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ASYMPTOTIC BEHAVIOR OF DIFFERENCE EQUATIONS UNDER RATIONAL EXPECTATIONS

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Abstract. In this paper, two different classes of difference equations under rational expectations are examined. The first one concerns the conditional expectational singular difference equations and the second one is about the time-varying conditional expectational difference equations. Asymptotic behaviors of the solutions of the latter are especially studied.

Keywords: Random sequence, singular difference equation, time-varying difference equation.

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1 Introduction

In recent years, solutions of various kinds of difference equations have been considerably investigated in lots of publications (see for instance [2, 3, 4, 10]) because of their significance and applications in physics, mechanics, and mathematical biology. Conditional expectational difference equations have received less studies about them.

This paper deals with the existence of solutions of different classes of conditional expectational difference equations. The first type is of the form

$$A\mathbf{E}_t[X(t+1)] + BX(t) = g(t),\ t \in \mathbb{Z}_+, \tag{1.1}$$

where A, B are $N \times N$ square matrices satisfying $\det A = \det B = 0$ and $g\colon \Omega \times \mathbb{Z}_+ \to \mathbb{R}^N$ is a bounded random function. The second type is of the form

$$\mathbf{E}_t[X(t+1)] = \mathcal{A}(t)X(t) + g(t),\ \ t \in \mathbb{Z}_+, \tag{1.2}$$

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where $\{\mathcal{A}(t)\}_{t\in\mathbb{Z}_+}$ is a family of $N \times N$ invertible random matrices and $g\colon \Omega \times \mathbb{Z}_+ \to \mathbb{R}^N$ is a bounded random function.

Such conditional expectational difference equations arise typically in the context of rational expectations models. Starting with the seminal paper by Blanchard and Kahn [5], an extensive literature developed which analyzes the existence and nature of their solutions. The most influential papers, at least for the present exposition, are Klein [8] and Sims [11], among others. Recently, Neusser [9] developed a comprehension theory for rational expectations models with time-varying (random) coefficients based on Lyapunov exponents defined as the asymptotic growth rates of trajectories.

In this paper we review the analytical aspects: boundedness, stability, and asymptotic behavior of equations (1.1)–(1.2). Our aim is twofold. First, we use Diagana and Pennequin [6] techniques to study the existence of solutions of conditional expectational singular difference equations. Second, we make use of dichotomy techniques to study the existence of solutions of time-varying conditional expectational difference equations.

The rest of this paper is organized as follows. In Section 2, notations and definitions are introduced. In Section 3, some stochastic nonsingular linear difference equations are discussed. In Section 4, we investigate solutions to some conditional expectational singular difference equations. Finally, the last section is devoted to the study of asymptotic behaviors of the solutions of time-varying conditional expectational difference equations.

2 Preliminaries

In this section we review some basic concepts and results which will be useful to prove our main results. To facilitate our task, we first introduce the notions needed in the sequel.

Let $(\mathbb{R}^N, \|\cdot\|)$ be the N-dimensional Euclidean space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Throughout the rest of the paper, $\mathbb{Z}_+$ denotes the set of all positive integers. Define $L^1(\Omega; \mathbb{R}^N)$ to be the space of all $\mathbb{R}^N$-valued random variables V such that

$$\mathbf{E}\|V\| := \left(\int_\Omega \|V(\omega)\| \, \mathrm{d}\mathbf{P}(\omega)\right) < \infty. \tag{2.1}$$

It is then routine to check that $L^1(\Omega; \mathbb{R}^N)$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_1$ defined by $\|V\|_1 := \mathbf{E}\|V\|$ for each $V \in L^1(\Omega, \mathbb{R}^N)$.

Let $X = \{X(t)\}_{t\in\mathbb{Z}_+}$ be a sequence of $\mathbb{R}^N$-valued random variables satisfying $\mathbf{E}\|X(t)\| < \infty$ for each $t \in \mathbb{Z}_+$. Thus, interchangeably we can, and do, speak of such a sequence as a function, which goes from $\mathbb{Z}_+$ into $L^1(\Omega; \mathbb{R}^N)$.

Definition 2.1 *We say that a sequence $X = \{X(t)\}_{t\in\mathbb{Z}_+}$ is bounded if there exists an $M > 0$ such that $\mathbf{E}\|X(t)\| \le M$ for all $t \in \mathbb{Z}_+$.*

Let $UB(\mathbb{Z}_+;\ L^1(\Omega; \mathbb{R}^N))$ denote the collection of all uniformly bounded $L^1(\Omega; \mathbb{R}^N)$-valued random sequences $X = \{X(t)\}_{t\in\mathbb{Z}_+}$. It is then easy to check that the space $UB(\mathbb{Z}_+;\ L^1(\Omega; \mathbb{R}^N))$ is a Banach space when it is equipped with the norm

$$\|X\|_\infty = \sup_{t\in\mathbb{Z}_+} \mathbf{E}\|X(t)\|.$$

We now state the following composition result.

Lemma 2.2 *Let $F : \mathbb{Z}_+ \times L^1(\Omega;\mathbb{R}^N) \to L^1(\Omega;\mathbb{R}^N)$, $(t,U) \mapsto F(t,U)$, be bounded in mean in $t \in \mathbb{Z}_+$ uniformly in $U \in L^1(\Omega;\mathbb{R}^N)$. Assume that there exists a function γ from $\mathbb{Z}_+$ into $\mathbb{R}_+$ such that $\mathbf{E}\|F(t,U) - F(t,V)\| \leq \gamma(t)\,\mathbf{E}\|U - V\|$ for all $\mathbb{R}^N$-valued random variables U, V with finite expectation and $t \in \mathbb{Z}_+$, and that*

$$F(t,0) = 0 \quad \textit{and} \quad \sum_{s=1}^{\infty} \gamma(s) < \infty.$$

Then, for any mean bounded random sequence $X = \{X(t)\}_{t\in\mathbb{Z}_+}$, the random sequence $Y(t) = F(t, X(t))$ is bounded in mean.

Proof. The proof of Lemma 2.2 is straightforward, and hence is omitted. □

Let $\{\mathcal{A}(t)\}_{t\in\mathbb{Z}_+}$ be a family of $N \times N$ invertible random matrices and consider the following conditional expectational first-order linear difference equation of type (1.2). Its corresponding homogeneous equation is given by

$$\mathbf{E}_t[X(t+1)] = \mathcal{A}(t)X(t), \;\; t \in \mathbb{Z}_+. \tag{2.2}$$

The sequence $\{\mathcal{A}(t), g(t)\}$ of random variables is defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Define $\mathcal{F}_t = \sigma\{(X(s), \mathcal{A}(s), g(s)) : s \leq t\}$ to be the smallest σ-algebra such that $(X(s), \mathcal{A}(s), g(s))$ is measurable for all $s \leq t$. Then, $\mathbf{E}_t[\,.\,] = \mathbf{E}[\,.\,|\mathcal{F}_t]$ denotes the conditional expectation with respect to $\mathcal{F}_t$.

Definition 2.3 *The matrix $U(t,a)$ which satisfies equation* (2.2) *and $U(a,a) = I$ is called principal fundamental matrix. We denote $U(t,a)$ by $U(t)U^{-1}(a)$.*

Theorem 2.4 *The following holds*

$$U(t)U^{-1}(a) = \begin{cases} \displaystyle\prod_{s=a}^{t-1} \mathcal{A}(a+t-1-s) & \textit{for all } a \leq t \in \mathbb{Z}_+, \\ \displaystyle\prod_{s=t}^{a-1} \mathcal{A}^{-1}(s) & \textit{for all } a \geq t \in \mathbb{Z}_+. \end{cases} \tag{2.3}$$

Proof. The proof of the theorem can be seen in Argawal [1] for instance. □

Definition 2.5 *Let $U(t)$ be the principal fundamental matrix of equation* (2.2)*. Equation* (2.2) *is said to possess a discrete exponential dichotomy if there exists a projection P, which commutes with $U(t)$, and $M > 0$ and $\beta \in (0,1)$ such that*

$$\|U(t)PU^{-1}(s)\| \leq M\beta^{t-s}, \;\; t \leq s,$$
$$\|U(t)[I-P]U^{-1}(s)\| \leq M\beta^{s-t}, \;\; s \leq t.$$

Definition 2.6 *A zero solution $X(t)$ of equation* (1.2) *is stable if for any positive integer t_0 and any $\varepsilon > 0$ there is a $\delta(t_0, \varepsilon) > 0$ such that $0 < \mathbf{E}\|X(t_0)\| < \delta(t_0, \varepsilon)$ implies $\mathbf{E}\|X(t)\| < \varepsilon$ for all $t \geq t_0$.*

Definition 2.7 *A zero solution $X(t)$ of equation* (1.2) *is asymptotically stable if for any positive integer t_0 and any $\varepsilon > 0$ there is a $\delta(t_0, \varepsilon) > 0$ such that $0 < \mathbf{E}\|X(t_0)\| < \delta(t_0, \varepsilon)$ implies $\mathbf{E}\|X(t)\| \to 0$ as $t \to \infty$.*

Definition 2.8 *Let $Y(t)$ be a solution of equation* (1.2). *Then, $Y(t)$ is called conditionally stable if there exists a sequence $\{Y_n(t)\}$ of solutions of equation* (1.2) *which converges to $Y(t)$ in $L^1(\Omega, \mathbb{R}^N)$ uniformly in $t \in \mathbb{Z}_+$.*

3 Solutions to some stochastic nonsingular linear difference equations

In this section we study the existence of solutions of the following systems of conditional expectational linear first-order stochastic difference equations of type

$$\mathbf{E}_t[X(t+1)] = MX(t) + g(t), \tag{3.1}$$

where M is a $N \times N$ square matrix, $X(t+1)$ is non-predeterminined at time t (that is, it is nondegenerate with respect to information available up to and including time t) and $g\colon \mathbb{Z}_+ \to \mathbb{C}^N$ is a mean bounded random sequence. In equation (3.1), $\mathbf{E}_t[\,.\,] = \mathbf{E}[\,.\,|\mathcal{F}_t]$ denotes the conditional expectation, where $\mathcal{F}_t = \sigma\{(X(l), g(l)) : l \leq t\}$ is the smallest σ-algebra such that $(X(l), g(l))$ is measurable for all $l \leq t$. We assume that g and $X(0)$ are independent. This assumption together with equation (3.1) imply that g is independent of the sequence $\{X(t)\}_{t \in \mathbb{Z}_+}$. For simplicity we assume that stochastic process $\{X(t),\ t \in \mathbb{Z}_+\}$ is mean stationary, that is, $\mathbf{E}[X(t)] = \mathbf{E}[X(t+r)]$ for all r.

We begin with the scalar case $M = \lambda$ and denote by $\mathbb{S}^1$ the unit circle on the complex plane, that is, $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. We can now solve equation (3.1) forward noting that

$$\begin{aligned}
\mathbf{E}_{t+j}X(t+j+1) &= \lambda X(t+j) + g(t+j),\\
\mathbf{E}_t[\mathbf{E}_{t+j}[X(t+j+1)]] &= \lambda \mathbf{E}_t[X(t+j)] + \mathbf{E}_t[g(t+j)],\\
\mathbf{E}_t[X(t+j+1)] &= \lambda \mathbf{E}_t[X(t+j)] + \mathbf{E}_t[g(t+j)],
\end{aligned}$$

where the last line follows from the Law of Iterated Expectations. Then, equation (3.1) implies that

$$\begin{aligned}
X(t) &= \lambda^{-1}\mathbf{E}_t[X(t+1)] - \lambda^{-1}[g(t)]\\
&= \lambda^{-1}\left[\lambda^{-1}\mathbf{E}_t[X(t+2)] - \lambda^{-1}\mathbf{E}_t[g(t+1)] - \lambda^{-1}[g(t)]\right]\\
&\ \vdots\\
&= \lambda^{-l}\mathbf{E}_t[X(t+l)] - \sum_{j=0}^{l-1} \lambda^{-(j+1)}\mathbf{E}_t[g(t+j)].
\end{aligned}$$

Letting $l \to \infty$, the last line becomes

$$X(t) = \lim_{l \to \infty} \lambda^{-l}\mathbf{E}_t[X(t+l)] - \sum_{j=0}^{\infty} \lambda^{-(j+1)}\mathbf{E}_t[g(t+j)]. \tag{3.2}$$

If $|\lambda| > 1$, for a bounded $\{X(t)\}$ we have

$$\lim_{l\to\infty} \lambda^{-l}\mathbf{E}_t[X(t+l)] = 0.$$

Hence,

$$X(t) = -\sum_{j=0}^{\infty} \lambda^{-(j+1)}\mathbf{E}_t[g(t+j)].$$

Remark 3.1 *Let us assume that* $|\lambda| < 1$. *Clearly,* $\lim_{l\to\infty} \lambda^{-l}\mathbf{E}_t[X(t+l)]$ *is not finite, so that equation* (3.2) *does not help us pin down a unique bounded solution. On the other hand, note that the set of solutions of equation* (3.1) *can also be described by*

$$X(t) = \lambda X(t-1) + g(t-1) + \nu(t), \tag{3.3}$$

where $\{\nu(t)\}$*, the sequence of expectational errors, is an arbitrary stochastic process that satisfies* $\mathbf{E}_t[\nu(t+1)] = 0$ *for all* t*. Hence, all solutions defined by equation* (3.3) *for a bounded random sequence satisfying* $\mathbf{E}_t[\nu(t+1)] = 0$ *for all* k*, are bounded solutions. That is, if* $|\lambda| < 1$*, one has an infinity of bounded solutions.*

Theorem 3.2 *Suppose that* $M := \lambda$ *with* $|\lambda| > 1$*. If* $g\colon \mathbb{Z}_+ \to \mathbb{C}^N$ *is bounded in mean, then there is a mean bounded solution of equation* (3.1) *given by*

$$\bar{X}(t) = -\mathbf{E}_t\left[\sum_{l=t}^{\infty} \lambda^{t-l-1}g(l)\right].$$

In addition, $\mathbf{E}\|\bar{X}(t)\| \to 0$ *as* $t \to \infty$.

Proof. Define $Y(t) = \lambda^{-t}$ and since $|\lambda| > 1$, we deduce from Remark 3.1 that

$$\sum_{l=0}^{\infty} |Y(l)| = \frac{|\lambda|}{|\lambda| - 1}.$$

It follows that X is bounded in mean. Next, we check that X is a solution of equation (3.1). We have

$$\begin{aligned} X(t+1) &= -\left[\sum_{l=t+1}^{\infty} \lambda^{t-l}\mathbf{E}_t[g(l)]\right] = -\sum_{l=t}^{\infty} \lambda^{t-l}\mathbf{E}_t[g(l)] + g(t) \\ &= -\lambda\sum_{l=t}^{\infty} \lambda^{t-l-1}\mathbf{E}_t[g(l)] + g(t). \end{aligned}$$

Taking $\mathbf{E}_t[\,.\,]$ on both sides, we obtain

$$\mathbf{E}_t[X(t+1)] = \lambda X(t) + g(t).$$

Moreover,

$$\mathbf{E}\|\bar{X}(t)\| \le \mathbf{E}\left[\sum_{l=t}^{\infty} |\lambda|^{t-l-1}\mathbf{E}_t\|g(l)\|\right] \le C\sum_{l=t}^{\infty} |\lambda|^{t-l-1},$$

where $C = \sup_{r\in\mathbb{Z}_+} \mathbf{E}\|g(r)\|$. Consequently, $\lim_{t\to\infty} \mathbf{E}\|\bar{X}(t)\| = 0$. □

As a consequence of the previous theorem, we obtain the following result in case of a nonsingular random matrix M.

Theorem 3.3 *Suppose M is a constant $N \times N$ nonsingular matrix with eigenvalues λ outside $\mathbb{S}^1$. Then, if g is bounded in mean, there is a mean bounded solution of equation* (3.1).

Proof. Our proof follows closely that of [2]. Since we are in stochastic case and for the sake of clarity, we reproduce it here with slight modifications. It is well-known that there exists a nonsingular random matrix S such that $S^{-1}MS = B$ is an upper triangular matrix. Setting $X(t) = SY(t)$, equation (3.1) becomes

$$\mathbf{E}_t[Y(t+1)] = BY(t) + S^{-1}g(t),\ t \in \mathbb{Z}_+. \tag{3.4}$$

Obviously, equation (3.4) is of the same type as equation (3.1). One can easily see that $S^{-1}g(t)$ is bounded in mean. The general case of an arbitrary random matrix M can now be reduced to the scalar case. Indeed, the last equation of (3.4) is of the form

$$\mathbf{E}_t[Z(t+1)] = \lambda Z(t) + d(t),\ t \in \mathbb{Z}_+, \tag{3.5}$$

where λ is an element of $\mathbb{C}$ with $|\lambda| > 1$ and $\{d(l)\}_{l\in\mathbb{Z}_+}$ is a mean bounded random sequence. Hence, all we need to show is that any solution $Z(t)$ of equation (3.5) is bounded in mean. But this is the content of Theorem 3.2. It then implies that the Nth component $Y_N(t)$ of the solution $Y(t)$ of equation (3.4) is bounded in mean. Then, substituting $Y_N(t)$ in the $(N-1)$th equation of (3.4) we obtain again an equation of the form (3.5) for $Y_{N-1}(t)$; and so on. The proof is complete. □

4 Solutions to some conditional expectational singular difference equations

4.1 Linear case

We are first interested in the case when the forcing term f does not depend on x. Namely, we study the existence of bounded solutions of equation (1.1). In equation (1.1), $\mathbf{E}_t[\,.\,] = \mathbf{E}[\,.\mid \mathcal{F}_t]$ represents the conditional expectation where $\mathcal{F}_t = \sigma\{(X(l), g(l)) : l \le t\}$ is the smallest σ-algebra such that $(X(l), g(l))$ is measurable for all $l \le t$. We assume that g is independent of $X(0)$. This assumption together with equation (1.1) imply that g is independent of the sequence $\{X(t)\}_{t\in\mathbb{Z}}$.

Set $\rho(A, B) = \{\lambda \in \mathbb{C} : \lambda A + B \text{ is invertible}\}$. In order to proceed, we adopt the following assumption.

(H.1) (*Saddle-point property*): The number of generalized eigenvalues with modulus larger than 1 equals the number of non-predetermined variables.

The condition **(H.1)** means that there are as many non-predetermined variables as there are eigenvalues outside $\mathbb{S}^1$. For more details, see Blanchard and Kahn [5].

We now state one of our main results.

Theorem 4.1 *Suppose that* $\mathbb{S}^1 \subset \rho(A,B)$ *and that* **(H.1)** *holds. Then, equation* (1.1) *has a unique mean bounded solution.*

Proof. We borrow the Diagana–Pennequin [6] proof and adapt it in stochastic case. We set

$$\widehat{A} = (A+B)^{-1}A,\ \ \widehat{B} = (A+B)^{-1}B \ \text{ and } \ \widehat{g}(t) = (A+B)^{-1}g(t).$$

We can easily show that equation (1.1) is equivalent to

$$\widehat{A}\mathbf{E}_t[X(t+1)] + \widehat{B}X(t) = \widehat{g}(t),\ t \in \mathbb{Z}_+. \tag{4.1}$$

Using the identity $\widehat{A} + \widehat{B} = I_N$, we deduce that $\widehat{A}\widehat{B} = \widehat{B}\widehat{A}$. Consequently, one can find common basis of trigonalization for $\widehat{A}$ and $\widehat{B}$. Thus, there exists an invertible matrix T such that

$$\widehat{A} = T^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T$$

and

$$\widehat{B} = T^{-1}\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} T,$$

where A_1, B_2 are invertible and A_2, B_1 are nilpotent.

Noting $A_i + B_i$ is the identity matrix of the same size as A_i and letting

$$TX(t) = \begin{bmatrix} W(t) \\ V(t) \end{bmatrix}$$

and

$$T\widehat{g}(t) = \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix},$$

where $\{\alpha(t)\}$ and $\{\beta(t)\}$ are bounded random sequences, equation (4.1) can be written as follows

$$\begin{cases} A_1\mathbf{E}_t[W(t+1)] + B_1W(t) = \alpha(t), \\ A_2\mathbf{E}_t[V(t+1)] + B_2V(t) = \beta(t). \end{cases} \tag{4.2}$$

Using the fact that both A_1 and B_2 are invertible, one can show that equation (4.1) is equivalent to

$$\begin{cases} \mathbf{E}_t[W(t+1)] + A_1^{-1}B_1W(t) = A_1^{-1}\alpha(t), \\ B_2^{-1}A_2\mathbf{E}_t[V(t+1)] + V(t) = B_2^{-1}\beta(t). \end{cases} \tag{4.3}$$

Now, let us focus the first equations appearing in (4.3), that is,

$$\mathbf{E}_t[W(t+1)] - (-A_1^{-1}B_1)W(t) = A_1^{-1}\alpha(t),\ t \in \mathbb{Z}_+. \tag{4.4}$$

Clearly, $\{A_1^{-1}\alpha(l)\}_{l\in\mathbb{Z}_+}$ is bounded in mean. We shall now prove that $-A_1^{-1}B_1$ has no eigenvalue that belongs to $\mathbb{S}^1$. From that we will deduce that equation (4.4) has a unique bounded solution. For that, let us consider a non-zero eigenvalue λ of $-A_1^{-1}B_1$. Let $U \neq \mathbf{0}$. We have $-A_1^{-1}B_1 U = \lambda U$. Consequently,

$$(\lambda A + B)U = 0,$$

from which we deduce that

$$(\lambda \widehat{A} + \widehat{B})T^{-1}\begin{bmatrix} U \\ 0 \end{bmatrix} = 0.$$

Since

$$T^{-1}\begin{bmatrix} U \\ 0 \end{bmatrix} \neq 0,$$

it follows that $\lambda\widehat{A} + \widehat{B}$ is not invertible and so is the case for $\lambda A + B$. With the assumptions made, this proves that $|\lambda| \neq 1$.

Using assumption **(H.1)** and Remark 3.1, we can assume that $|\lambda| > 1$ and conclude that there exists a unique bounded solution $\{W(t)\}_{t\in\mathbb{Z}_+}$ to the first equation of (4.3).

For the second equation appearing in (4.3), setting $Y(t) = V(t+1)$ it becomes

$$B_2^{-1}A_2\mathbf{E}_t[Y(t)] + Y(t-1) = B_2^{-1}\beta(t). \tag{4.5}$$

Using similar arguments as before, we can prove that equation (4.5) has a unique bounded solution $\{Y(t)\}_{t\in\mathbb{Z}_+}$, so the second equation appearing in (4.3) has also a unique bounded solution $\{V(t)\}_{t\in\mathbb{Z}_+}$. Since equations (4.3) and (1.1) are equivalent, we obtain the existence and uniqueness of a bounded solution of equation (1.1). □

4.2 Nonlinear case

We now study the existence of bounded solutions to the following equation

$$A\mathbf{E}_t[X(t+1)] + BX(t) = h(t, X(t)),\ t \in \mathbb{Z}_+, \tag{4.6}$$

where A, B are $N \times N$ square matrices satisfying $\det A = \det B = 0$ and $h\colon \mathbb{Z}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a bounded sequence. Our setting requires the following assumption.

(H.2) The function $(t, w) \mapsto h(t, w)$ is bounded in mean in $t \in \mathbb{Z}_+$ uniformly in w in $\mathcal{O}$, where $\mathcal{O} = \{y \in \mathbb{R}^N : \|y\| \leq \eta\}$ for a fixed $\eta > 0$. In addition, we assume that there exists a constant $L > 0$ such that

$$\mathbf{E}\|h(t,U) - h(t,V)\| \leq L \cdot \mathbf{E}\|U - V\|_{\mathbb{R}^N} \text{ for all } U, V \in L^1(\Omega, \mathcal{O}),\ t \in \mathbb{Z}_+.$$

Theorem 4.2 *Suppose that $\mathbb{S}^1 \subset \rho(A, B)$ and that* **(H.1)** *and* **(H.2)** *hold. Then, for sufficiently small L, equation* (4.6) *has a unique bounded solution.*

Proof. Letting $g(t) = h(t, X(t))$ and using Lemma 2.2 together with similar arguments presented above, one can easily obtain the existence and uniqueness of a bounded solution to equation (4.6). □

4.3 Application to some conditional expectational second-order singular difference equations

In this section, we study the existence of solutions to second-order conditional expectational difference equation of type

$$A\mathbf{E}_t[X(t+1)] + BX(t) + CX(t-1) = h(t, X(t)),\ t \in \mathbb{Z}_+, \tag{4.7}$$

where A, B, C are $N \times N$ square matrices satisfying $\det A = \det B = \det C = 0$ and $h\colon \mathbb{Z} \times L^1(\Omega, \mathbb{R}^N) \to L^1(\Omega, \mathbb{R}^N)$ is mean bounded in the first variable uniformly in the second variable. In equation (4.7), $\mathbf{E}_t[\,.\,] = \mathbf{E}[\,.\mid \mathcal{F}_t]$, where $\mathcal{F}_t = \sigma\{(X(l), h(l,.)) : l \le t\}$ is the smallest σ-algebra such that $(X(l), h(l,.))$ is measurable for all $l \le t$.

In order to study the existence of bounded solution to equation (4.7), we make extensive use of the results obtained in the previous section. For that, we rewrite equation (4.7) as follows

$$\mathcal{L}\mathbf{E}_t[W(t+1)] + MW(t) = H(t, W(t)),\ t \in \mathbb{Z}_+, \tag{4.8}$$

where

$$\mathcal{L} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad M = \begin{bmatrix} B & C \\ -I & 0 \end{bmatrix}, \quad H = \begin{bmatrix} h \\ 0 \end{bmatrix} \quad \text{and} \quad W(t) = \begin{bmatrix} X(t) \\ X(t-1) \end{bmatrix}$$

with 0 and I being the $N \times N$ zero and identity matrices.

Lemma 4.3 *$\lambda\mathcal{L} + M$ is invertible if and only if $\lambda^2 A + \lambda B + C$ is invertible.*

Proof. The $2N \times N$ square matrix $\lambda\mathcal{L} + M$ is given by

$$\lambda\mathcal{L} + M = \begin{bmatrix} \lambda A + B & C \\ -I & \lambda \end{bmatrix}.$$

Consequently, solving

$$(\lambda\mathcal{L} + M)\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

yields $(\lambda A + B)U + CV = X$ and $-U + \lambda V = Y$. If $\lambda^2 A + \lambda B + C$ is invertible, then from $U = \lambda V - Y$ it follows that $(\lambda^2 A + \lambda B + C)V = X + (\lambda A + B)Y$ which yields

$$V = [\lambda^2 A + \lambda B + C]^{-1}(X + (\lambda A + B)Y),$$
$$U = \lambda\big\{[\lambda^2 A + \lambda B + C]^{-1}(X + (\lambda A + B)Y)\big\} - Y.$$

The latter implies that $\lambda\mathcal{L} + M$ is invertible.

The converse can be proven using similar arguments as before, and hence is omitted. □

Set $\rho(A, B, C) = \big\{\lambda \in \mathbb{C} : \lambda^2 A + \lambda B + C \text{ is invertible}\big\}$. Using Lemma 4.3 and Theorem 4.2, we obtain the following result.

Theorem 4.4 *Suppose that $\mathbb{S}^1 \subset \rho(A, B, C)$ and that* **(H.1)** *and* **(H.2)** *hold. Then, for sufficiently small L, equation* (4.7) *has a unique bounded solution.*

5 Asymptotic behavior of some time-varying conditional expectational difference equations

5.1 Time-varying conditional expectational first order difference equations

We begin with the following lemma which plays an important role in the sequel.

Lemma 5.1 *Let a be a positive integer, P be a projection* (i.e., $P^2 = P$)*, and $U(t)$ be an $N \times N$ invertible matrix defined for all $t \geq a+1$. Assume that there exists a constant $K > 1$ such that*

$$\sum_{s=a}^{t-1} \|U(t)PU^{-1}(s)\| \leq K \text{ for all } t \geq a+1.$$

Then, there exists a constant K_1 such that

$$\|U(t)P\| \leq K\|U(a+1)\| \left(\frac{K-1}{K}\right)^{t-a-1} \text{ for all } t \geq a+1. \tag{5.1}$$

Moreover, we have $\lim_{t\to\infty} \mathbf{E}\|U(t)P\| = 0$.

Proof. See Agarwal [1] or Schinas [10]. □

For conditional expectational linear difference equation, we have the following theorem.

Theorem 5.2 *Suppose that the conditional expectational difference equation* (2.2) *corresponding to equation* (1.2) *has a regular discrete dichotomy and that $g\colon Z_+ \to \mathbb{R}^N$ is bounded in mean. Then, equation* (1.2) *has a mean bounded solution which is given by*

$$\begin{aligned}\bar{X}(t) = &\sum_{j=-\infty}^{t-1} U(t)PU^{-1}(j+1)g(j) \\ &- \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)g(j)\right],\end{aligned} \tag{5.2}$$

where $U(t)PU^{-1}(j) = 0$ for $j > t$ and $g(j) = 0$ for $j < 0$.

Proof. We first show that $\bar{X}(t)$ defined by (5.2) is a solution of equation (1.2). For that, let $U(t)U^{-1}(s)$ be the fundamental matrix of (2.2). Then,

$$\begin{aligned}\mathbf{E}_t[\bar{X}(t+1)] - \mathcal{A}(t)\bar{X}(t) = &\sum_{j=-\infty}^{t} U(t+1)PU^{-1}(j+1)g(j)] \\ &- \mathbf{E}_t\left[\sum_{j=t+1}^{\infty} U(t+1)[I-P]U^{-1}(j+1)g(j)\right] \\ &- \mathcal{A}(t)\sum_{j=-\infty}^{t-1} U(t)PU^{-1}(j+1)g(j)\end{aligned}$$

$$+\mathcal{A}(t)\mathbf{E}_t\Bigg[\sum_{j=t}^{\infty}U(t)[I-P]U^{-1}(j+1)g(j)\Bigg].$$

Now, using the fact that $\mathcal{A}(t)U(t)U^{-1}(j+1)=U(t+1)U^{-1}(j+1)$, we can write

$$\begin{aligned}
\mathbf{E}_t[X(t+1)]-\mathcal{A}(t)X(t) &= \sum_{j=-\infty}^{t}U(t+1)PU^{-1}(j+1)g(j)\\
&\quad-\mathbf{E}_t\Bigg[\sum_{j=t+1}^{\infty}U(t+1)[I-P]U^{-1}(j+1)g(j)\Bigg]\\
&\quad-\sum_{j=-\infty}^{t-1}U(t+1)PU^{-1}(j+1)g(j)\\
&\quad+\mathbf{E}_t\Bigg[\sum_{j=t}^{\infty}U(t+1)[I-P]U^{-1}(j+1)g(j)\Bigg]\\
&=\sum_{j=-\infty}^{t-1}U(t+1)PU^{-1}(j+1)g(j)\\
&\quad+U(t+1)PU^{-1}(j+1)g(t)\\
&\quad-\mathbf{E}_t\Bigg[\sum_{j=t}^{\infty}U(t+1)[I-P]U^{-1}(j+1)g(j)\Bigg]\\
&\quad+\mathbf{E}_t\big[U(t+1)[I-P]U^{-1}(t+1)g(j)\big]\\
&\quad-\sum_{j=-\infty}^{t-1}U(t+1)PU^{-1}(j+1)g(j)\\
&\quad+\mathbf{E}_t\Bigg[\sum_{j=t}^{\infty}U(t+1)[I-P]U^{-1}(j+1)g(j)\Bigg]\\
&=Pg(j)+\mathbf{E}_t\big[[1-P]g(j)\big].
\end{aligned}$$

Taking the conditional expectation $\mathbf{E}_t[\,.\,]$ on both sides, we conclude that

$$\mathbf{E}_t[X(t+1)]-\mathcal{A}(t)X(t)=\big[\mathbf{E}_t[P]+\mathbf{E}_t[1-P]\big]g(t)=g(t).$$

It remains to prove that $\bar{X}$ is bounded in mean. By the discrete dichotomy assumption and boundedness of g, we have

$$\begin{aligned}
\mathbf{E}\|\bar{X}(t)\| &\le \sum_{r=-\infty}^{t-1}\mathbf{E}\|U(t)PU^{-1}(r+1)g(r)\|+\sum_{r=t}^{\infty}\mathbf{E}\|U(t)[I-P]U^{-1}(r+1)g(r)\|\\
&\le\left\{\sum_{r=-\infty}^{t-1}M\beta^{t-r-1}+\sum_{r=t}^{\infty}M\beta^{t-r-1}\right\}\sup_{s\in\mathbb{Z}_+}\mathbf{E}\|g(s)\|=M\frac{1+\beta}{1-\beta}\,\|g\|_\infty.
\end{aligned}$$

Thus, the desired result will follow. □

In order to state similar results for the nonlinear case, we set $\mathcal{O} = \left\{y \in \mathbb{R}^N : \|y\| \leq \eta\right\}$ for a fixed $\eta > 0$ and take a function $F\colon Z_+ \times L^1(\Omega, \mathcal{O}) \to L^1(\Omega, \mathbb{R}^N)$, $(t, X) \mapsto F(t, X)$, with $F(t, 0) = 0$. Let us consider the following conditional expectational nonlinear difference equation

$$\mathbf{E}_t[X(t+1)] = \mathcal{A}(t)X(t) + F(t, X(t)), \;\; t \in \mathbb{Z}_+, \tag{5.3}$$

where F satisfies the following conditions.

(H.3) There exists a function γ from $\mathbb{Z}_+$ to $\mathbb{R}_+$ such that

$$\mathbf{E}\|F(t, U) - F(t, V)\| \leq \gamma(t)\, \mathbf{E}\|U - V\|$$

for all $\mathcal{O}$-valued random variables U, V with finite expectation and $t \in \mathbb{Z}_+$. In addition, we assume that

$$\sum_{s=1}^{\infty} \gamma(s) < \infty.$$

Theorem 5.3 *Suppose that the conditional expectational difference equation* (2.2) *corresponding to equation* (5.3) *has a regular discrete dichotomy with positive constants $M > 0$ and $\beta \in (0, 1)$ and that F satisfies* **(H.3)**. *Then, equation* (5.3) *has a unique mean bounded solution which is given by*

$$\begin{aligned} \bar{X}(t) = &\sum_{j=-\infty}^{t-1} U(t)PU^{-1}(j+1)F(j, X(j)) \\ &- \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)F(j, X(j))\right], \end{aligned} \tag{5.4}$$

where $U(t)PU^{-1}(j) = 0$ for $j > t$ and $F(j, .) = 0$ for $j < 0$ provided that

$$\frac{M\Gamma(\beta + 1)}{1 - \beta} < 1 \;\; \textit{with} \;\; \Gamma = \sup_{t \in \mathbb{Z}_+} \gamma(t)\,.$$

In addition, $\mathbf{E}\|\bar{X}(t)\| \to 0$ as $t \to \infty$.

Proof. Consider the Banach space $UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N))$ with the supremum norm. Setting $g(r) := F(r, \varphi(r))$, it follows from Theorem 2.2 that if $\varphi \in UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N))$, then $g \in UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N))$. Now, define $\Lambda\colon UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N)) \to UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N))$ to be the nonlinear operator given by

$$\begin{aligned} (\Lambda\varphi)(t) := &\sum_{j=-\infty}^{t-1} U(t)PU^{-1}(j+1)F(j, \varphi(j)) \\ &- \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)F(j, \varphi(j))\right]. \end{aligned}$$

Using the proof of Theorem 5.2, one can easily show that $\Lambda(UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N))) \subset UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{R}^N))$, and hence Λ is well-defined. Moreover, for $\varphi,\ \psi \in UB(\mathbb{Z}_+; L^1(\Omega, \mathbb{B}))$ having the same property as X defined in equation (5.3), we have

$$\begin{aligned}
&\mathbf{E}\|(\Lambda\varphi)(t) - (\Lambda\psi)(t)\| \\
&\quad = \mathbf{E}\Bigg\| \sum_{j=-\infty}^{t-1} U(t)PU^{-1}(j+1)[F(j,\varphi(j)) - F(j,\psi(j))] \\
&\qquad - \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)[F(j,\varphi(j)) - F(j,\psi(j))]\right]\Bigg\| \\
&\quad \le \sum_{j=-\infty}^{t-1} M\beta^{t-j-1}\gamma(j)\,\mathbf{E}\big\|\varphi(j) - \psi(j)\big\| \\
&\qquad + \sum_{j=t}^{\infty} M\beta^{j+1-t}\gamma(j)\,\mathbf{E}\big\|\varphi(j) - \psi(j)\big\| \\
&\quad \le \left\{\sum_{j=-\infty}^{t-1} M\beta^{t-j-1} + \sum_{j=t}^{\infty} M\beta^{j+1-t}\right\}\Gamma \sup_{r\in\mathbb{Z}_+} \mathbf{E}\|\varphi(r) - \psi(r)\| \\
&\quad \le \Gamma ML\,\frac{\beta+1}{1-\beta}\,\|\varphi - \psi\|_\infty,
\end{aligned}$$

where $\Gamma = \sup_{r\in\mathbb{Z}_+} |\gamma(r)|$. Thus,

$$\|\Lambda\varphi - \Lambda\psi\|_\infty \le \Gamma M\,\frac{\beta+1}{1-\beta}\,\|\varphi - \psi\|_\infty\,.$$

This means that Λ is a contraction provided that $\frac{M\Gamma(\beta+1)}{1-\beta} < 1$. Using the Banach fixed point theorem, we deduce that Λ has a unique fixed point $\bar{X}$, which is the unique bounded solution of equation (5.3).

We now prove that $\lim_{t\to\infty} \mathbf{E}\|\bar{X}(t)\| = 0$. To this end, fix a positive integer t_0. For $t \ge t_0$, using independence we have

$$\begin{aligned}
\mathbf{E}\|\bar{X}(t)\| \le{}& \mathbf{E}\|U(t)P\|\,\mathbf{E}\|U^{-1}(t_0)\|\,\mathbf{E}\|X(t_0)\| \\
&+ \sum_{j=t_0}^{t-1} \mathbf{E}\|U(t)PU^{-1}(j+1)\|\,\gamma(j)\mathbf{E}\|X(j)\| \\
&+ \sum_{j=t}^{\infty} \mathbf{E}\|U(t)[I-P]U^{-1}(j+1)\|\,\gamma(j)\mathbf{E}\|X(j)\|.
\end{aligned}$$

Now, fix $\varepsilon > 0$ and choose t_1 large enough so that $M\eta\sum_{j=t}^{\infty}\gamma(j) < \varepsilon$ for any $t \ge t_1$. Also, since equation (2.2) has a regular discrete dichotomy, we know that

$$\mathbf{E}\|U(t)P\,U^{-1}(j+1)\| \le M$$

and

$$\mathbf{E}\|U(t)[I-P]\,U^{-1}(j+1)\| \le M.$$

We can then write

$$\mathbf{E}\|\bar{X}(t)\| \leq \mathbf{E}\|U(t)P\|\Big\{\mathbf{E}\|U^{-1}(t_0)\|\,\mathbf{E}\|X(t_0)\| + \eta\sum_{j=t_0}^{t_1-1}\mathbf{E}\|U^{-1}(j+1)\|\,\gamma(j)\Big\} + 2\eta M\sum_{j=t_1}^{\infty}\gamma(j).$$

Hence,

$$\mathbf{E}\|\bar{X}(t)\| \leq \mathbf{E}\|U(t)P\|\Big\{\mathbf{E}\|U^{-1}(t_0)\|\,\mathbf{E}\|X(t_0)\| + \eta\sum_{j=t_0}^{t_1-1}\mathbf{E}\|U^{-1}(j+1)\|\,\gamma(j)\Big\} + 2\varepsilon$$

for any $t \geq t_1$. Letting $\varepsilon \to 0$, we get

$$\mathbf{E}\|\bar{X}(t)\| \leq \mathbf{E}\|U(t)P\|\Big\{\mathbf{E}\|U^{-1}(t_0)\|\,\mathbf{E}\|X(t_0)\| + \eta\sum_{j=t_0}^{t_1-1}\mathbf{E}\|U^{-1}(j+1)\|\,\gamma(j)\Big\}$$

for any $t \geq t_1$. Now, using Lemma 5.1, we conclude that $\lim_{t\to\infty}\mathbf{E}\|\bar{X}(t)\| = 0$. □

To study the conditional stability, let us consider the following conditional expectational difference equations

$$\mathbf{E}_t[Z(t+1)] = \mathcal{A}(t)Z(t) + f_n(t),\ \ t \in \mathbb{Z}_+,\ \ Z(t_0) = z_0, \tag{5.5}$$

and

$$\mathbf{E}_t[Y(t+1)] = \mathcal{A}(t)Y(t) + F(t, Y(t)) + f_n(t),\ \ t \in \mathbb{Z}_+,\ \ Y(t_0) = y_0, \tag{5.6}$$

where F satisfies **(H.3)** and f_n is a non-random function: $t \mapsto f_n(t)$ from $\mathbb{Z}_+$ to $\mathbb{R}^N$ satisfying

(H.4) $f_n(0) = 0$ and $f_n(t) \to 0$ as $n \to \infty$ uniformly in t.

Theorem 5.4 *Suppose that the conditional expectational difference equation* (2.2) *corresponding to equation* (5.6) *has a regular discrete dichotomy with positive constants $M > 0$ and $\beta \in (0,1)$ and that F and f_n satisfy* **(H.3)** *and* **(H.4)***, respectively. Let $\bar{X}(t)$ be a solution of equation* (5.3) *and let $\bar{X}_n(t)$ be a solution of equation* (5.6)*. Assume that $\mathbf{E}\|\bar{X}_n(0) - \bar{X}(0)\| \to 0$ as $n \to \infty$. Then, the following holds*

(a) *$\bar{X}_n(t)$ can be uniquely decomposed as follows: $\bar{X}_n(t) = \bar{X}(t) + \bar{Z}_n(t)$, where $\bar{Z}_n(t)$ is a solution of equation* (5.5)*,*

(b) $\mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| \to 0$ *as $n \to \infty$ uniformly in t,*

(c) $\mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| \to 0$ *as $t \to \infty$ for all n.*

Proof. We first show that $\bar{X}_n(t) - \bar{X}(t)$ is a solution of (5.5). We have

$$
\begin{aligned}
&\bar{X}_n(t+1) - \bar{X}(t+1) \\
&= U(t+1)PX_n(t_0) + \sum_{j=t_0+1}^{t} U(t+1)PU^{-1}(j+1)[F(j,X(j)) + f_n(j)] \\
&\quad - \mathbf{E}_t\left[\sum_{j=t+1}^{\infty} U(t+1)[I-P]U^{-1}(j+1)[F(j,X(j)) + f_n(j)]\right] \\
&\quad - U(t+1)PX_n(t_0) + \sum_{j=t_0+1}^{t} U(t+1)PU^{-1}(j+1)[F(j,X(j))] \\
&\quad + \mathbf{E}_t\left[\sum_{j=t+1}^{\infty} U(t+1)[I-P]U^{-1}(j+1)F(j,X(j))\right] \\
&= U(t+1)P[X_n(t_0) - X(t_0)] + \sum_{j=t_0+1}^{t} U(t+1)PU^{-1}(j+1)f_n(j) \\
&\quad - \mathbf{E}_t\left[\sum_{j=t+1}^{\infty} U(t+1)[I-P]U^{-1}(j+1)f_n(j)\right] \\
&= \mathcal{A}(t)\Bigg\{U(t)P[X_n(t_0) - X(t_0)] + \sum_{j=t_0+1}^{t} U(t)PU^{-1}(j+1)f_n(j) \\
&\quad - \mathbf{E}_t\left[\sum_{j=t+1}^{\infty} U(t)[I-P]U^{-1}(j+1)f_n(j)\right]\Bigg\} \\
&= \mathcal{A}(t)\Bigg\{U(t)P[X_n(t_0) - X(t_0)] + \sum_{j=t_0+1}^{t-1} U(t)PU^{-1}(j+1)f_n(j) \\
&\quad - \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)f_n(j)\right]\Bigg\} \\
&\quad + U(t+1)PU^{-1}(t+1)f_n(j) + \mathbf{E}_t\big[U(t+1)[I-P]U^{-1}(t+1)f_n(j)\big] \\
&= \mathcal{A}(t)[X_n(t) - X(t)] + \big[P + \mathbf{E}_t[I-P]\big]f_n(t).
\end{aligned}
$$

Taking the conditional expectation $\mathbf{E}_t[\,.\,]$ on both sides, we conclude that

$$\mathbf{E}_t[\bar{X}_n(t+1) - \bar{X}(t+1)] = \mathcal{A}(t)[\bar{X}_n(t) - \bar{X}(t)] + f_n(t).$$

Moreover, keeping in mind that $\bar{X}(t)$ is a solution of equation (5.3) and that $\bar{Z}_n(t)$ is a solution of equation (5.5), $X_n(t)$ can be decomposed as follows

$$
\begin{aligned}
\bar{X}_n(t) = U(t)P[X_n(t_0) - X(t_0)] &+ \sum_{j=t_0+1}^{t-1} U(t)PU^{-1}(j+1)f_n(j) \\
&- \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)f_n(j)\right]
\end{aligned}
$$

$$+ U(t)PX(t_0) + \sum_{j=t_0+1}^{t-1} U(t)PU^{-1}(j+1)F(j,X(j))$$
$$- \mathbf{E}_t\left[\sum_{j=t}^{\infty} U(t)[I-P]U^{-1}(j+1)F(j,X(j))\right].$$

Now, let $\mathcal{Z}$ be the set of solutions of equation (5.5) and $\mathcal{X}$ be the set of solutions of equation (5.3).

Letting $Z_n(t_0) = X_n(t_0) - X(t_0)$, we obtain $\bar{X}_n(t) = \bar{Z}_n(t) + \bar{X}(t)$, and since $\mathcal{Z} \cap \mathcal{X} = \emptyset$, this decomposition is unique.

Next, let us compute $\lim_{n\to\infty} \mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\|$. Using the independence and the property of conditional expectation, we have

$$\begin{aligned}
\mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| &\\
\le \mathbf{E}\|U(t)P\|\,\mathbf{E}\|X_n(t_0) - X(t_0)\| &\\
+ \sum_{j=t_0+1}^{t-1} \mathbf{E}\|U(t)PU^{-1}(j+1)\|\,\|f_n(j)\| &\\
+ \sum_{j=t}^{\infty} \mathbf{E}\|U(t)[I-P]U^{-1}(j+1)\|\,\|f_n(j)\| &\\
\le \mathbf{E}\|U(t)P\|\,\mathbf{E}\|X_n(t_0) - X(t_0)\| + \sum_{j=t_0+1}^{t-1} M\beta^{t-1-j}\,\|f_n(j)\| &\\
+ \sum_{j=t}^{\infty} M\beta^{j+1-t}\,\|f_n(j)\|. &
\end{aligned}$$

This implies that $\lim_{n\to\infty} \mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| = 0$ for all $t \ge t_0$.

We now prove that $\lim_{t\to\infty} \mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| = 0$ for each n. To this end, fix $\varepsilon > 0$ and choose $B > 0$ such that $\|f_n(t)\| \le B$ for all n and t, and pick t_1 large enough so that $\sum_{j=t_1}^{\infty} \beta^{j+1-t_1} < \frac{\varepsilon}{2B}$. Then, for any $t \ge t_1 > t_0$, we have

$$\begin{aligned}
\mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| &\le \mathbf{E}\|U(t)P\|\left[\mathbf{E}\|X_n(t_0) - X(t_0)\| + \sum_{j=t_0+1}^{t_1-1} \mathbf{E}\|U^{-1}(j+1)\|\,\|f_n(j)\|\right]\\
&\quad + \sum_{j=t_1}^{\infty} \mathbf{E}\|U(t)PU^{-1}(j+1)\|\,\|f_n(j)\|\\
&\quad + \sum_{j=t_1}^{\infty} \mathbf{E}\|U(t)[I-P]U^{-1}(j+1)\|\,\|f_n(j)\|\\
&\le \mathbf{E}\|U(t)P\|\left[2\eta + \sum_{j=t_0+1}^{t_1-1} \mathbf{E}\|U^{-1}(j+1)\|\,\|f_n(j)\|\right]\\
&\quad + 2\sum_{j=t_1}^{\infty} M\beta^{j-t_1}\,\|f_n(j)\|\\
&\le \mathbf{E}\|U(t)P\|\left[2\eta + \sum_{j=t_0+1}^{t_1-1} \mathbf{E}\|U^{-1}(j+1)\|\,B\right] + \varepsilon.
\end{aligned}$$

Hence, $\lim_{t\to\infty} \mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| = 0$ for each n. We can conclude that the assertion of the above theorem shows us the conditional stability of the solution $\bar{X}$ in the sense that if $\bar{X}(t)$ is a solution of equation (5.3), there exists a sequence $\bar{X}_n(t)$ of solutions of equation (5.6) such that

$$\lim_{n\to\infty} \mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| = 0$$

uniformly in $t \geq t_0$. In addition, $\lim_{t\to\infty} \mathbf{E}\|\bar{X}_n(t) - \bar{X}(t)\| = 0$ for each n. □

Example 5.5 *In this example, the Euclidean space $\mathbb{R}^2$ is equipped with the norm $\| \, . \, \|$, which is defined for all $u = (u_1, u_2) \in \mathbb{R}^2$ by $\|u\| = |u_1| + |u_2|$.*

Let α be a random variable taking its values in $(0, 1)$ and let $\xi_t = (\xi_t^1, \xi_t^2)$, $t \in \mathbb{Z}_+$, be a sequence of random vectors in $\mathbb{R}^2$ such that $\sup_{t\in\mathbb{Z}_+} \mathbf{E}\|\xi_t\| \leq L$ for some $L > 0$. Set

$$\mathcal{A}(t) = \alpha \begin{bmatrix} \beta^t & 0 \\ 0 & \beta^{-t} \end{bmatrix}$$

and define $F\colon \mathbb{Z}_+ \times L^1(\Omega, \mathcal{O}) \to \mathbb{R}^2$ by

$$F(t, X) = \begin{bmatrix} \beta^t \, \xi_t^1 \, \tan^{-1} X_1 \\ \beta^t \, \xi_t^2 \, \tan^{-1} X_2 \end{bmatrix},$$

where $\mathcal{O} = \{y \in \mathbb{R}^2 : \|y\| \leq \eta\}$ for any $\eta > 0$ fixed and $X = (X_1, X_2)^T$. Then, if the projection P is given by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} \|U(t) \, P \, U^{-1}(s)\| &\leq \beta^{\frac{(t-s)(t+s-1)}{2}} \leq \beta^{t-s} \text{ for all } t \geq s, \\ \|U(t) \, [I - P] \, U^{-1}(s)\| &\leq \beta^{\frac{(s-t)(t+s-1)}{2}} \leq \beta^{t-s} \text{ for all } s \geq t, \\ \mathbf{E}\|F(t, X) - F(t, Y)\| &\leq L \, \beta^t \, \mathbf{E}\|X - Y\| = \gamma(t)\mathbf{E}\|X - Y\|, \end{aligned}$$

and

$$\sum_{s=1}^{\infty} \beta^t = \frac{\beta}{1 - \beta} < \infty.$$

Therefore, all hypotheses of Theorems 5.3 *and* 5.4 *are satisfied.*

5.2 Time-varying conditional expectational second order difference equations

Let $\mathbb{B} = \mathbb{R}^k$ be the k-dimensional space of real numbers equipped with Euclidean topology. This subsection deals with time-varying conditional expectational second-order difference equations on $\mathbb{R}^k$ of the form

$$\mathcal{E}_t[X(t+1)] + A_1(t)X(t) + A_0(t)X(t-1) = f(t, X(t)), \; t \in \mathbb{Z}_+, \tag{5.7}$$

where $A_0(t)$ and $A_1(t)$ are invertible $k \times k$ random matrices, and the forcing term $f\colon \mathbb{Z}_+ \times \mathbb{R}^k \to \mathbb{R}^k$ is bounded in mean and satisfies **(H.3)**. Here $\mathcal{E}_t[\, . \,] = \mathbf{E}[\, . \mid \mathcal{G}_t]$ with $\mathcal{G}_t =$

$\sigma\{(X(s), A_0(s), A_1(s), F(s,.)) : s \leq t\}$. We assume that $A_0(t)$ and $A_1(t)$ are independent and independent of $X(0)$. This assumption together with equation (5.7) imply that the sequence $(A_0(t), A_1(t))$ is independent of the sequence $\{X(t)\}_{t\in\mathbb{Z}_+}$.

We now study the existence and uniqueness of solutions of equation (5.7). For that, the main idea consists in rewriting equation (5.7) as a conditional expectational first-order difference equations on $(\mathbb{R}^k)^2 = \mathbb{R}^k \times \mathbb{R}^k$. Indeed, setting $Z(t) := (X(t), X(t-1))^T$, where the symbol T stands for the transpose operation and if I denotes the identity matrix of $\mathbb{R}^k$, then equation (5.7) can be rewritten in $(\mathbb{R}^k)^2$ in the following form

$$\mathcal{E}_t[Z(t+1)] = \mathcal{A}(t)Z(t) + F(t, Z(t)), \ \ t \in \mathbb{Z}_+, \tag{5.8}$$

and its corresponding homogeneous equation

$$\mathcal{E}_t[Z(t+1)] = \mathcal{A}(t)Z(t), \ \ t \in \mathbb{Z}_+, \tag{5.9}$$

where $\mathcal{A}(t)$ is the family of time-dependent sequence matrices defined by

$$\mathcal{A}(t) = \begin{pmatrix} -A_1(t) & -A_0(t) \\ I & 0 \end{pmatrix}$$

and the function F appearing in equation (5.8) is defined by $F(t, Z) = (f(t, X), 0)^T$.

Corollary 5.6 *Under assumption* **(H.3)***, if the conditional expectational difference equation* (5.9) *corresponding to equation* (5.8) *has a discrete regular dichotomy with dichotomy constants* $M > 0$ *and* $\beta \in (0, 1)$*, then equation* (5.7) *has a unique bounded solution* $\bar{X}$ *provided that*

$$\frac{M\Gamma(\beta+1)}{1-\beta} < 1 \ \ \textit{with} \ \ \Gamma = \sup_{t\in\mathbb{Z}_+} \gamma(t).$$

In addition, $\mathbf{E}\|\bar{X}(t)\| \to 0$ *as* $t \to \infty$.

Proof. Following the same lines as in the proof of Theorem 5.3 it follows that equation (5.8) has a unique bounded solution given by the mapping

$$t \mapsto Z(t) := (X(t), X(t-1))^T,$$

provided that

$$\frac{M\Gamma(\beta+1)}{1-\beta} < 1 \ \ \text{with} \ \ \Gamma = \sup_{t\in\mathbb{Z}_+} \gamma(t).$$

In addition, $\mathbf{E}\|\bar{Z}(t)\| \to 0$ as $t \to \infty$. Therefore, equation (5.7) has a unique bounded solution $\bar{X}(t)$. □

Example 5.7 *To illustrate Corollary* 5.6*, we study the existence of solutions to some conditional expectational second-order difference equation on* $\mathbb{R}$ *of the form*

$$\mathcal{E}_t[X(t+1)] + b(t)X(t) + a(t)X(t-1) = f(t, X(t)), \ t \in \mathbb{Z}_+, \tag{5.10}$$

where f *satisfies* **(H.3)***. We also assume that the sequences* $a = \{a(t), t \in \mathbb{Z}_+\}$ *and* $b = \{b(t), t \in \mathbb{Z}_+\}$ *of real random variables appearing in equation* (5.10) *are independent and independent of*

$X(0)$. *This assumption together with equation* (5.10) *imply that* (a, b) *is independent of the sequence* $X = \{X(t)\}_{t\in\mathbb{Z}_+}$.

Setting $Z(t) := (X(t), X(t-1))^T$, *note that equation* (5.10) *can be rewritten in* $\mathbb{R}^2$ *as follows*

$$\mathcal{E}_t[Z(t+1)] = \mathcal{A}(t)Z(t) + F(t, Z(t)),\ \ t \in \mathbb{Z}_+, \tag{5.11}$$

and its corresponding homogeneous equation

$$\mathcal{E}_t[Z(t+1)] = \mathcal{A}(t)Z(t),\ \ t \in \mathbb{Z}_+, \tag{5.12}$$

where $\mathcal{A}(t)$ *is the family of time-dependent random matrices defined by*

$$\mathcal{A}(t) = \begin{pmatrix} -b(t) & -a(t) \\ 1 & 0 \end{pmatrix}$$

and the function F *appearing in equation* (5.11) *is defined by* $F(t, Z) = (f(t, X), 0)^T$.

We adopt the following assumptions.

(H.5) *There exist* $a_0, b_0 > 0$ *such that* $\inf_{t\in\mathbb{Z}_+} a(t) = a_0$ *and* $\inf_{t\in\mathbb{Z}_+} b(t) = b_0$, *almost surely.*

(H.6) $b(t) \neq 2\sqrt{a(t)}$ *for all* $t \in \mathbb{Z}_+$, *almost surely.*

Next, we show that equation (5.12) *has a regular discrete dichotomy. For that, let us compute the eigenvalues of* $\mathcal{A}(t)$. *We have*

$$P_t(\lambda) = \det(\mathcal{A}(t) - \lambda I_{\mathbb{R}^2}) = \lambda^2 + b(t)\lambda + a(t)$$

for all $t \in \mathbb{Z}_+$. *Then, the characteristic equation is given by*

$$\lambda^2 + b(t)\lambda + a(t) = 0$$

with discriminant given by $D(t) = b^2(t) - 4a(t)$ *for all* $t \in \mathbb{Z}_+$. *Clearly,* **(H.6)** *yields either* $D(t) > 0$ *or* $D(t) < 0$ *for all* $t \in \mathbb{Z}_+$. *Under assumptions* **(H.5)** *and* **(H.6)**, *we have two cases.*

1. *If* $D(t) > 0$ *for all* $t \in \mathbb{Z}_+$, *then the eigenvalues of* $\mathcal{A}(t)$ *are given by*

$$\lambda_1(t) = \frac{-b(t) + \sqrt{b^2(t) - 4a(t)}}{2} \quad \text{and} \quad \lambda_1(t) = \frac{-b(t) - \sqrt{b^2(t) - 4a(t)}}{2}.$$

   Moreover, it can be shown easily that $\lambda_1(t), \lambda_2(t) < 0$ *for all* $t \in \mathbb{Z}_+$.

2. *If* $D(t) < 0$ *for all* $t \in \mathbb{Z}_+$, *then the eigenvalues of* $\mathcal{A}(t)$ *are given by*

$$\lambda_1(t) = \frac{-b(t) + i\sqrt{4a(t) - b^2(t)}}{2} \quad \text{and} \quad \lambda_1(t) = \frac{-b(t) - i\sqrt{4a(t) - b^2(t)}}{2}.$$

   Moreover, it can be shown easily that $\mathcal{R}e\lambda_1(t), \mathcal{R}e\lambda_2(t) < 0$ *for all* $t \in \mathbb{Z}_+$.

In view of the above, it follows that

$$U(t)U^{-1}(s) = \prod_{r=s}^{t-1} \mathcal{A}(r) = \mathcal{A}(s)\mathcal{A}(s+1)\mathcal{A}(s+2)\ldots\mathcal{A}(t-2)\mathcal{A}(t-1)$$

for all $(t,s) \in \mathcal{T}$*, where* $\mathcal{T} = \{(t,s) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : t \geq s\}$*, has exponential dichotomy which yields* (*see Henry* [7]) *that equation* (5.12) *has discrete dichotomy.*

We can now conclude that under assumptions **(H.3)**–**(H.6)***, all hypotheses of Corollary* 5.6 *are satisfied.*

Remark 5.8 *The solution process to equation* (1.2) *has a nice predictable compensator. One may use Burkholder–Davis Gundy inequalities for martingale obtained as a difference between the solution and the compensator to analyze the asymptotic behavior.*

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