York, 2001.

[16] T. J. Xiao, J. Liang, J. Zhang, *Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces*, Semigroup Forum **76** (2008), 518–524.

[17] I. Zabsonré, H. Touré, *Pseudo almost periodic and pseudo almost automorphic solutions of class r under the light of measure theory*, African Diaspora Journal of Mathematics **19** (2016), no. 1, 58–86.

[18] R. Zhang, Y. K. Chang, G. M. N'Guérékata, *New composition theorems of Stepanov-like almost automorphic functions and applications to nonautonomous evolution equations*, Nonlinear Analysis. Real World Applications **13** (2012), no. 6, 2866–2879.