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ON THE SOLUTIONS OF A FRACTIONAL DIFFERENTIAL INCLUSION OF CAPUTO–FABRIZIO TYPE

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Abstract. We study a Cauchy problem associated to a fractional differential inclusion defined by Caputo–Fabrizio fractional derivative. We establish an existence result, we obtain a continuous selection of the solution set and we prove the arcwise connectedness of the solution set of the considered problem.

Keywords: Differential inclusion, fractional derivative, measurable selection.

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1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([5, 13, 15, 18] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [7] allows to use Cauchy conditions which have physical meaning.

Recently, a new fractional order derivative with regular kernel has been introduced by Caputo and Fabrizio [8]. The Caputo–Fabrizio operator is useful for modelling several classes of problems with the dynamics having the exponential decay law. This new definition is able to describe better

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heterogeneousness, systems with different scales with memory effects, the wave movement on surface of shallow water, the heat transfer model, mass-spring-damper model etc. ([1, 2, 17, 20]). Another good property of this new definition is that using Laplace transform of the fractional derivative, the fractional differential equation turns into a classical differential equation of integer order. Properties of this definition have been studied in *e.g.* [3, 8, 9, 19]. Some recent qualitative results for fractional differential equations defined by Caputo–Fabrizio fractional derivative were studied in *e.g.* [21, 22, 23].

In the present paper we study an initial value problem associated to a differential inclusion defined by Caputo–Fabrizio operator. More exactly, we consider the following problem

$$D^{\sigma}_{CF}x(t) \in F(t, x(t)) \text{ a.e. in } [0, T], \quad x(0) = x_0,\ x'(0) = x_1, \tag{1.1}$$

where $F(.,.)\colon [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map, $x_0, x_1 \in \mathbb{R}$, D^{σ}_{CF} denotes Caputo–Fabrizio's fractional derivative of order $\sigma \in (1, 2)$ and $\mathcal{P}(\mathbb{R})$ is the family of all non-empty subsets of $\mathbb{R}$.

Our goal is to present several qualitative results for solutions of problem (1.1). First, we show that Filippov's ideas [14] can be suitably adapted in order to obtain the existence of solutions for problem (1.1). We recall that for a differential inclusion defined by a Lipschitz continuous set-valued map with non-convex values, Filippov's theorem [14] consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion. Secondly, we prove the existence of a continuous selection of the set of solutions of the considered problem. The key tool in this approach is a result of Bressan and Colombo concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. Finally, using a result from [16] concerning the arcwise connectedness of the fixed point set of a class of set-valued contractions, we obtain the arcwise connectedness of the solution set of problem (1.1).

The common point of all these results is that the set-valued map that defines the problem has non-convex values. Instead of convexity, our assumption is that F is Lipschitz continuous in the state variable. We also note that even if similar results exist in the literature for other problems defined by fractional differential inclusions (e.g., [10, 11, 12] etc.), our results are new in the framework of Caputo–Fabrizio fractional differential inclusions.

The paper is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel. Section 3 is devoted to the Filippov type existence theorem. In Section 3 we obtain a continuous version of the result from the previous section and in Section 4 we obtain the arcwise connectedness of the solution set.

2 Preliminaries

Let us denote by I the interval $[0, T]$, $T > 0$, and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.)\colon I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.)\colon I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\, \mathrm{d}t$.

Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all non-empty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$, then

$\chi_A(.)\colon I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote its closure by $\overline{A}$.

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \setminus A$. We denote by $\mathcal{D}(I,X)$ the family of all decomposable closed subsets of $L^1(I,X)$.

Recall that the Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by the formula $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, where $d^*(A,B) = \sup\{d(a,B) : a \in A\}$ and $d(x,B) = \inf_{y \in B} d(x,y)$.

The next definitions were introduced in [8].

Definition 2.1

(a) *The Caputo–Fabrizio integral of order $\alpha \in (0,1)$ of a function $f \in AC_{loc}([0,\infty),\mathbb{R})$ (which means that $f'(.)$ is integrable on $[0,T]$ for any $T > 0$) is defined by*

$$I^{\alpha}_{CF} f(t) = (1-\alpha) f(t) + \alpha \int_0^t f(s)\,\mathrm{d}s.$$

(b) *The Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$ of f is defined for $t \geq 0$ by*

$$D^{\alpha}_{CF} f(t) = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)\,\mathrm{d}s.$$

(c) *The Caputo–Fabrizio fractional derivative of order $\sigma = \alpha + n$ of f, where $\alpha \in (0,1)$ and $n \in \mathbb{N}$, is defined by*

$$D^{\sigma}_{CF} f(t) = D^{\alpha}_{CF}(D^{n}_{CF} f(t)).$$

In particular, if $\sigma = \alpha + 1$, then $\alpha \in (0,1)$ and $D^{\sigma}_{CF} f(t) = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f''(s)\,\mathrm{d}s$.

Definition 2.2 *A mapping $x(.) \in AC(I,\mathbb{R})$ is called a solution of problem* (1.1) *if there exists a function $f(.) \in L^1(I,\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. (I), $D^{\alpha}_{CF} x(t) = f(t)$, $t \in I$, and $x(0) = x_0$, $x'(0) = x_1$.*

We shall use the following notation for the solution sets of (1.1):

$$\mathcal{S}(x_0,x_1) = \{x(.) : x(.) \text{ is a solution of (1.1)}\}.$$

The next result is proved in [22] (Theorem 3.4 and Remark 3.5).

Lemma 2.3 *For $\sigma = \alpha + 1$, where $\alpha \in (0,1)$, and $f(.) \in L^1(I,\mathbb{R})$ the initial value problem*

$$D^{\sigma}_{CF} x(t) = f(t), \quad x(0) = x_0,\ x'(0) = x_1,$$

has a unique solution given by

$$x(t) = x_0 + x_1 t + (1-\alpha) \int_0^t f(s)\,\mathrm{d}s + \alpha \int_0^t (t-s) f(s)\,\mathrm{d}s. \tag{2.1}$$

Remark 2.4 *If we define $\mathcal{G}(t,s) = (1-\alpha) + \alpha(t-s)$, then the solution in* (2.1) *may be written as $x(t) = x_0 + x_1 t + \int_0^t \mathcal{G}(t,s) f(s)\,\mathrm{d}s$. Moreover, for any $s,t \in I$ we have $|\mathcal{G}(t,s)| \leq (1-\alpha) + \alpha T =: M$.*

3 An existence result

First we recall a selection result established in [4] which is a version of the celebrated Kuratowski and Ryll–Nardzewski selection theorem.

Lemma 3.1 *Let X be a separable Banach space, B be the closed unit ball in X, $H\colon I \to \mathcal{P}(X)$ be a set-valued map with non-empty closed values and $g\colon I \to X$, $L\colon I \to \mathbb{R}_+$ be measurable functions. If $H(t) \cap (g(t) + L(t)B) \neq \emptyset$ a.e. (I), then the set-valued map $t \mapsto H(t) \cap (g(t) + L(t)B)$ has a measurable selection.*

In the sequel we assume the following conditions on F.

Hypothesis H1

(i) $F\colon I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has non-empty closed values and $F(.,x)$ is measurable for every $x \in \mathbb{R}$.*

(ii) *There exists $L \in L^1(I,\mathbb{R})$ such that for almost all $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz continuous in the sense that $d_H(F(t,x),F(t,y)) \leq L(t)|x-y|$ for all $x,y \in \mathbb{R}$.*

Theorem 3.2 *Assume that hypothesis* H1 *is satisfied and let $y(.)\colon I \to \mathbb{R}$ be such that there exists $q(.) \in L^1(I,\mathbb{R})$ with $d(D^{\sigma}_{CF}y(t), F(t,y(t))) \leq q(t)$ a.e. (I) and $y(0) = y_0$, $y'(0) = y_1$. Then, there exists a solution $x(.)\colon I \to \mathbb{R}$ of problem* (1.1) *satisfying for all $t \in I$ the following estimate*

$$|x(t) - y(t)| \leq \big(|x_0 - y_0| + t|x_1 - y_1| + M|q|_1\big)e^{M\int_0^t L(s)\,\mathrm{d}s}.$$

Proof. The set-valued map $t \mapsto F(t,y(t))$ is measurable with closed values and the hypothesis that $d(D^{\sigma}_{CF}y(t), F(t,y(t))) \leq q(t)$ a.e. (I) is equivalent to $F(t,y(t)) \cap \{D^{\sigma}_{CF}y(t) + q(t)[-1,1]\} \neq \emptyset$ a.e. (I). It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t,y(t))$ a.e. (I) such that

$$|f_1(t) - D^{\sigma}_{CF}y(t)| \leq q(t) \text{ a.e. } (I). \tag{3.1}$$

Define $x_1(t) = x_0 + x_1 t + \int_0^t \mathcal{G}(t,s)f_1(s)\,\mathrm{d}s$. One has

$$\begin{aligned}
|x_1(t) - y(t)| &= \left|x_0 + x_1 t - y_0 - y_1 t + \int_0^t \mathcal{G}(t,s)(f_1(s) - D^{\sigma}_{CF}y(s))\,\mathrm{d}s\right| \\
&\leq |x_0 + x_1 t - y_0 - y_1 t| + \int_0^t |\mathcal{G}(t,s)|q(s)\,\mathrm{d}s \\
&\leq |x_0 - y_0| + t|x_1 - y_1| + M|q|_1.
\end{aligned}$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I,\mathbb{R})$ and $f_n(.) \in L^1(I,\mathbb{R})$, where $n \geq 1$, with the following properties:

$$x_n(t) = x_0 + x_1 t + \int_0^t \mathcal{G}(t,s)f_n(s)\,\mathrm{d}s,\ t \in I, \tag{3.2}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \text{ a.e. } (I),\ n \geq 1, \tag{3.3}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \text{ a.e. } (I),\ n \geq 1. \tag{3.4}$$

If this construction is realized, then from (3.3)–(3.4) for almost all $t \in I$ we have

$$\begin{aligned}|x_{n+1}(t) - x_n(t)| &\le \int_0^t |\mathcal{G}(t,t_1)|.|f_{n+1}(t_1) - f_n(t_1)|\,\mathrm{d}t_1\\ &\le M\int_0^t L(t_1)|x_n(t_1) - x_{n-1}(t_1)|\,\mathrm{d}t_1\\ &\le M\int_0^t L(t_1)\int_0^{t_1} |\mathcal{G}(t_1,t_2)|.|f_n(t_2) - f_{n-1}(t_2)|\,\mathrm{d}t_2\,\mathrm{d}t_1\\ &\le M^n\int_0^t L(t_1)\int_0^{t_1} L(t_2)\dots\int_0^{t_{n-1}} L(t_n)|x_1(t_n) - y(t_n)|\,\mathrm{d}t_n\\ &\le \frac{\left(M\int_0^t L(s)\,\mathrm{d}s\right)^n}{n!}\big(|x_0-y_0| + t|x_1-y_1| + M|q|_1\big).\end{aligned}$$

Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $C(I,\mathbb{R})$, and hence it converges uniformly to some $x \in C(I,\mathbb{R})$. Therefore, by (3.4), for almost all $t \in I$ the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Let f be the pointwise limit of f_n.

Moreover, one has

$$\begin{aligned}|x_n(t) - y(t)| &\le |x_1(t) - y(t)| + \sum_{i=1}^{n-1}|x_{i+1}(t) - x_i(t)|\\ &\le |x_0-y_0| + t|x_1-y_1| + M|q|_1\\ &\quad + \sum_{i=1}^{n-1}\frac{\left(M\int_0^t L(s)\,\mathrm{d}s\right)^i}{i!}\big(|x_0-y_0| + t|x_1-y_1| + M|q|_1\big)\\ &\le \big(|x_0-y_0| + t|x_1-y_1| + M|q|_1\big)e^{M\int_0^t L(s)\,\mathrm{d}s}.\end{aligned} \tag{3.5}$$

On the other hand, from (3.1), (3.4) and (3.5) for almost all $t \in I$ we obtain

$$\begin{aligned}|f_n(t) - D^\sigma_{CF}y(t)| &\le \sum_{i=1}^{n-1}|f_{i+1}(t) - f_i(t)| + |f_1(t) - D^\sigma_{CF}y(t)|\\ &\le L(t)\big(|x_0-y_0| + t|x_1-y_1| + M|q|_1\big)e^{M\int_0^t L(s)\,\mathrm{d}s} + q(t).\end{aligned}$$

Hence, the sequence f_n is integrably bounded, and therefore $f \in L^1(I,\mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.2), (3.3) we deduce that x is a solution of (1.1). Finally, passing to the limit in (3.5) we obtain the desired estimate on x.

It remains to construct the sequences x_n, f_n with the properties described in (3.2)–(3.4). The construction will be done by induction. Since the first step is already realized, assume that for some $N \ge 1$ we already constructed $x_n \in C(I,\mathbb{R})$ and $f_n \in L^1(I,\mathbb{R})$, $n = 1,2,\dots,N$, satisfying (3.2), (3.4) for $n = 1,2,\dots,N$ and (3.3) for $n = 1,2,\dots,N-1$. The set-valued map $t \mapsto F(t,x_N(t))$ is measurable. Moreover, the map $t \mapsto L(t)|x_N(t) - x_{N-1}(t)|$ is measurable. By the Lipschitz continuity of $F(t,.)$ for almost all $t \in I$ we have

$$F(t,x_N(t)) \cap \big\{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1,1]\big\} \ne \emptyset.$$

From Lemma 3.1 there exists a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.))$ such that

$$|f_{N+1}(t) - f_N(t)| \le L(t)|x_N(t) - x_{N-1}(t)| \text{ a.e. } (I).$$

We define x_{N+1} as in (3.2) with $n = N + 1$. Thus, f_{N+1} satisfies (3.3) and (3.4) and the proof is complete. □

The assumptions in Theorem 3.2 are satisfied with $y = 0$, $q = L$, and therefore we obtain the following consequence of Theorem 3.2.

Corollary 3.3 *Assume that hypothesis* H1 *is satisfied and* $d(0, F(t, 0)) \leq L(t)$ *a.e.* (I). *Then, there exists a solution* $x(.)\colon I \to \mathbb{R}$ *of problem* (1.1) *satisfying for all* $t \in I$ *the estimate*

$$|x(t)| \leq \big(|x_0| + t|x_1| + M|q|_1\big)e^{M\int_0^t L(s)\,\mathrm{d}s}.$$

4 A continuous selection of the solution set

In what follows (S, d) is a separable metric space and X is a real separable Banach space. We recall that a set-valued map $G(\cdot)\colon S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the set $\{s \in S : G(s) \subset C\}$ is closed. The proof of the next two lemmas may be found in [6].

Lemma 4.1 *Let* $F^*(.,.)\colon I \times S \to \mathcal{P}(X)$ *be a closed-valued* $\mathcal{L}(I) \otimes \mathcal{B}(S)$ *measurable set-valued map such that* $F^*(t,.)$ *is l.s.c. for any* $t \in I$. *Then, the set-valued map* $G(.)\colon S \to \mathcal{D}(I, X)$ *defined by*

$$G(s) = \{v \in L^1(I, X) : v(t) \in F^*(t, s) \text{ a.e. } (I)\}$$

is l.s.c. with non-empty closed values if and only if there exists a continuous mapping $p(.)\colon S \to L^1(I, X)$ *such that for every* $s \in S$ *we have* $d(0, F^*(t, s)) \leq p(s)(t)$ *a.e.* (I).

Lemma 4.2 *Let* $G(.)\colon S \to \mathcal{D}(I, X)$ *be a l.s.c. set-valued map with closed decomposable values and let* $\phi(.)\colon S \to L^1(I, X)$, $\psi(.)\colon S \to L^1(I, \mathbb{R})$ *be continuous such that the set-valued map* $H(.)\colon S \to \mathcal{D}(I, X)$ *defined by*

$$H(s) = \mathrm{cl}\{v \in G(s) : |v(t) - \phi(s)(t)| < \psi(s)(t) \text{ a.e. } (I)\}$$

has non-empty values. Then, H *has a continuous selection,* i.e.*, there exists a continuous mapping* $h\colon S \to L^1(I, X)$ *such that* $h(s) \in H(s)$ *for every* $s \in S$.

Hypothesis H2

(i) S is a separable metric space and $a(.), b(.)\colon S \to \mathbb{R}$, $c(.)\colon S \to (0, \infty)$ are continuous mappings.

(ii) There exists continuous mappings $y(.)\colon S \to AC(I, \mathbb{R})$ and $p(.)\colon S \to \mathbb{R}$ such that

$$d(D(y(s))^{\sigma}_{CF}(t), F(t, y(s)(t)) \leq p(s)(t) \text{ a.e. } (I),\ s \in S.$$

Let us introduce the following notation that will be needed in the sequel; set

$$\xi(s)(t) = Me^{Mm(t)}\big[tc(s) + |a(s) - y(s)(0)| + T|b(s) - (y(s))'(0)|\big] + \int_0^t p(s)(u)e^{M(m(t)-m(u))}\,\mathrm{d}u$$

and

$$m(t) = \int_0^t L(s)\,\mathrm{d}s.$$

Theorem 4.3 *Assume that hypotheses* H1 *and* H2 *are satisfied. Then, there exist a continuous mapping* $x(.)\colon S \to C(I, \mathbb{R})$ *such that for any* $s \in S$, $x(s)(.)$ *is a solution of the problem*

$$D_{CF}^{\sigma} z(t) \in F(t, z(t)), \quad x(0) = a(s), \quad x'(0) = b(s)$$

and

$$|x(s)(t) - y(s)(t)| \le \xi(s)(t), \quad (t, s) \in I \times S.$$

Proof. For $n \ge 0$ let us denote $\varepsilon_n(s) = c(s)\frac{n+1}{n+2}$, and let $d(s) = |a(s) - y(s)(0)| + T|b(s) - (y(s))'(0)|$. Moreover, for $n \ge 1$ set

$$p_n(s)(t) = M^n \left[\int_0^t p(s)(u) \frac{(m(t) - m(u))^{n-1}}{(n-1)!}\,\mathrm{d}u + \frac{(m(t))^{n-1}}{(n-1)!} t \varepsilon_n(s) \right] + M^{n-1} d(s).$$

Set also $x_0(s)(t) = y(s)(t)$ and $f_0(s)(t) = g(s)(t)$, where $s \in S$.

We consider the set-valued maps $G_0(.)$, $H_0(.)$ defined, respectively, by

$$\begin{aligned} G_0(s) &= \{v \in L^1(I, \mathbb{R}) : v(t) \in F(t, y(s)(t)) \text{ a.e.}(I)\}, \\ H_0(s) &= \mathrm{cl}\{v \in G_0(s) : |v(t) - g(s)(t)| < p(s)(t) + \varepsilon_0(s)\}. \end{aligned}$$

Since $d(g(s)(t), F(t, y(s)(t))) \le p(s)(t) < p(s)(t) + \varepsilon_0(s)$, according to Lemma 3.1, the set $H_0(s)$ is not-empty. Set $F_0^*(t, s) = F(t, y(s)(t))$ and note that $d(0, F_0^*(t, s)) \le |g(s)(t)| + p(s)(t) = p^*(s)(t)$ and that $p^*(.)\colon S \to L^1(I, \mathbb{R})$ is continuous. Applying now Lemmas 4.1 and 4.2 we obtain the existence of a continuous selection f_0 of H_0, i.e. such that

$$\begin{gathered} f_0(s)(t) \in F(t, y(s)(t)) \text{ a.e. } (I),\ s \in S, \\ |f_0(s)(t) - g(s)(t)| \le p_0(s)(t) = p(s)(t) + \varepsilon_0(s),\ s \in S, t \in I. \end{gathered}$$

We define $x_1(s)(t) = a(s) + tb(s) + \int_0^t \mathcal{G}(t, u) f_0(s)(u)\,\mathrm{d}u$. One has

$$\begin{aligned} &|x_1(s)(t) - x_0(s)(t)| \\ &\quad \le |a(s) - y(s)(0)| + T|b(s) - (y(s))'(0)| + M \int_0^t |f_0(s)(u) - g(s)(u)|\,\mathrm{d}u \\ &\quad \le d(s) + M \int_0^t (p(s)(u) + \varepsilon_0(s))\,\mathrm{d}u = p_1(s)(t). \end{aligned}$$

We construct two sequences of approximations $f_n(.)\colon S \to L^1(I, \mathbb{R})$ and $x_n(.)\colon S \to C(I, \mathbb{R})$ with the following properties:

(a) $f_n(.)\colon S \to L^1(I, \mathbb{R})$ and $x_n(.)\colon S \to C(I, \mathbb{R})$ are continuous,

(b) $f_n(s)(t) \in F(t, x_n(s)(t))$ a.e. (I), $s \in S$,

(c) $|f_n(s)(t) - f_{n-1}(s)(t)| \le L(t) p_n(s)(t)$ a.e. (I), $s \in S$,

(d) $x_{n+1}(s)(t) = a(s) + tb(s) + \int_0^t \mathcal{G}(t,u) f_n(s)(u)\,\mathrm{d}u$ for all $t \in I, s \in S$.

Suppose we have already constructed $f_i(.), x_i(.), i = 1, \dots, n$, satisfying (a)–(c) and define $x_{n+1}(.)$ as in (d). From (c) and (d) one has

$$\begin{aligned} |x_{n+1}(s)(t) - x_n(s)(t)| &\leq M \int_0^t |f_n(s)(u) - f_{n-1}(s)(u)|\,\mathrm{d}u \\ &\leq M \int_0^t L(u) p_n(s)(u)\,\mathrm{d}u \\ &< p_{n+1}(s)(t). \end{aligned} \tag{4.1}$$

Consider the following set-valued maps, defined for any $s \in S$,

$$\begin{aligned} &G_{n+1}(s) = \{v \in L^1(I,\mathbb{R}) : v(t) \in F(t, x_{n+1}(s)(t)) \text{ a.e. } (I)\}, \\ &H_{n+1}(s) = \mathrm{cl}\{v \in G_{n+1}(s) : |v(t) - f_n(s)(t)| < L(t) p_n(s)(t) \text{ a.e. } (I)\}. \end{aligned}$$

To prove that $H_{n+1}(s)$ is non-empty we note first that the real function

$$t \mapsto r_n(s)(t) = c(s) \frac{M^{n+1} t L(t) (m(t))^n}{(n+2)(n+3)n!}$$

is measurable and strictly positive for any s. One has $d(f_n(s)(t), F(t, x_{n+1}(s)(t))) \leq L(t)|x_n(s)(t) - x_{n+1}(s)(t)| \leq L(t) p_n(s)(t) - r_n(s)(t)$, and therefore according to Lemma 3.1 there exists $v(.) \in L^1(I,\mathbb{R})$ such that $v(t) \in F(t, x_{n+1}(s)(t))$ a.e. (I) and

$$|v(t) - f_n(s)(t)| < d(f_n(s)(t), F(t, x_{n+1}(s)(t))) + r_n(s)(t).$$

Hence, $H_{n+1}(s)$ is not empty. Set $F^*_{n+1}(t,s) = F(t, x_{n+1}(s)(t))$ and note that we may write $d(0, F^*_{n+1}(t,s)) \leq |f_n(s)(t)| + L(t) p_{n+1}(s)(t) = p^*_{n+1}(s)(t)$ a.e. (I) and $p^*_{n+1}(.) \colon S \to L^1(I,\mathbb{R})$ is continuous. By Lemmas 4.1 and 4.2 there exists a continuous map $f_{n+1}(.) \colon S \to L^1(I,\mathbb{R})$ such that for any $s \in S$,

$$\begin{gathered} f_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t)) \text{ a.e. } (I), \\ |f_{n+1}(s)(t) - f_n(s)(t)| \leq L(t) p_{n+1}(s)(t) \text{ a.e. } (I). \end{gathered}$$

From (4.1) and (d) we obtain

$$\begin{aligned} |x_{n+1}(s)(.) - x_n(s)(.)|_C &\leq M |f_{n+1}(s)(.) - f_n(s)(.)|_1 \\ &\leq \frac{(Mm(T))^n}{n!} (M|p(s)(.)|_1 + MTc(s) + d(s)). \end{aligned} \tag{4.2}$$

Therefore, $f_n(s)(.)$, $x_n(s)(.)$ are Cauchy sequences in the Banach space $L^1(I,\mathbb{R})$ and $C(I,\mathbb{R})$, respectively. Let $f(.) \colon S \to L^1(I,\mathbb{R})$ and $x(.) \colon S \to C(I,\mathbb{R})$ be their limits. The function $s \mapsto M|p(s)(.)|_1 + MTc(s) + d(s)$ is continuous, hence locally bounded. Therefore, (4.2) implies that for every $s' \in S$ the sequence $f_n(s')(.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighbourhood of s. Hence, $s \mapsto f(s)(.)$ is continuous from S into $L^1(I,\mathbb{R})$. From (4.2), as before, $x_n(s)(.)$ is Cauchy in $C(I,\mathbb{R})$ locally uniformly with respect to s. So, $s \mapsto x(s)(.)$ is continuous from S into $C(I,\mathbb{R})$. On the other hand, since $x_n(s)(.)$ converges uniformly to $x(s)(.)$ and

$$d(f_n(s)(t), F(t, x(s)(t))) \leq L(t)|x_n(s)(t) - x(s)(t)| \quad \text{a.e. } (I),\ s \in S,$$

passing to the limit with a subsequence of $f_n(.)$ converging pointwise to $f(.)$ we obtain

$$f(s)(t) \in F(t, x(s)(t)) \quad \text{a.e. } (I),\ s \in S.$$

Passing to the limit in (d) we obtain

$$x(s)(t) = a(s) + tb(s) + \int_0^t \mathcal{G}(t,u) f(s)(u)\,\mathrm{d}u.$$

By adding inequalities (c) for all n, together with (4.1), we get

$$|x_{n+1}(s)(t) - y(s)(t)| \le \sum_{l=0}^{n} p_l(s)(t) \le \xi(s)(t). \tag{4.3}$$

Finally, passing to the limit in (4.3) we obtain the estimate in the statement of the theorem. □

Theorem 4.3 allows to obtain the next corollary which is a general result concerning continuous selections of the solution set of problem (1.1).

Hypothesis H3. Hypothesis H1 is satisfied and there exists $r(.) \in L^1(I, \mathbb{R})$ such that $d(0, F(t, 0)) \le r(t)$ a.e. (I).

Theorem 4.4 *Assume that hypothesis* H3 *is satisfied. Then, there exists a function* $x(.,.) \colon I \times \mathbb{R}^2 \to \mathbb{R}$ *such that*

(a) $x(., (\xi, \eta)) \in \mathcal{S}(\xi, \eta)$ *for every* $(\xi, \eta) \in \mathbb{R}^2$,

(b) $(\xi, \eta) \to x(., (\xi, \eta))$ *is continuous from* $\mathbb{R}^2$ *into* $C(I, \mathbb{R})$.

Proof. We take $S = \mathbb{R} \times \mathbb{R}$, $a(\xi, \eta) = \xi$, $b(\xi, \eta) = \eta$, $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$, $c(.) \colon \mathbb{R} \times \mathbb{R} \to (0, \infty)$ an arbitrary continuous function, $g(.) = 0$, $y(.) = 0$, $p(\xi, \eta)(t) = r(t)$, $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$, $t \in I$, and we apply Theorem 4.3 in order to obtain the conclusion of the theorem. □

5 Arcwise connectedness of the solution set

In this section we are concerned with the more general problem

$$D^{\sigma}_{CF} x(t) \in F(t, x(t), H(t, x(t))) \text{ a.e. } ([0, T]), \quad x(0) = x_0,\ x'(0) = x_1, \tag{5.1}$$

where $F \colon I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $H \colon I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$.

We assume that F and H are closed-valued multifunctions Lipschitz continuous with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the differential inclusion in (5.1) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set to (5.1). When F does not depend on the last variable, (5.1) reduces to (1.1) and the result remains valid for problem (1.1).

Let Z be a metric space with the distance d_Z. In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces Z_i, $i = 1, 2$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1, z_2), (z_1', z_2')) = \sum_{i=1}^{2} d_{Z_i}(z_i, z_i')$.

Let X be a non-empty set and let $F\colon X \to \mathcal{P}(Z)$ be a set-valued map with non-empty closed values. The range of F is the set $F(X) = \bigcup_{x\in X} F(x)$. The multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x, x_0) < \delta$ implies $D_Z(F(x), F(x_0)) < \varepsilon$.

Consider a finite, positive, non-atomic measure space $(T, \mathcal{F}, \mu)$ and let $(X, |.|_X)$ be a Banach space. We recall that a set $A \in \mathcal{F}$ is called an atom of μ if $\mu(A) \neq 0$ and for any $B \in \mathcal{F}, B \subset A$ one has $\mu(B) = 0$ or $\mu(B) = \mu(A)$. The measure μ is called non-atomic if $\mathcal{F}$ does not contain atoms of μ. For example, Lebesgue's measure on a given interval in $\mathbb{R}$ is a non-atomic measure.

Next we recall some preliminary results that are the main tools in the proof of our theorem. To simplify the notation we write E in place of $L^1(T, X)$. The next two lemmas are proved in [16].

Lemma 5.1 *Assume that $\phi\colon S \times E \to \mathcal{P}(E)$ and $\psi\colon S \times E \times E \to \mathcal{P}(E)$ are Hausdorff continuous multifunctions with non-empty, closed, decomposable values, satisfying the following conditions*:

(a) *there exists $L \in [0, 1)$ such that $d_H(\phi(s, u), \phi(s, u')) \leq L|u - u'|_E$ for every $s \in S$ and every $u, u' \in E$,*

(b) *there exists $M \in [0, 1)$ such that $L + M < 1$ and*

$$d_H(\psi(s, u, v), \psi(s, u', v')) \leq M(|u - u'|_E + |v - v'|_E)$$

for every $s \in S$ and every $(u, v), (u', v') \in E \times E$.

Set $\mathrm{Fix}(\Gamma(s, .)) = \{u \in E : u \in \Gamma(s, u)\}$, *where* $\Gamma(s, u) = \psi(s, u, \phi(s, u))$, $(s, u) \in S \times E$. *Then, for every $s \in S$ the set* $\mathrm{Fix}(\Gamma(s, .))$ *is non-empty and arcwise connected.*

Lemma 5.2 *Let $U : T \to \mathcal{P}(X)$ and $V : T \times X \to \mathcal{P}(X)$ be two non-empty closed-valued multifunctions satisfying the following conditions*:

(a) *U is measurable and there exists $r \in L^1(T)$ such that $D_X(U(t), \{0\}) \leq r(t)$ for almost all $t \in T$,*

(b) *the multifunction $t \mapsto V(t, x)$ is measurable for every $x \in X$,*

(c) *the multifinction $x \mapsto V(t, x)$ is Hausdorff continuous for all $t \in T$.*

Let $v\colon T \to X$ be a measurable selection of $t \mapsto V(t, U(t))$. Then, there exists a selection $u \in L^1(T, X)$ of $U(.)$ such that $v(t) \in V(t, u(t))$, $t \in T$.

Hypothesis H4. Let $F\colon I \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ and $H\colon I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be two set-valued maps with non-empty closed values, satisfying the following assumptions:

(i) the set-valued maps $t \mapsto F(t, u, v)$ and $t \mapsto H(t, u)$ are measurable for all $u, v \in \mathbb{R}$,

(ii) there exists $l \in L^1(I, \mathbb{R}_+)$ such that for every $u, u' \in \mathbb{R}$ we have

$$d_H(H(t, u), H(t, u')) \leq l(t)|u - u'| \text{ a.e. } (I),$$

(iii) there exist $m \in L^1(I, \mathbb{R}_+)$ and $\theta \in [0, 1)$ such that for every $u, v, u', v \in \mathbb{R}$ we have

$$d_H(F(t, u, v), F(t, u', v')) \leq m(t)|u - u'| + \theta|v - v'| \text{ a.e. } (I),$$

(iv) there exist $f, g \in L^1(I, \mathbb{R}_+)$ such that

$$d(0, F(t, 0, 0)) \leq f(t) \text{ and } d(0, H(t, 0)) \leq g(t) \text{ a.e. } (I).$$

For $(x_0, x_1) \in \mathbb{R}^2$ we denote by $S(x_0, x_1)$ the solution set of (5.1). In what follows $N(t) := \max\{l(t), m(t)\}$, $t \in I$.

Theorem 5.3 *Assume that hypothesis* H4 *is satisfied and* $2M \int_0^T N(s)\,\mathrm{d}s + \theta < 1$. *Then, for every* $(x_0, x_1) \in \mathbb{R}^2$, *the solution set* $S(x_0, x_1)$ *of* (5.1) *is non-empty and arcwise connected in the space* $C(I, \mathbb{R})$.

Proof. For $\xi = (x_0, x_1) \in \mathbb{R}^2$ and $u \in L^1(I, \mathbb{R})$ set $P_\xi(t) = x_0 + tx_1$ and

$$u_\xi(t) = P_\xi(t) + \int_0^T \mathcal{G}(t, s)u(s)\,\mathrm{d}s, \quad t \in I.$$

We prove that the multifunctions $\phi\colon \mathbb{R}^2 \times L^1(I, \mathbb{R}) \to \mathcal{P}(L^1(I, \mathbb{R}))$ and $\psi\colon \mathbb{R}^2 \times L^1(I, \mathbb{R}) \times L^1(I, \mathbb{R}) \to \mathcal{P}(L^1(I, \mathbb{R}))$ given by

$$\phi(\xi, u) = \{v \in L^1(I, \mathbb{R}) : v(t) \in H(t, u_\xi(t)) \text{ a.e. } (I)\},$$
$$\psi(\xi, u, v) = \{w \in L^1(I, \mathbb{R}) : w(t) \in F(t, u_\xi(t), v(t)) \text{ a.e. } (I)\}$$

satisfy the hypotheses of Lemma 5.1. Since u_ξ is measurable and H satisfies hypothesis H4 (i) and (ii), the multifunction $t \mapsto H(t, u_\xi(t))$ is measurable and has non-empty closed values. Hence, it has a measurable selection. Therefore, due to Hypothesis H4 (iv), the set $\phi(\xi, u)$ is non-empty. The fact that the set $\phi(\xi, u)$ is closed and decomposable follows by simple computation. In the same way we obtain that $\psi(\xi, u, v)$ is a non-empty closed decomposable set.

Pick $(\xi, u), (\xi_1, u_1) \in \mathbb{R}^2 \times L^1(I, \mathbb{R})$, $\xi_1 = (y_0, y_1)$ and choose $v \in \phi(\xi, u)$. For each $\varepsilon > 0$ there exists $v_1 \in \phi(\xi_1, u_1)$ such that, for every $t \in I$, one has

$$\begin{aligned}|v(t) - v_1(t)| &\leq d_H(H(t, u_\xi(t)), H(t, u_{\xi_1}(t))) + \varepsilon \\ &\leq N(t)\left[|P_\xi(t) - P_{\xi_1}(t)| + \int_0^T |\mathcal{G}(t, s)|.|u(s) - u_1(s)|\,\mathrm{d}s\right] + \varepsilon.\end{aligned}$$

Hence, there exists $M_0 \geq 0$ such that

$$|v - v_1|_1 \leq M_0|\xi - \xi_1|.\int_0^T N(t)\,\mathrm{d}t + M\int_0^T N(t)\,\mathrm{d}t|u - u_1|_1 + T\varepsilon$$

for any $\varepsilon > 0$. This implies that

$$d_{L^1(I,\mathbb{R})}(v, \phi(\xi_1, u_1)) \leq M_0|\xi - \xi_1|.\int_0^T N(t)\,\mathrm{d}t + M\int_0^T N(t)\,\mathrm{d}t|u - u_1|_1$$

for all $v \in \phi(\xi, u)$. Consequently,

$$d_H(\phi(\xi,u),\phi(\xi_1,u_1)) \le M_0|\xi-\xi_1|.\int_0^T N(t)\,\mathrm{d}t + M\int_0^T N(t)\,\mathrm{d}t|u-u_1|_1,$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 5.1.

Pick $(\xi,u,v),(\xi_1,u_1,v_1) \in \mathbb{R}^2 \times L^1(I,\mathbb{R}) \times L^1(I,\mathbb{R})$ and choose $w \in \psi(\xi,u,v)$. Then, as before, for each $\varepsilon > 0$ there exists $w_1 \in \psi(\xi_1,u_1,v_1)$ such that for every $t \in I$,

$$\begin{aligned}
&|w(t)-w_1(t)|\\
&\quad\le d_H(F(t,u_\xi(t),v(t)),F(t,u_{\xi_1}(t),v_1(t)))+\varepsilon\\
&\quad\le N(t)|u_\xi(t)-u_{\xi_1}(t)|+\theta|v(t)-v_1(t)|+\varepsilon\\
&\quad\le N(t)\left[|P_\xi(t)-P_{\xi_1}(t)|+\int_0^T|\mathcal{G}(t,s)|.|u(s)-u_1(s)|\,\mathrm{d}s\right]+\theta|v(t)-v_1(t)|+\varepsilon\\
&\quad\le N(t)[M_0|\xi-\xi_1|+M|u-u_1|_1]+\theta|v(t)-v_1(t)|+\varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|w-w_1|_1\\
&\quad\le M_0|\xi-\xi_1|\int_0^T N(t)\,\mathrm{d}t+M\int_0^T N(t)\,\mathrm{d}t|u-u_1|_1+\theta|v-v_1|_1+T\varepsilon\\
&\quad\le M_0|\xi-\xi_1|\int_0^T N(t)\,\mathrm{d}t+\left(M\int_0^T N(t)\,\mathrm{d}t+\theta\right)d_{L^1(I,\mathbb{R})\times L^1(I,\mathbb{R})}((u,v),(u_1,v_1))+T\varepsilon.
\end{aligned}$$

As above, we deduce that

$$\begin{aligned}
&d_H(\psi(\xi,u,v),\psi(\xi_1,u_1,v_1))\\
&\qquad\le M_0|\xi-\xi_1|\int_0^T N(t)\,\mathrm{d}t+\left(M\int_0^T N(t)\,\mathrm{d}t+\theta\right)d_{L^1(I,\mathbb{R})\times L^1(I,\mathbb{R})}((u,v),(u_1,v_1)).
\end{aligned}$$

The multifunction ψ is thus Hausdorff continuous and satisfies the hypothesis of Lemma 5.1.

Define $\Gamma(\xi,u)=\psi(\xi,u,\phi(\xi,u))$, $(\xi,u)\in\mathbb{R}^2\times L^1(I,\mathbb{R})$. According to Lemma 5.1, the set $\mathrm{Fix}(\Gamma(\xi,.))=\{u\in L^1(I,\mathbb{R}):u\in\Gamma(\xi,u)\}$ is non-empty and arcwise connected in $L^1(I,\mathbb{R})$. We shall prove that

$$\mathrm{Fix}(\Gamma(\xi,.))=\{u\in L^1(I,\mathbb{R}):u(t)\in F(t,u_\xi(t),H(t,u_\xi(t)))\text{ a.e. }(I)\}. \tag{5.2}$$

Denote by $A(\xi)$ the right-hand side of (5.2). If $u\in\mathrm{Fix}(\Gamma(\xi,.))$, then there is $v\in\phi(\xi,v)$ such that $u\in\psi(\xi,u,v)$. Therefore, $v(t)\in H(t,u_\xi(t))$ and

$$u(t)\in F(t,u_\xi(t),v(t))\subset F(t,u_\xi(t),H(t,u_\xi(t)))\quad\text{a.e. }(I),$$

so that $\mathrm{Fix}(\Gamma(\xi,.))\subset A(\xi)$. Let now $u\in A(\xi)$. By Lemma 5.2, there exists a selection $v\in L^1(I,\mathbb{R})$ of the multifunction $t\mapsto H(t,u_\xi(t))$ satisfying $u(t)\in F(t,u_\xi(t),v(t))$ a.e. (I). Hence, $v\in\phi(\xi,v)$, $u\in\psi(\xi,u,v)$, and thus $u\in\Gamma(\xi,u)$. This completes the proof of (5.2).

Finally, the function $\mathcal{T}\colon L^1(I,\mathbb{R})\to C(I,\mathbb{R})$ given by $\mathcal{T}(u)(t):=\int_0^T\mathcal{G}(t,s)u(s)\,\mathrm{d}s$ for $t\in I$ is continuous and one has

$$S(\xi)=P_\xi(.)+\mathcal{T}(\mathrm{Fix}(\Gamma(\xi,.))),\quad \xi\in\mathbb{R}^2.$$

Since $\mathrm{Fix}(\Gamma(\xi,.))$ is non-empty and arcwise connected in $L^1(I,\mathbb{R})$, the set $S(\xi)$ has the same properties in $C(I,\mathbb{R})$. □

6 Conclusions

In this paper we obtained an existence result for fractional differential inclusions involving Caputo–Fabrizio fractional derivative in the situation when the values of the set-valued map are not convex employing a method originally introduced by Filippov. Also, we found a continuous selection of the solution set and we proved the arcwise connectedness of the solution set of the considered problem. Such kind of results may be useful in order to obtain qualitative results concerning the solutions of fractional differential inclusions defined by Caputo–Fabrizio fractional derivative such as: controllability along a reference trajectory, differentiability of solutions with respect to the initial conditions.

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