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ANALYSIS OF A QUASISTATIC THERMO-VISCOELASTIC PIEZOELECTRIC CONTACT PROBLEM

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Abstract. This paper deals with the study of a quasistatic problem of friction contact between two thermo-viscoelastic piezoelectric bodies with long-term memory. The contact is modelled with a version of normal compliance condition and the associated Coulomb's law of friction in which the adhesion of contact surfaces is taken into account. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of evolutionary variational equalities, a classical existence and uniqueness result on parabolic equalities, differential equations and a fixed point theorem.

Keywords: thermo-viscoelastic piezoelectric material, adhesion, Coulomb's law of friction, normal compliance, fixed point.

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1 Introduction

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general Mathematical Theory of Contact Mechanics is currently maturing. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws (*i.e.*, different materials), varied geometries and settings, and different contact conditions, see for instance [9, 19, 20] and the references therein. The theory's aim is to provide a sound, clear and rigorous background for the

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constructions of models for contact between deformable bodies; proving existence, uniqueness and regularity results; assigning precise meaning to solutions; and the necessary setting for finite element approximations of the solutions. There is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see for instance [2, 5, 6, 8, 15, 17] and the references therein. The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metal and rocks polymers. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage field. See for examples [1, 14] for the case of hardening, temperature and other internal state variables and the references [4, 7] for the case adhesion field which is denoted in this paper by β, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. General models for piezoelectric materials can be found in [10]. Frictional contact problems for piezoelectric materials were studied in [8, 20], under the assumption that the foundation is insulated. Part of these results were extended recently in [15, 16] in the case of an electrically conductive foundation. There, the material behavior was described with an electro-viscoelastic constitutive law and the process was assumed to be quasistatic. The unique solvability of the corresponding problems was obtained by using arguments of hemivariational inequalities. A quasistatic problem with normal compliance for electro-viscoelastic materials in frictional contact with a conductive foundation was investigated in [11]. There, the variational formulation of the corresponding problem was derived and the existence of a unique weak solution was obtained, under a smallness assumption on the data. The proof was based on arguments of evolutionary variational inequalities with monotone operators and a fixed point theorem.

Quasistatic friction contact problems for viscoelastic materials with temperature can be found in [1]. Electro-viscoelastic friction contact problems can be found in [15]. Contact problems for electro-viscoelastic materials with long-term memory and friction can be found in [8]. Contact problems for viscoelastic materials with friction and normal compliance can be found in [3]. Contact problems for viscoelastic materials with adhesion can be found in [4]. The recent paper extends the above mentioned works as the contact is modelled with a version of normal compliance, friction and adhesion between two thermo-electro-viscoelastic bodies with long-term memory.

The aim of this paper is to study a quasistatic frictional contact problem with adhesion between two thermo-viscoelastic piezoelectric bodies with long-term memory. We use a thermo-viscoelastic piezoelectric constitutive law with long-term memory given by

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \int_0^t \mathcal{F}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s)\big)\,\mathrm{d}s, \tag{1.1}$$

where $\boldsymbol{u}^\ell$ represents the displacement field, $\boldsymbol{\sigma}^\ell$ and $\varepsilon(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively, θ^ℓ represents the absolute temperature. Here $\mathcal{A}^\ell$ is a given nonlinear operator, $\mathcal{F}^\ell$ is the relaxation operator, and $\mathcal{G}^\ell$ represents the elasticity operator. $E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, $\mathcal{E}^\ell$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In (1.1) and everywhere in this paper the dot above a variable represents the derivative with respect to the time variable t. It follows from (1.1) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}^\ell_V(t) + \boldsymbol{\sigma}^\ell_E(t) + \boldsymbol{\sigma}^\ell_R(t)$, where $\boldsymbol{\sigma}^\ell_V(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}^\ell_E(t) = (\mathcal{E}^\ell)^* \nabla\varphi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}^\ell_R(t)$ satisfies a rate-type thermoelastic relation

$$\boldsymbol{\sigma}^\ell_R(t) = \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + \int_0^t \mathcal{F}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s)\big)\,\mathrm{d}s.$$

Note also that when $\mathcal{F}^\ell = 0$ the constitutive law (1.1) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation. The normal compliance contact condition was first considered in [13] in the study of problems with linearly elastic and viscoelastic materials and then it was used in various references, see *e.g.* [3, 7, 16]. This condition allows the interpenetration between bodies and it was justified by considering the interpenetration and deformation of surface asperities.

The paper is organized as follows. In Sect.2 we describe the mathematical models for the frictional contact problem between two thermo-viscoelastic piezoelectric bodies with long-term memory. The contact is modelled with normal compliance and adhesion. In Sect.3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. We state our main result, the existence of a unique weak solution to the model in Theorem 4.1. The proof of the theorem is provided in Sect.4, where it is carried out in several steps and is based on arguments of evolutionary variational equalities, a classical existence and uniqueness result on parabolic equalities, differential equations and Banach fixed point theorem.

2 Problem Statement

The physical setting is as follows. Let us consider two electro-thermo-viscoelastic bodies with long-term memory, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0, T)$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0, T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times (0, T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The bodies is in adhesive contact with an obstacle, over the contact surface Γ_3. With the assumption above, the classical formulation of the friction contact problem with adhesion between two electro-thermo-viscoelastic bodies with long-term memory is given by.

Problem P. For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}$, a temperature $\theta^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}$, a bonding field $\beta : \Gamma_3 \times [0, T] \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times [0, T] \longrightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \int_0^t \mathcal{F}^\ell\big(t - s, \varepsilon(\boldsymbol{u}^\ell(s)), \theta^\ell(s)\big)\, \mathrm{d}s \text{ in } \Omega^\ell \times (0, T), \quad (2.1)$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0, T), \quad (2.2)$$

$$\dot{\theta}^\ell - \kappa_0^\ell \Delta \theta^\ell = \Theta^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \theta^\ell\big) + \rho^\ell \quad \text{in } \Omega^\ell \times (0, T), \quad (2.3)$$

$$\operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0, T), \quad (2.4)$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0, T), \quad (2.5)$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0, T), \quad (2.6)$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0, T), \quad (2.7)$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \beta^2 R_\nu([u_\nu]) \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \quad (2.8)$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \left\|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\right\| \le \mu p_\nu([u_\nu]), \\ \left\|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\right\| < \mu p_\nu([u_\nu]) \Rightarrow [\boldsymbol{u}_\tau] = 0, \qquad \text{on } \Gamma_3 \times (0,T), \\ \left\|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\right\| = \mu p_\nu([u_\nu]) \Rightarrow \exists \lambda \ge 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) = -\lambda [\boldsymbol{u}_\tau] \end{cases} \tag{2.9}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu (R_\nu([u_\nu]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\right|^2\big) - \varepsilon_a\Big)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{2.11}$$

$$\boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{2.12}$$

$$\kappa_0^\ell \frac{\partial^\ell \theta^\ell}{\partial \nu^\ell} + \alpha^\ell \theta^\ell = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.13}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \;\; \theta^\ell(0) = \theta_0^\ell \quad \text{in } \Omega^\ell, \tag{2.14}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.15}$$

First, equations (2.1) and (2.2) represent the electro-viscoelastic constitutive law with long-term memory and thermal effects. Equation (2.3) represents the energy conservation, where Θ^ℓ is a non-linear constitutive function which represents the heat generated by the work of internal forces and ρ^ℓ is a given volume heat source. Equations (2.4) and (2.5) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Next, the equations (2.6) and (2.7) represent the displacement and traction boundary condition, respectively. Condition (2.8) represents the normal compliance conditions with adhesion, where γ_ν is a given adhesion coefficient, p_ν is a given positive function which will be described below, and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction, in this condition the interpenetrability between two bodies, that is $[u_\nu]$ can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \beta^2 R_\nu([u_\nu])$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu \beta^2 L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (2.9) is a non local Coulomb's friction law conditions coupled with adhesive, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

Next, the equation (2.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [4], see also [20] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (2.10), $\dot{\beta} \leq 0$. (2.11) and (2.12) represent the electric boundary conditions. The relation (2.13) represents a Fourier boundary condition for the temperature on Γ^ℓ. (2.14) represents the initial displacement field, the initial temperature. Finally, (2.15) represents the initial condition in which β_0 is the given initial bonding field.

3 Variational formulation and the main result

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represents the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}^\ell.\boldsymbol{v}^\ell = u_i^\ell.v_i^\ell, \quad \left|\boldsymbol{v}^\ell\right| = (\boldsymbol{v}^\ell.\boldsymbol{v}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad \left|\boldsymbol{\tau}^\ell\right| = (\boldsymbol{\tau}^\ell.\boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in L^2(\Omega^\ell)\}, \qquad H_1^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in H^1(\Omega^\ell)\},$$
$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell);\ \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\}, \quad \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell;\ \mathrm{div}\boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell \,\mathrm{d}x, \quad (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell \,\mathrm{d}x + \int_{\Omega^\ell} \nabla\boldsymbol{u}^\ell.\nabla\boldsymbol{v}^\ell \,\mathrm{d}x,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell \,\mathrm{d}x, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell \,\mathrm{d}x + \int_{\Omega^\ell} \mathrm{div}\,\boldsymbol{\sigma}^\ell.\,\mathrm{Div}\,\boldsymbol{\tau}^\ell \,\mathrm{d}x$$

and the associated norms $\|.\|_{H^\ell}$, $\|.\|_{H_1^\ell}$, $\|.\|_{\mathcal{H}^\ell}$, and $\|.\|_{\mathcal{H}_1^\ell}$ respectively. For every element $\boldsymbol{v}^\ell \in H_1^\ell$, we also use the notation $\boldsymbol{v}^\ell$ for the trace of $\boldsymbol{v}^\ell$ on Γ^ℓ and we denote by v_ν^ℓ and $\boldsymbol{v}_\tau^\ell$ the *normal* and the *tangential* components of $\boldsymbol{v}^\ell$ on the boundary Γ^ℓ given by

$$v_\nu^\ell = \boldsymbol{v}^\ell.\nu^\ell, \quad \boldsymbol{v}_\tau^\ell = \boldsymbol{v}^\ell - v_\nu^\ell \boldsymbol{\nu}^\ell.$$

Denote by σ_ν^ℓ and $\boldsymbol{\sigma}_\tau^\ell$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$, respectively. If $\boldsymbol{\sigma}^\ell$ is continuously differentiable on $\Omega^\ell \cup \Gamma^\ell$, then

$$\sigma_\nu^\ell = (\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell).\boldsymbol{\nu}^\ell, \quad \boldsymbol{\sigma}_\tau^\ell = \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell - \sigma_\nu^\ell \boldsymbol{\nu}^\ell,$$
$$(\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell, \boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.\boldsymbol{v}^\ell \,\mathrm{d}a$$

for all $\boldsymbol{v}^\ell \in H_1^\ell$, where $\mathrm{d}a$ is the surface measure element.

To obtain the variational formulation of the problem (2.1)–(2.15), we introduce for the bonding field the set

$$\mathcal{Z} = \left\{ \varsigma \in L^\infty\big(0,T; L^2(\Gamma_3)\big);\ 0 \le \varsigma \le 1 \text{ a.e. on } \Gamma_3 \right\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \left\{ \boldsymbol{v}^\ell \in H_1^\ell;\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_1^\ell \right\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality holds :

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \ge c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell}, \quad \forall \boldsymbol{v}^\ell \in V^\ell, \tag{3.1}$$

where the constant c_K denotes a positive constant which may depends only on Ω^ℓ, Γ_1^ℓ (see [18]). Over the space V^ℓ we consider the inner product given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell, \tag{3.2}$$

and let $\|.\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (3.1) that the norms $\|.\|_{H_1^\ell}$ and $\|.\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|.\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.2), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}^\ell\|_{V^\ell}, \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{3.3}$$

We also introduce the spaces

$$E_0^\ell = L^2(\Omega^\ell), \quad E_1^\ell = H^1(\Omega^\ell), \quad W^\ell = \left\{ \psi^\ell \in E_1^\ell;\ \psi^\ell = 0 \text{ on } \Gamma_a^\ell \right\},$$
$$\mathcal{W}^\ell = \left\{ \boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \operatorname{div} \boldsymbol{D}^\ell \in L^2(\Omega^\ell) \right\}.$$

Since $meas\Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^\ell\|_{W^\ell} \ge c_F \|\psi^\ell\|_{H^1(\Omega^\ell)}, \quad \forall \psi^\ell \in W^\ell, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ.
Over the space W^ℓ, we consider the inner product given by

$$(\varphi^\ell, \psi^\ell)_{W^\ell} = \int_{\Omega^\ell} \nabla\varphi^\ell . \nabla\psi^\ell \, \mathrm{d}x \tag{3.5}$$

and let $\|.\|_{W^\ell}$ be the associated norm. It follows from (3.4) that $\|.\|_{H^1(\Omega^\ell)}$ and $\|.\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|.\|_{W^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\mathbf{c}_0$, depending only on Ω^ℓ, Γ_a^ℓ and Γ_3, such that

$$\|\zeta^\ell\|_{L^2(\Gamma_3)} \le \mathbf{c}_0 \|\zeta^\ell\|_{W^\ell}, \quad \forall \zeta^\ell \in W^\ell. \tag{3.6}$$

The space $\mathcal{W}^\ell$ is a real Hilbert space with the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Phi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell . \boldsymbol{\Phi}^\ell \, \mathrm{d}x + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{D}^\ell . \operatorname{div} \boldsymbol{\Phi}^\ell \, \mathrm{d}x,$$

where div $\boldsymbol{D}^\ell = (\boldsymbol{D}^\ell_{i,i})$, and the associated norm $\|.\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V} = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,$$
$$\mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \quad E_0 = E_0^1 \times E_0^2, \quad E_1 = E_1^1 \times E_1^2, \quad W = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, E_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_{E_1}$, $(.,.)_W$ and $(.,.)_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\boldsymbol{V}}$, $\|.\|_{E_1}$, $\|.\|_W$ and $\|.\|_{\mathcal{W}}$, respectively.

In the study of the Problem **P**, we consider the following assumptions:
The *viscosity function* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2)| \le L_{\mathcal{A}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|, \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \qquad (\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2, \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \text{ for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{0}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{3.7}$$

The *elasticity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{G}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2)| \le L_{\mathcal{G}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|, \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \text{ for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}^\ell. \end{cases} \tag{3.8}$$

The *relaxation function* $\mathcal{F}^\ell : \Omega^\ell \times (0,T) \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{F}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_1, r_1) - \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_2, r_2)| \le L_{\mathcal{F}^\ell}\big(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |r_1 - r_2|\big), \\ \qquad \text{for all } t \in (0,T), \ \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \ r_1, r_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \text{ is Lebesgue measurable in } \Omega^\ell, \\ \qquad \text{for any } t \in (0,T), \ \boldsymbol{\xi} \in \mathbb{S}^d, \ r \in \mathbb{R}. \\ \text{(c) The mapping } t \mapsto \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \text{ is continuous in } (0,T), \\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d, \ r \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{0}, 0) \text{ belongs to } \mathcal{H}^\ell, \text{ for all } t \in (0,T). \end{cases} \tag{3.9}$$

The *energy function* $\Theta^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\Theta^\ell} > 0 \text{ such that} \\ \quad |\Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1, \alpha_1) - \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2, \alpha_2)| \le L_{\Theta^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|\big), \\ \quad \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d \text{ and } \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \text{ belongs to } L^2(\Omega^\ell). \\ \text{(d) } \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha) \text{ is bounded for all } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d, \alpha \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{3.10}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x}, \tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell), \ 1 \le i, j, k \le d. \end{cases} \tag{3.11}$$

Recall also that the transposed operator $(\mathcal{E}^\ell)^*$ is given by $(\mathcal{E}^\ell)^* = (e^{\ell,*}_{ijk})$, where $e^{\ell,*}_{ijk} = e^\ell_{kij}$ and the following equality holds

$$\mathcal{E}^\ell \sigma.\boldsymbol{v} = \sigma.(\mathcal{E}^\ell)^*\boldsymbol{v}, \quad \forall \sigma \in \mathbb{S}^d,\ \forall \boldsymbol{v} \in \mathbb{R}^d.$$

The *electric permittivity operator* $\mathcal{B}^\ell = (b^\ell_{ij}) : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

$$\begin{cases} \text{(a) } \mathcal{B}^\ell(\boldsymbol{x}, \mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j), \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } b^\ell_{ij} = b^\ell_{ji},\ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i, j \le d. \\ \text{(c) There exists } m_{\mathcal{B}^\ell} > 0 \text{ such that } \mathcal{B}^\ell\mathbf{E}.\mathbf{E} \ge m_{\mathcal{B}^\ell}|\mathbf{E}|^2,\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{3.12}$$

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists\, L_\nu > 0 \text{ such that } |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu |r_1 - r_2|, \quad \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0, \quad \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\ \forall r \in \mathbb{R}. \\ \text{(d) } p_\nu(\boldsymbol{x}, r) = 0, \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.13}$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{aligned} &\mathbf{f}^\ell_0 \in C(0, T; L^2(\Omega^\ell)^d), \quad \mathbf{f}^\ell_2 \in C(0, T; L^2(\Gamma^\ell_2)^d), \\ &q^\ell_0 \in C(0, T; L^2(\Omega^\ell)), \quad q^\ell_2 \in C(0, T; L^2(\Gamma^\ell_b)), \\ &\rho^\ell \in C(0, T; L^2(\Omega^\ell)). \end{aligned} \tag{3.14}$$

The adhesion coefficients γ_ν, γ_τ and ε_a satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3),\ \ \varepsilon_a \in L^2(\Gamma_3),\ \gamma_\nu, \gamma_\tau, \varepsilon_a \ge 0, \text{ a.e. on } \Gamma_3. \tag{3.15}$$

The energy coefficient κ^ℓ_0 satisfies

$$\kappa^\ell_0 > 0. \tag{3.16}$$

Finally, the friction coefficient and the initial data satisfy

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \ge 0, \quad \text{a.e. on } \Gamma_3 \tag{3.17}$$

$$\boldsymbol{u}^\ell_0 \in \boldsymbol{V}^\ell,\ \ \theta^\ell_0 \in E^\ell_1,\ \ \beta_0 \in L^2(\Gamma_3),\ 0 \le \beta_0 \le 1, \text{ a.e. on } \Gamma_3. \tag{3.18}$$

Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0, T] \to \boldsymbol{V}$ and $q = (q^1, q^2) : [0, T] \to W$ as follows:

$$(\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}^\ell_0(t) \cdot \boldsymbol{v}^\ell \, \mathrm{d}x + \sum_{\ell=1}^{2} \int_{\Gamma^\ell_2} \mathbf{f}^\ell_2(t) \cdot \boldsymbol{v}^\ell \, \mathrm{d}a, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.19}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q^\ell_0(t)\zeta^\ell \, \mathrm{d}x - \sum_{\ell=1}^{2} \int_{\Gamma^\ell_b} q^\ell_2(t)\zeta^\ell \, \mathrm{d}a, \quad \forall \zeta \in W. \tag{3.20}$$

Next, we define the mappings $a_0 : E_1 \times E_1 \to \mathbb{R}$, $j_{ad} : L^2(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, $j_{\nu c} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{fr} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, respectively, by

$$a_0(\zeta,\xi) = \sum_{\ell=1}^{2} \kappa_0^\ell \int_{\Omega^\ell} \nabla\zeta^\ell.\nabla\xi^\ell \,\mathrm{d}x + \sum_{\ell=1}^{2} \alpha^\ell \int_{\Gamma^\ell} \zeta^\ell \xi^\ell \,\mathrm{d}a, \tag{3.21}$$

$$j_{ad}(\beta,\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]).[\boldsymbol{v}_\tau] \Big) \,\mathrm{d}a, \tag{3.22}$$

$$j_{\nu c}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \,\mathrm{d}a, \tag{3.23}$$

$$j_{fr}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} \mu p_\nu([u_\nu]) \big\|[v_\tau]\big\| \,\mathrm{d}a \tag{3.24}$$

for all $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}$ and $t \in [0,T]$. We note that conditions (3.14) imply

$$\mathbf{f} \in C(0,T;\boldsymbol{V}), \quad q \in C(0,T;W). \tag{3.25}$$

We now turn to derive a variational formulation of the mechanical problem P. To that end we assume that $\{\boldsymbol{u}^\ell, \boldsymbol{\sigma}^\ell, \varphi^\ell, \theta^\ell, \beta, \boldsymbol{D}^\ell\}$ are sufficiently smooth functions satisfying (2.1)–(2.15) and let $v = (v^1, v^2) \in V$, and $t \in [0,T]$. First, using Green's formula and (2.4) we have

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell,\ \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)))_{\mathcal{H}^\ell}$$
$$= \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t).(\boldsymbol{v}^\ell - \dot{\boldsymbol{u}}^\ell(t)) \,\mathrm{d}x + \sum_{\ell=1}^{2} \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.(\boldsymbol{v}^\ell - \dot{\boldsymbol{u}}^\ell(t)) \,\mathrm{d}a, \tag{3.26}$$

and by (2.6), (2.7) and (3.19) we find

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell,\ \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)))_{\mathcal{H}^\ell} = (\mathbf{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}^\ell(t))_V + \sum_{\ell=1}^{2} \int_{\Gamma_3} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.(\boldsymbol{v}^\ell - \dot{\boldsymbol{u}}^\ell(t)) \,\mathrm{d}a. \tag{3.27}$$

Using now (2.8) and (2.9), it follows that

$$\sum_{\ell=1}^{2} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.(\boldsymbol{v}^\ell - \dot{\boldsymbol{u}}^\ell(t)) = \sum_{\ell=1}^{2} \sigma_\nu^\ell (\boldsymbol{v}_\nu^\ell - \dot{\boldsymbol{u}}_\nu^\ell(t)) + \sum_{\ell=1}^{2} \boldsymbol{\sigma}_\tau^\ell.(\boldsymbol{v}_\tau^\ell - \dot{\boldsymbol{u}}_\tau^\ell(t))$$
$$= \sigma_\nu([\boldsymbol{v}_\nu] - [\dot{\boldsymbol{u}}_\nu(t))]) + \boldsymbol{\sigma}_\tau.([\boldsymbol{v}_\tau] - [\dot{\boldsymbol{u}}_\tau(t)])$$
$$= -p_\nu([u_\nu])([\boldsymbol{v}_\nu] - [\dot{\boldsymbol{u}}_\nu(t))]) + \gamma_\nu \beta^2 R_\nu([u_\nu])([\boldsymbol{v}_\nu] - [\dot{\boldsymbol{u}}_\nu(t))])$$
$$+ (\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])).([\boldsymbol{v}_\tau] - [\dot{\boldsymbol{u}}_\tau(t)]) - \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]).([\boldsymbol{v}_\tau] - [\dot{\boldsymbol{u}}_\tau(t)])$$

and use (3.22) and (3.23) to obtain

$$\sum_{\ell=1}^{2} \int_{\Gamma_3} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.(\boldsymbol{v}^\ell - \dot{\boldsymbol{u}}^\ell(t)) \,\mathrm{d}a \geq -j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) - j_{\nu c}(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))$$
$$- \int_{\Gamma_3} \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\| (\|[\boldsymbol{v}_\tau]\| - \|[\dot{\boldsymbol{u}}_\tau(t)]\|) \,\mathrm{d}a. \tag{3.28}$$

Finally, we combine (2.9), (3.27) and (3.28) and use the definition (3.24) to deduce that

$$\sum_{\ell=1}^{2}(\boldsymbol{\sigma}^\ell,\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)))_{\mathcal{H}^\ell}+j_{ad}(\beta(t),\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))$$
$$+j_{\nu c}(\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))+j_{fr}(\boldsymbol{u}(t),\boldsymbol{v})-j_{fr}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\geq(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_{\boldsymbol{V}}. \tag{3.29}$$

On the other hand, applying Green's formula, (2.3) and (2.13), with $\xi=(\xi^1,\xi^2)\in E_1$, it follows that

$$\begin{aligned}
&\sum_{\ell=1}^{2}\Big(\Theta^\ell\big(\boldsymbol{\sigma}^\ell(t)+\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)),\theta^\ell(t)\big)+\rho^\ell(t),\xi^\ell\Big)_{L^2(\Omega^\ell)}\\
&=\sum_{\ell=1}^{2}(\dot{\theta}^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}-\sum_{\ell=1}^{2}\int_{\Omega^\ell}\kappa_0^\ell\Delta\theta^\ell(t)\xi^\ell\,\mathrm{d}x\\
&=\sum_{\ell=1}^{2}(\dot{\theta}^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}+\sum_{\ell=1}^{2}\int_{\Omega^\ell}\kappa_0^\ell\nabla\theta^\ell(t)\nabla\xi^\ell\,\mathrm{d}x-\sum_{\ell=1}^{2}\int_{\Gamma^\ell}\kappa_0^\ell\frac{\partial^\ell\theta^\ell(t)}{\partial\nu^\ell}\xi^\ell\,\mathrm{d}a\\
&=\sum_{\ell=1}^{2}(\dot{\theta}^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}+\sum_{\ell=1}^{2}\int_{\Omega^\ell}\kappa_0^\ell\nabla\theta^\ell(t)\nabla\xi^\ell\,\mathrm{d}x+\sum_{\ell=1}^{2}\int_{\Gamma^\ell}\alpha^\ell\theta^\ell(t)\xi^\ell\,\mathrm{d}a.
\end{aligned}$$

We use now (3.21) in the previous equality to obtain

$$\begin{aligned}
&\sum_{\ell=1}^{2}(\dot{\theta}^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}+a_0(\theta(t),\xi)\\
&=\sum_{\ell=1}^{2}\Big(\Theta^\ell\big(\boldsymbol{\sigma}^\ell(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)),\theta^\ell(t)\big),\xi^\ell\Big)_{L^2(\Omega^\ell)}+\sum_{\ell=1}^{2}(\rho^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}.
\end{aligned}\tag{3.30}$$

By similar arguments, from (2.2) and (3.20), with $\phi=(\phi^1,\phi^2)\in W$, it follows that

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi^\ell(t),\nabla\phi^\ell)_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell(t)),\nabla\phi^\ell)_{H^\ell}=(q(t),\phi)_W. \tag{3.31}$$

We now gather the constitutive laws (2.1)-(2.2), the boundary condition (2.10), the initial condition (2.14), inequality (3.29) and equalities (3.30), (3.31) to obtain the following variational formulation of the mechanical problem **P.**

Problem PV. Find a displacement field $\boldsymbol{u}=(\boldsymbol{u}^1,\boldsymbol{u}^2):[0,T]\to\boldsymbol{V}$, a stress field $\boldsymbol{\sigma}=(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2):[0,T]\to\mathcal{H}$, an electric potential field $\varphi=(\varphi^1,\varphi^2):[0,T]\to W$, a temperature $\theta=(\theta^1,\theta^2):[0,T]\to E_1$, a bonding field $\beta:[0,T]\to L^\infty(\Gamma_3)$ and an electric displacement field $\boldsymbol{D}=(\boldsymbol{D}^1,\boldsymbol{D}^2):[0,T]\to\mathcal{W}$ such that

$$\boldsymbol{\sigma}^\ell=\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell)+\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)+(\mathcal{E}^\ell)^*\nabla\varphi^\ell+\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell(s)),\theta^\ell(s)\big)\,\mathrm{d}s,\ \text{in }\Omega^\ell\times(0,T), \tag{3.32}$$

$$\boldsymbol{D}^\ell=\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell)-\mathcal{B}^\ell\nabla\varphi^\ell\quad\text{in }\Omega^\ell\times(0,T), \tag{3.33}$$

$$\begin{aligned}
&\sum_{\ell=1}^{2}(\boldsymbol{\sigma}^\ell,\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)))_{\mathcal{H}^\ell}+j_{ad}(\beta(t),\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))\\
&+j_{\nu c}(\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))+j_{fr}(\boldsymbol{u}(t),\boldsymbol{v})-j_{fr}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\\
&\geq(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_{\boldsymbol{V}},\ \ \forall\boldsymbol{v}\in\boldsymbol{V},\ \text{a.e. }t\in(0,T),
\end{aligned}\tag{3.34}$$

$$\sum_{\ell=1}^{2}(\dot{\theta}^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta(t),\xi) = \sum_{\ell=1}^{2}\Big(\Theta^\ell\big(\boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)),\theta^\ell(t)\big),\xi^\ell\Big)_{L^2(\Omega^\ell)}$$

$$+\sum_{\ell=1}^{2}(\rho^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}, \quad \forall \xi\in E_1, \text{ a.e. } t\in(0,T), \tag{3.35}$$

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi^\ell(t),\nabla\phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell(t)),\nabla\phi^\ell)_{H^\ell} = (q(t),\phi)_W, \forall\phi\in W, \text{ a.e. } t\in(0,T), \tag{3.36}$$

$$\dot{\beta}(t) = -\Big(\beta(t)\big(\gamma_\nu(R_\nu([u_\nu(t)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_\tau(t)])|^2\big) - \varepsilon_a\Big)_+, \text{ a.e. } (0,T), \tag{3.37}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \theta(0)=\theta_0, \ \zeta(0)=\zeta_0, \ \beta(0)=\beta_0. \tag{3.38}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a temperature, a bonding field and an electric displacement field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

Remark 3.1 *We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction* $0\le\beta\le 1$. *Indeed, equation* (3.37) *guarantees that* $\beta(x,t)\le\beta_0(x)$ *and, therefore, assumption* (3.18) *shows that* $\beta(x,t)\le 1$ *for* $t\ge 0$, *a.e.* $x\in\Gamma_3$. *On the other hand, if* $\beta(x,t_0)=0$ *at time* t_0, *then it follows from* (3.37) *that* $\dot\beta(x,t)=0$ *for all* $t\ge t_0$ *and therefore,* $\beta(x,t)=0$ *for all* $t\ge t_0$, *a.e.* $x\in\Gamma_3$. *We conclude that* $0\le\beta(x,t)\le 1$ *for all* $t\in[0,T]$, *a.e.* $x\in\Gamma_3$.

Below in this section β,β_1,β_2 denote elements of $L^2(\Gamma_3)$ such that $0\le\beta,\beta_1,\beta_2\le 1$ a.e. $x\in\Gamma_3$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{v}$ represent elements of $\boldsymbol{V}$ and $C>0$ represents generic constants which may depend on Ω^ℓ, Γ_3, p_ν, γ_ν, γ_τ and L. First, we note that the functional j_{ad} and $j_{\nu c}$ are linear with respect to the last argument and, therefore,

$$j_{ad}(\beta,\boldsymbol{u},-\boldsymbol{v}) = -j_{ad}(\beta,\boldsymbol{u},\boldsymbol{v}), \qquad j_{\nu c}(\boldsymbol{u},-\boldsymbol{v}) = -j_{\nu c}(\boldsymbol{u},\boldsymbol{v}). \tag{3.39}$$

Next, (3.23) and (3.13)(b) imply

$$j_{\nu c}(\boldsymbol{u}_1,\boldsymbol{v}_2) - j_{\nu c}(\boldsymbol{u}_1,\boldsymbol{v}_1) + j_{\nu c}(\boldsymbol{u}_2,\boldsymbol{v}_1) - j_{\nu c}(\boldsymbol{u}_2,\boldsymbol{v}_2) \le 0, \quad \forall \boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2\in\boldsymbol{V}, \tag{3.40}$$

and using (3.24), (3.13)(a), keeping in mind (3.3), we obtain

$$\begin{aligned} &j_{fr}(\boldsymbol{u}_1,\boldsymbol{v}_2) - j_{fr}(\boldsymbol{u}_1,\boldsymbol{v}_1) + j_{fr}(\boldsymbol{u}_2,\boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_2,\boldsymbol{v}_2)\\ &\le c_0^2 L_\nu\|\mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{V}}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{\boldsymbol{V}}, \quad \forall \boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2\in\boldsymbol{V}. \end{aligned} \tag{3.41}$$

Inequalities (3.39)–(3.41) will be used in various places in the rest of the paper. Our main existence and uniqueness result that we state now and prove in the next section is the following.

Theorem 3.2 (Existence and uniqueness) *Assume that* (3.7)–(3.18) *hold. Then there exists a unique solution of Problem **PV**. Moreover, the solution satisfies*

$$\boldsymbol{u}\in C^1(0,T;\boldsymbol{V}), \tag{3.42}$$

$$\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1), \tag{3.43}$$

$$\varphi\in C(0,T;W), \tag{3.44}$$

$$\theta\in L^2(0,T;E_1)\cap H^1(0,T;E_0), \tag{3.45}$$

$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap\mathcal{Z}, \tag{3.46}$$

$$\boldsymbol{D}\in C(0,T;\mathcal{W}). \tag{3.47}$$

The functions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, θ, β and $\boldsymbol{D}$ which satisfy (3.32)-(3.38) are called a weak solution of the contact Problem **P**. We conclude that, under the assumptions (3.7)– (3.18), the mechanical problem (2.1)–(2.15) has a unique weak solution satisfying (3.42)–(3.47).

4 Proof of Theorem 3.2

The proof of Theorem 3.2 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(.,.)_X$ and the associated norm $\|.\|_X$, and consider the problem of finding $\boldsymbol{u} : [0,T] \to X$ such that

$$\begin{cases} (A\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_X + (B\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_X + j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \geq (f(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_X, \quad \forall \boldsymbol{v} \in X,\ t \in [0,T], \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{4.1}$$

To study problem (4.1) we need the following assumptions: The operator $A : X \to X$ is Lipschitz continuous and strongly monotone, *i.e.*,

$$\begin{cases} \text{(a) There exists } L_A > 0 \text{ such that} \\ \qquad \|A\boldsymbol{u}_1 - A\boldsymbol{u}_2\|_X \leq L_A \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X, \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in X, \\ \text{(b) There exists } m_A > 0 \text{ such that} \\ \qquad (A\boldsymbol{u}_1 - A\boldsymbol{u}_2,\ \boldsymbol{u}_1 - \boldsymbol{u}_2)_X \geq m_A \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X, \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in X. \end{cases} \tag{4.2}$$

The nonlinear operator $B : X \to X$ is Lipschitz continuous, *i.e.*, there exists $L_B > 0$ such that

$$\|B\boldsymbol{u}_1 - B\boldsymbol{u}_2\|_X \leq L_B \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X, \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in X. \tag{4.3}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(\boldsymbol{u}, .) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b) There exists } m_j > 0 \text{ such that} \\ \qquad j(\boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ \qquad\quad \leq m_j \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_X, \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in X. \end{cases} \tag{4.4}$$

Finally, we assume that

$$f \in C(0,T;X) \tag{4.5}$$

and

$$\boldsymbol{u}_0 \in X. \tag{4.6}$$

The following existence, uniqueness result and regularity was proved in [9, Theorem 11.3].

Theorem 4.1 *Let* (4.2)–(4.6) *hold. Then:*

1. *There exists a unique solution* $\boldsymbol{u} \in C^1(0,T;X)$ *of Problem* (4.1).
2. *If, moreover,* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *are two solutions of* (4.1) *corresponding to the data* $f_1, f_2 \in C(0,T;X)$, *then there exists* $c > 0$ *such that*

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_X \leq c\big(\|f_1(t) - f_2(t)\|_X + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_X\big), \tag{4.7}$$

for all $t \in [0,T]$.

We turn now to the proof of Theorem 3.2 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in what follows that (3.7)–(3.18) hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\mathcal{B}^\ell$, $\mathcal{G}^\ell$, $\mathcal{F}^\ell$, $\mathcal{E}^\ell$, γ_ν, γ_τ, Θ^ℓ, κ_0^ℓ, and T with $\ell = 1, 2$, but does not depend on t nor of the rest of input data, and whose value may change from place to place. Let $\eta = (\eta^1, \eta^2) \in C(0,T;\boldsymbol{V})$ be given. In the first step we consider the following variational problem.

Problem PV$_\eta^u$. Find a displacement field $\boldsymbol{u}_\eta = (\boldsymbol{u}_\eta^1, \boldsymbol{u}_\eta^2) : [0,T] \to \boldsymbol{V}$ such that

$$\begin{aligned}
&\sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta^\ell),\ \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta^\ell(t)))_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}(\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^\ell),\ \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta^\ell(t)))_{\mathcal{H}^\ell} \\
&+ j_{\nu c}(\boldsymbol{u}_\eta(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)) + j_{fr}(\boldsymbol{u}_\eta(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}_\eta(t), \dot{\boldsymbol{u}}_\eta(t)) \qquad (4.8)\\
&+ (\eta(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_{\boldsymbol{V}} \geq (\mathbf{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_{\boldsymbol{V}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V},\ t \in (0,T),
\end{aligned}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \qquad (4.9)$$

We have the following result for the problem **PV**$_\eta^u$.

Lemma 4.2 *(1) There exists a unique solution $\boldsymbol{u}_\eta \in C^1(0,T;\boldsymbol{V})$ to the problem* (4.8) *and* (4.9)*.*

(2) If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of (4.8) *and* (4.9) *corresponding to the data η_1, $\eta_2 \in C(0,T;\boldsymbol{V})$, then there exists $c > 0$ such that*

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_{\boldsymbol{V}} \leq c\left(\|\eta_1(t) - \eta_2(t)\|_{\boldsymbol{V}} + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}}\right), \quad \forall\, t \in [0,T]. \qquad (4.10)$$

Proof. We apply Theorem 4.1 where $X = \boldsymbol{V}$, with the inner product $(\cdot,\cdot)_{\boldsymbol{V}}$ and the associated norm $\|\cdot\|_{\boldsymbol{V}}$. We use the Riesz representation theorem to define the operators $A : \boldsymbol{V} \to \boldsymbol{V}$, and $B : \boldsymbol{V} \to \boldsymbol{V}$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \qquad (4.11)$$

$$(B\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2}(\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \qquad (4.12)$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, and define the functions $\mathbf{f}_\eta : [0,T] \to \boldsymbol{V}$, $j : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}(t), \quad \forall t \in [0,T], \qquad (4.13)$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}) + j_{fr}(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \qquad (4.14)$$

Assumptions (3.7) and (3.8) imply that the operators A and B satisfy conditions (4.2) and (4.3), respectively.

It follows from (3.13), (3.17), (3.23) and (3.24) that the functional j, (4.14), satisfies condition (4.4)(a). We use again (3.40), (3.41) and (4.14) to find

$$\begin{aligned}
&j(\boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{u}_2, \boldsymbol{v}_2) \\
&\leq c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{V}}, \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V}, \qquad (4.15)
\end{aligned}$$

which shows that the functional j satisfies condition (4.4)(b) on $X = \boldsymbol{V}$. Moreover, using (3.25) and, keeping in mind that $\boldsymbol{\eta} \in C(0,T;\boldsymbol{V})$, we deduce from (4.13) that $\mathbf{f}_\eta \in C(0,T;\boldsymbol{V})$, *i.e.*, $\mathbf{f}_\eta$ satisfies (4.5). Finally, we note that (3.18) shows that condition (4.6) is satisfied. Using now (4.11)–(4.14) we find that Lemma 4.2 is a direct consequence of Theorem 4.1. □

In the second step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.2 and we consider the following variational problem.

Problem $\mathbf{PV}_\eta^\varphi$. Find the electric potential field $\varphi_\eta : [0,T] \to W$ such that

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell \nabla\varphi_\eta^\ell(t), \nabla\phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2}(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_\eta^\ell(t)), \nabla\phi^\ell)_{H^\ell} = (q(t),\phi)_W, \ \forall\phi \in W, \ \text{a.e. } t \in (0,T). \quad (4.16)$$

We have the following result.

Lemma 4.3 *Problem $\boldsymbol{PV}_\eta^\varphi$ has a unique solution φ_η which satisfies the regularity* (3.44).

Proof. We define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi,\phi) = \sum_{\ell=1}^{2}(\mathcal{B}^\ell \nabla\varphi^\ell, \nabla\phi^\ell)_{H^\ell}, \quad \forall\varphi,\phi \in W. \quad (4.17)$$

We use (3.4), (3.5), (3.12) and (4.17) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on W, moreover using (3.20) and the Riesz representation theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t),\phi)_W = (q(t),\phi)_W + \sum_{\ell=1}^{2}(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_\eta^\ell(t)), \nabla\phi^\ell)_{H^\ell}, \quad \forall\phi \in W,\ t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t),\phi) = (q_\eta(t),\phi)_W, \quad \forall\phi \in W. \quad (4.18)$$

We conclude that φ_η is a solution of Problem $\mathbf{PV}_\eta^\varphi$. Let $t_1, t_2 \in [0,T]$, it follows from (4.16) that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \leq C\big(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big). \quad (4.19)$$

We also note that assumptions (3.25) and $\boldsymbol{u}_\eta \in C^1(0,T;\boldsymbol{V})$, inequality (4.19) implies that $\varphi_\eta \in C(0,T;W)$. □

In the third step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma4.2 and we consider the following initial-value problem.

Problem $\mathbf{PV}_\eta^\beta$. Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot\beta_\eta(t) = -\Big(\beta_\eta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}(t)])|^2\big) - \varepsilon_a\Big)_+, \quad \text{a.e. } t \in (0,T), \quad (4.20)$$

$$\beta_\eta(0) = \beta_0. \quad (4.21)$$

We have the following result.

Lemma 4.4 *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem $\boldsymbol{PV}_\eta^\beta$.*

Proof. For the simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\Big(\beta\big[\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}(t)])|^2\,\big] - \varepsilon_a\Big)_+,$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz theorem (see, [20, p. 48]) we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ as a solution to the Problem $\mathbf{PV}_\eta^\beta$. Also, the arguments used in Remark 3.1 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\beta_\eta \in \mathcal{Z}$, which concludes the proof of the lemma. □

In the fourth step we let $\lambda = (\lambda^1, \lambda^2) \in C(0,T;E_0)$ be given and consider the following initial-value problem for the temperature.

Problem $\mathbf{PV}_\lambda^\theta$. *Find the temperature* $\theta_\lambda = (\theta_\lambda^1, \theta_\lambda^2) : [0,T] \to E_0$ such that

$$\sum_{\ell=1}^{2}(\dot\theta_\lambda^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta_\lambda^\ell(t),\xi) = \sum_{\ell=1}^{2}(\lambda^\ell(t)+\rho^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)},\ \forall\xi\in E_0,\ \text{a.e. } t\in(0,T), \tag{4.22}$$

$$\theta_\lambda(0) = \theta_0. \tag{4.23}$$

Lemma 4.5 *There exists a unique solution θ_λ to the auxiliary problem $\boldsymbol{PV}_\lambda^\theta$ satisfying* (3.45).

Proof. We observe that the expression (4.22) is uncoupled. By using some arguments of evolutionary variational equations (see, *e.g.*, [12]), it follows that there exists a unique solution to (4.22) satisfying (4.23) and the regularity (3.45). □

Finally as a consequence of these results and using the properties of the operator $\mathcal{E}^\ell$, the operator $\mathcal{F}^\ell$, the functional j_{ad}, and the functional Θ^ℓ, for $t \in [0,T]$, we consider the element

$$\Lambda(\eta,\lambda)(t) = \big(\Lambda^1(\eta,\lambda)(t),\ \Lambda^2(\eta,\lambda)(t)\big) \in \boldsymbol{V} \times E_0, \tag{4.24}$$

defined by the equations

$$\begin{aligned}(\Lambda^1(\eta,\lambda)(t),\boldsymbol{v})_{\boldsymbol{V}} &= \sum_{\ell=1}^{2}\Big(\int_0^t \mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_\eta^\ell(s)),\theta_\lambda^\ell(s)\big)\,\mathrm{d}s,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell}\\ &\quad+\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\varphi_\eta^\ell,\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{ad}(\beta_\eta(t),\boldsymbol{u}_\eta(t),\boldsymbol{v}),\ \ \forall\boldsymbol{v}\in\boldsymbol{V},\end{aligned} \tag{4.25}$$

$$\Lambda^2(\eta,\lambda)(t) = \Big(\Theta^1\big(\boldsymbol{\sigma}_{\eta\lambda}^1(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^1(t)),\theta_\lambda^1(t)\big),\ \Theta^2\big(\boldsymbol{\sigma}_{\eta\lambda}^2(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^2(t)),\theta_\lambda^2(t)\big)\Big). \tag{4.26}$$

Here, for every $(\eta,\lambda)\in C(0,T;\boldsymbol{V}\times E_0)$, $\boldsymbol{u}_\eta$, φ_η and β_η, θ_λ represent the displacement field, the potential electric field, the bonding field and the temperature field obtained in Lemmas 4.2, 4.3, 4.4 and 4.5 respectively, and $\boldsymbol{\sigma}^\ell_{\eta\lambda}$ denotes by

$$\boldsymbol{\sigma}^\ell_{\eta\lambda}(t)=\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(t))+(\mathcal{E}^\ell)^*\nabla\varphi^\ell_\eta+\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_\eta(s)),\theta^\ell_\lambda(s)\big)\,\mathrm{d}s\ \text{ in }\Omega^\ell\times(0,T). \quad (4.27)$$

We have the following result.

Lemma 4.6 *There exists a unique* $(\eta^*,\lambda^*)\in C(0,T;\boldsymbol{V}\times E_0)$, *such that* $\Lambda(\eta^*,\lambda^*)=(\eta^*,\lambda^*)$.

Proof. Let $(\eta_1,\lambda_1),(\eta_2,\lambda_2)\in C(0,T;\boldsymbol{V}\times E_0)$ and denote by $\boldsymbol{u}_i$, φ_i, β_i, θ_i and $\boldsymbol{\sigma}_i$, the functions obtained in Lemmas 4.2, 4.3, 4.4, 4.5 and the relation (4.27) respectively, for $(\eta,\lambda)=(\eta_i,\lambda_i)$, $i=1,2$. Let $t\in[0,T]$. Using (3.9), (3.11), (3.22) and the definition of R_ν, $\boldsymbol{R}_\tau$, we have

$$\begin{aligned}
&\|\Lambda^1(\eta_1,\lambda_1)(t)-\Lambda^1(\eta_2,\lambda_2)(t)\|^2_{\boldsymbol{V}}\le\sum_{\ell=1}^2\|(\mathcal{E}^\ell)^*\nabla\varphi^\ell_1(t)-(\mathcal{E}^\ell)^*\nabla\varphi^\ell_2(t)\|^2_{\mathcal{H}^\ell}\\
&+\sum_{\ell=1}^2\int_0^t\big\|\mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_1(s)),\theta^\ell_1(s)\big)-\mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_2(s)),\theta^\ell_2(s)\big)\big\|^2_{\mathcal{H}^\ell}\,\mathrm{d}s\\
&+C\|\beta_1^2(t)R_\nu([u_{1\nu}(t)])-\beta_2^2(t)R_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)}\\
&+C\|\beta_1^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)])-\beta_2^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|^2_{L^2(\Gamma_3)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\Lambda^1(\eta_1,\lambda_1)(t)-\Lambda^1(\eta_2,\lambda_2)(t)\|^2_{\boldsymbol{V}}\le C\bigg(\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,\mathrm{d}s\\
&+\int_0^t\|\theta_1(s)-\theta_2(s))\|^2_{E_0}\,\mathrm{d}s\ +\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\bigg).
\end{aligned} \quad (4.28)$$

Recall that above $u^\ell_{\eta\nu}$ and $\boldsymbol{u}^\ell_{\eta\tau}$ denote the normal and the tangential component of the function $\boldsymbol{u}^\ell_\eta$ respectively. By similar arguments, from (4.26), (4.27) and (3.10) it follows that

$$\begin{aligned}
&\|\Lambda^2(\eta_1,\lambda_1)(t)-\Lambda^2(\eta_2,\lambda_2)(t)\|^2_{E_0}\\
&\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,\mathrm{d}s+\|\theta_1(t)-\theta_2(t))\|^2_{E_0}\\
&+\int_0^t\|\theta_1(s)-\theta_2(s))\|^2_{E_0}\,\mathrm{d}s+\|\varphi_1(t)-\varphi_2(t)\|^2_W\bigg).
\end{aligned} \quad (4.29)$$

It follows now from (4.28) and (4.29) that

$$\begin{aligned}
&\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\|^2_{\boldsymbol{V}\times E_0}\\
&\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\boldsymbol{V}}\,\mathrm{d}s+\|\theta_1(t)-\theta_2(t)\|^2_{E_0}\\
&+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{E_0}\,\mathrm{d}s+\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\bigg).
\end{aligned} \quad (4.30)$$

Also, since

$$\boldsymbol{u}_i^\ell(t) = \int_0^t \dot{\boldsymbol{u}}_i^\ell(s)\,\mathrm{d}s + \boldsymbol{u}_0^\ell(t), \quad t \in [0,T], \ \ \ell = 1,2,$$

we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \le \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_{\boldsymbol{V}}\,\mathrm{d}s$$

and using this inequality in (4.10) yields

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \le C\Big(\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\boldsymbol{V}}\,\mathrm{d}s + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{\boldsymbol{V}}\,\mathrm{d}s\Big). \tag{4.31}$$

Next, we apply Gronwall's inequality to deduce

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \le C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\boldsymbol{V}}\,\mathrm{d}s, \quad \forall t \in [0,T]. \tag{4.32}$$

On the other hand, from the Cauchy problem (4.20)–(4.21) we can write

$$\beta_i(t) = \beta_0 - \int_0^t \Big(\beta_i(s)\big(\gamma_\nu(R_\nu([u_{i\nu}(s)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}(s)])|^2\,\big) - \varepsilon_a\Big)_+\mathrm{d}s$$

and then

$$\begin{aligned}\big\|\beta_1(t) - \beta_2(t)\big\|_{L^2(\Gamma_3)} &\le C\int_0^t \big\|\beta_1(s)R_\nu([u_{1\nu}(s)])^2 - \beta_2(s)R_\nu([u_{2\nu}(s)])^2\big\|_{L^2(\Gamma_3)}\,\mathrm{d}s\\ &\quad +\ C\int_0^t \big\|\beta_1(s)\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(s)])|^2 - \beta_2(s)\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(s)])|^2\,\big\|_{L^2(\Gamma_3)}\,\mathrm{d}s.\end{aligned}$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\big\|\beta_1(t) - \beta_2(t)\big\|_{L^2(\Gamma_3)} \le C\Big(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\,\mathrm{d}s + \int_0^t \big\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\big\|_{L^2(\Gamma_3)^d}\,\mathrm{d}s\Big). \tag{4.33}$$

Next, we apply Gronwall's inequality to deduce

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d}\,\mathrm{d}s,$$

and from the relation (3.3) we obtain

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V\,\mathrm{d}s. \tag{4.34}$$

We use now (4.16), (3.4), (3.11) and (3.12) to find

$$\|\varphi_1(t) - \varphi_2(t)\|^2_W \le C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V. \tag{4.35}$$

From (4.22) we deduce that

$$(\dot\theta_1 - \dot\theta_2, \theta_1 - \theta_2)_{E_0} + a_0(\theta_1 - \theta_2, \theta_1 - \theta_2) + \big(\lambda_1 - \lambda_2, \theta_1 - \theta_2\big)_{E_0} = 0, \text{ a.e. } t \in (0,T).$$

We integrate this equality with respect to time, using the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$ and inequality $a_0(\theta_1 - \theta_2, \theta_1 - \theta_2) \ge 0$, to find

$$\frac{1}{2}\|\theta_1(t) - \theta_2(t)\|^2_{E_0} \le \int_0^t \big(\lambda_1(s) - \lambda_2(s), \theta_1(s) - \theta_2(s)\big)_{E_0}\,\mathrm{d}s,$$

which implies that

$$\|\theta_1(t)-\theta_2(t)\|^2_{E_0} \le \int_0^t \|\lambda_1(s)-\lambda_2(s)\|^2_{E_0}\,\mathrm{d}s + \int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{E_0}\,\mathrm{d}s.$$

This inequality combined with Gronwall's inequality leads to

$$\|\theta_1(t)-\theta_2(t)\|^2_{E_0} \le C\int_0^t \|\lambda_1(s)-\lambda_2(s)\|^2_{E_0}\,\mathrm{d}s, \quad \forall t\in[0,T]. \tag{4.36}$$

We substitute (4.32), (4.34), (4.35) and (4.36) in (4.30) to obtain

$$\|\Lambda(\eta_1,\lambda_1)(t)-\Lambda(\eta_2,\lambda_2)(t)\|^2_{\boldsymbol{V}\times E_0} \le C\int_0^t \|(\eta_1,\lambda_1)(s)-(\eta_2,\lambda_2)(s)\|^2_{\boldsymbol{V}\times E_0}\,\mathrm{d}s.$$

Reiterating this inequality m times we obtain

$$\|\Lambda^m(\eta_1,\lambda_1)-\Lambda^m(\eta_2,\lambda_2)\|^2_{C(0,T;\boldsymbol{V}\times E_0)} \le \frac{C^mT^m}{m!}\|(\eta_1,\lambda_1)-(\eta_2,\lambda_2)\|^2_{C(0,T;\boldsymbol{V}\times E_0)}.$$

Thus, for m sufficiently large, $\Lambda^m(.,.)$ is a contraction on the Banach space $C(0,T;\boldsymbol{V}\times E_0)$, and so $\Lambda(.,.)$ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 3.2.

Proof. *Existence.* Let $(\eta^*,\lambda^*)\in C(0,T;\boldsymbol{V}\times E_0)$ be the fixed point of $\Lambda(.,.)$ and denote

$$\boldsymbol{u}_* = \boldsymbol{u}_{\eta^*},\;\; \varphi_* = \varphi_{\eta^*},\;\; \beta_* = \beta_{\eta^*},\;\; \theta_* = \theta_{\lambda^*}, \tag{4.37}$$

$$\boldsymbol{\sigma}^\ell_* = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*) + \mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*) + (\mathcal{E}^\ell)^*\nabla\varphi^\ell_* + \int_0^t \mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\theta^\ell_*(s)\big)\,\mathrm{d}s, \tag{4.38}$$

$$\boldsymbol{D}^\ell_* = \mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_*) - \mathcal{B}^\ell\nabla\varphi^\ell_*. \tag{4.39}$$

We prove that $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (3.32)–(3.38) and the regularites (3.42)–(3.47). Indeed, we write (4.8) for $\eta=\eta^*$ and use (4.37) to find

$$\begin{aligned}&\sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*(t)))_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}(\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*(t)))_{\mathcal{H}^\ell}\\ &+ j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t)) + j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v}) - j_{fr}(\boldsymbol{u}_*(t),\dot{\boldsymbol{u}}_*(t))\\ &+ (\eta^*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))_{\boldsymbol{V}} \ge (\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))_{\boldsymbol{V}},\ \ \forall\boldsymbol{v}\in\boldsymbol{V},\ \ \text{a.e. } t\in[0,T].\end{aligned} \tag{4.40}$$

We use equalities $\Lambda^1(\eta^*,\lambda^*)=\eta^*$ and $\Lambda^2(\eta^*,\lambda^*)=\lambda^*$, it follows from (4.25) and (4.26) that

$$\begin{aligned}(\eta^*(t),\boldsymbol{v})_{\boldsymbol{V}} &= \sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\varphi^\ell_*(t),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{ad}(\beta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})\\ &+ \sum_{\ell=1}^{2}\left(\int_0^t \mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\theta^\ell_*(s)\big)\,\mathrm{d}s,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell},\quad \forall\boldsymbol{v}\in\boldsymbol{V},\ \ \text{a.e. } t\in(0,T),\end{aligned} \tag{4.41}$$

$$\lambda^\ell_*(t) = \Theta^\ell\big(\boldsymbol{\sigma}^\ell_*(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)),\theta^\ell_*(t)\big),\ \ \text{a.e. } t\in(0,T),\ \ell=1,2. \tag{4.42}$$

We now substitute (4.41) in (4.40) to obtain

$$\begin{aligned}
&\sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell)(t),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)))_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}(\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell)(t),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)))_{\mathcal{H}^\ell}\\
&+\sum_{\ell=1}^{2}\bigg(\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_*^\ell(s)),\theta_*^\ell(s)\big)\,\mathrm{d}s,\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t))\bigg)_{\mathcal{H}^\ell}\\
&+j_{ad}(\beta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))+j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})\\
&-j_{fr}(\boldsymbol{u}_*(t),\dot{\boldsymbol{u}}_*(t))+\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\varphi_*^\ell(t),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t))\big)_{\mathcal{H}^\ell}\\
&\geq(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))_{\boldsymbol{V}},\ \ \forall\boldsymbol{v}\in\boldsymbol{V},\quad \text{a.e. } t\in[0,T],
\end{aligned}\tag{4.43}$$

and we substitute (4.42) in (4.22) to obtain

$$\sum_{\ell=1}^{2}(\dot{\theta}_*^\ell(t),\xi^\ell)_{L^2(\Omega^\ell)}+a_0(\theta_*^\ell(t),\xi)=\sum_{\ell=1}^{2}\big(\lambda_*^\ell(t)+\rho^\ell(t),\xi^\ell\big)_{L^2(\Omega^\ell)},\ \forall\xi\in E_0,\ \text{a.e. } t\in(0,T).\tag{4.44}$$

We write now (4.16) for $\eta=\eta^*$ and use (4.37) to see that

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi_*^\ell(t),\nabla\phi^\ell)_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}=(q(t),\phi)_W,\ \ \forall\phi\in W,\ t\in[0,T].\tag{4.45}$$

Additionally, we use $\boldsymbol{u}_{\eta^*}$ in (4.20) and (4.37) to find

$$\dot{\beta}_*(t)=-\Big(\beta_*(t)\big(\gamma_\nu(R_\nu([u_{*\nu}(t)]))^2+\gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}(t)])|^2\big)-\varepsilon_a\Big)_+,\ \text{a.e. } t\in[0,T].\tag{4.46}$$

The relations (4.37)-(4.46) allow us to conclude now that $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\theta_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (3.32)–(3.37). Next, (3.38) and the regularity (3.42), (3.44)–(3.46) follow from Lemmas 4.2, 4.3, 4.4, and 4.5. Since $\boldsymbol{u}_*$, φ_*, β_* and θ_* satisfies (3.42), (3.44), (3.46) and (3.45) , respectively, it follows from (4.38) that

$$\boldsymbol{\sigma}_*\in C(0,T;\mathcal{H}).\tag{4.47}$$

For $\ell=1,2$, we choose $\boldsymbol{v}=\dot{\boldsymbol{u}}\pm\phi$ in (4.43), with $\phi=(\phi^1,\phi^2)$, $\phi^\ell\in D(\Omega^\ell)^d$ and $\phi^{3-\ell}=0$, to obtain

$$\operatorname{Div}\boldsymbol{\sigma}_*^\ell(t)=-\boldsymbol{f}_0^\ell(t),\quad\forall t\in[0,T],\ \ \ell=1,2,\tag{4.48}$$

where $D(\Omega^\ell)$ is the space of infinitely differentiable real functions with a compact support in Ω^ℓ. The regularity (3.43) follows from (3.14), (4.47) and (4.48). Let now $t_1,t_2\in[0,T]$, by (3.11), (3.12), (3.4) and (4.39), we deduce that

$$\|\boldsymbol{D}_*(t_1)-\boldsymbol{D}_*(t_2)\|_H\leq C\left(\|\varphi_*(t_1)-\varphi_*(t_2)\|_W+\|\boldsymbol{u}_*(t_1)-\boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (3.42) and (3.44) implies

$$\boldsymbol{D}_*\in C(0,T;H).\tag{4.49}$$

For $\ell=1,2$, we choose $\phi=(\phi^1,\phi^2)$ with $\phi^\ell\in D(\Omega^\ell)^d$ and $\phi^{3-\ell}=0$ in (4.45) and using (3.20) we find

$$\operatorname{div}\boldsymbol{D}_*^\ell(t)=q_0^\ell(t),\quad\forall t\in[0,T],\ \ \ell=1,2.\tag{4.50}$$

Property (3.47) follows from (3.14), (4.49) and (4.50).

Finally we conclude that the weak solution $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \theta_*, \beta_*, \boldsymbol{D}_*\}$ of the piezoelectric contact Problem **PV** has the regularity (3.42)–(3.47), which concludes the existence part of Theorem 3.2.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Lambda(.,.)$ defined by (4.25)-(4.26) and the unique solvability of the Problems PV_η^u, PV_η^φ, PV_η^β, and $\mathrm{PV}_\lambda^\theta$. □

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