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NONLINEAR PARABOLIC PROBLEM WITH VARIABLE EXPONENT AND MEASURE DATA

STANISLAS OUARO[*], **URBAIN TRAORE**[†]
Laboratoire de Mathématique et Informatique (LAMI), UFR,
Sciences Exactes et Appliquées, Université Joseph KI-ZERBO
03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso

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Abstract. In this paper we prove the existence and uniqueness of renormalized solutions to nonlinear parabolic equations with variable exponent and measure data. The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

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1 Introduction and main result

Let Ω be a bounded and open domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, and let T be a positive number. In this paper we study the existence and uniqueness of a renormalized solution for the following nonlinear parabolic problem

$$(P_\mu)\begin{cases} u_t - \operatorname{div}(a(x, \nabla u)) = \mu & \text{in } Q = (0, T) \times \Omega, \\ u = u_0 & \text{on } \{0\} \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

where μ is a bounded measure on Q which does not charge sets of null $p(.)$-capacity and $u_0 \in L^1(\Omega)$. In the case where $a(x, \nabla\xi) = |\xi|^{p(x)-2}\xi$, the operator $-\operatorname{div}(a(x, \nabla u))$ is called the $p(x)$-Laplacian.

[*]e-mail address: ouaro@yahoo.fr
[†]e-mail address: urbain.traore@yahoo.fr

As a model example the problem (P_μ) includes the p-Laplace evolution equation:

$$\begin{cases} u_t - \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f & \text{in } Q = (0,T)\times\Omega, \\ u = u_0 & \text{on } \{0\}\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{1.1}$$

The existence and uniqueness of entropy and renormalized solutions to problem (1.1) with $f \in L^1(Q)$ is nowadays well-known and was established by Zhang and Zhou in [18] and by Bendahmane *et al.* in [1]. Our main goal here is to extend this existence and uniqueness result to a larger class of measures which includes the L^1 case.

Let us recall that the existence and uniqueness of parabolic variational inequalities with variable exponent was studied in [6, 12]. In [14], the authors proved the existence and uniqueness of entropy solutions to a nonlinear parabolic equation with variable exponent and L^1-data.

The importance of the measures not charging sets of null capacity was first observed in the stationary case in [3], where the authors proved that every diffuse measure μ, *i.e.*, a measure which does not charge sets of null p-capacity, belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. That allowed them to prove the existence and uniqueness of an entropy solution for the following problem

$$\begin{cases} A(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the context of variable exponent, a similar approach was used in [13] for the elliptic problem

$$\begin{cases} \nabla.a(x,\nabla u) + \beta(u) \ni \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a diffuse measure. In [13], the authors used the ideas of [3] to prove that for every diffuse measure μ there exists $f \in L^1(\Omega)$ and $g \in W^{-1,p(.)}$ such that $\mu = f + g$. This allowed them to prove the existence and uniqueness of an entropy solution.

In order to use a similar approach in the evolution case, we developed in [15] the theory of $p(.)$-parabolic capacity and then investigated the relationships between time space dependent measures and capacity.

Let us recall that the connection between parabolic capacity and measures was first introduced in [8] by J. Droniou, A. Porretta and A. Prignet in the context of Sobolev spaces with constant exponent. In their works, they proved a decomposition theorem for measures (in space and time) that do not charge sets of null capacity. Using this result, they proved the existence and uniqueness of renormalized solutions for nonlinear parabolic initial boundary-value problems with such measures as right-hand sides.

Inspired by the approach developed in [8] our main goal here is to prove the existence and uniqueness results in a larger class of measures which includes the L^1 case, in the context of Sobolev spaces with variable exponent. Namely, we prove (in the framework of renormalized solutions) that the problem (P_μ) has a unique solution for every $u_0 \in L^1(\Omega)$ and for every measure μ which does not charge sets of null $p(.)$-capacity. The notion of a $p(.)$-parabolic capacity and its connection with measures are suitably developed in our previous work (cf. [15]).

The plan of the paper is the following. In the next section, we recall some basic notations and properties of Lebesgue and Sobolev spaces with variable exponent. In section 3, we introduce our notion of a renormalized solution for the problem (P_μ). We also give some properties of the solution and prove the existence and uniqueness of a renormalized solution.

2 Preliminaries

In this paper, we assume that

$$p(.)\colon \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that } 1 < p_- \le p_+ < +\infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is the set of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x$$

is finite (see [7]). If the exponent is bounded, *i.e.*, if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \le 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $\big(L^{p(\cdot)}(\Omega), \|.\|_{p(.)}\big)$ is a separable Banach space. Moreover, if $1 < p_- \le p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for $x \in \Omega$. Finally, we have the Hölder type inequality

$$\left|\int_\Omega uv\ \mathrm{d}x\right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{p(.)}\|v\|_{p'(.)}, \tag{2.2}$$

which holds for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Let $W^{1,p(\cdot)}(\Omega) := \big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\big\}$ and let $\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$. The space $\big(W^{1,p(\cdot)}(\Omega), \|.\|_{1,p(.)}\big)$ is a separable and reflexive Banach space.

By $C_c^\infty(X)$ we denote the space of continuous functions with compact support in X. We also set $W_0^{1,p(\cdot)}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1 (see [9, 19]) *Let $u\colon \Omega \to \mathbb{R}$ be a measurable function. Then, the following statements hold*:

(i) $\min\Big\{\|u\|_{p(.)}^{p_-},\ \|u\|_{p(.)}^{p_+}\Big\} < \rho_{p(.)}(u) < \max\Big\{\|u\|_{p(.)}^{p_-},\ \|u\|_{p(.)}^{p_+}\Big\}$ *if* $\|u\|_{p(.)} < +\infty$;

(ii) $\min\Big\{\rho_{p(.)}^{\frac{1}{p_-}}(u),\ \rho_{p(.)}^{\frac{1}{p_+}}(u)\Big\} < \rho_{p(.)}(u) < \max\Big\{\rho_{p(.)}^{\frac{1}{p_-}}(u),\ \rho_{p(.)}^{\frac{1}{p_+}}(u)\Big\}$ *if* $\rho_{p(.)}(u) < +\infty$;

(iii) $\rho_{p(.)}\big(u/\|u\|_{p(.)}\big) = 1$ *if* $0 < \|u\|_{p(.)} < +\infty$.

Following [1], we extend a variable exponent $p\colon \overline{\Omega} \to [1,+\infty)$ to $\overline{Q} = [0,T] \times \overline{\Omega}$ by setting $p(t,x) = p(x)$ for all $(t,x) \in \overline{Q}$. We may also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q) = \left\{ u\colon Q \to \mathbb{R} : u \text{ is measurable and such that } \iint_Q |u(t,x)|^{p(x)}\,\mathrm{d}(t,x) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)} := \inf\left\{ \lambda > 0 : \iint_Q \left| \frac{u(t,x)}{\lambda} \right|^{p(x)} \mathrm{d}(t,x) < 1 \right\},$$

which shares the same properties as $L^{p(\cdot)}(\Omega)$.

For the vector field $a(.,.)$ we assume that $a(x,\xi)\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (*i.e.*, $a(.,\xi)$ is measurable on Ω for every $\xi \in \mathbb{R}^N$ and $a(x,.)$ is continuous on $\mathbb{R}^N$ for almost every x in Ω) such that the following conditions hold.

- There exists a positive constant C_1 such that

$$|a(x,\xi)| \le C_1\Big(j(x) + |\xi|^{p(x)-1} \Big) \tag{2.3}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a non-negative function in $L^{p'(\cdot)}(\Omega)$.

- The following inequalities hold

$$\big(a(x,\xi) - a(x,\eta)\big).(\xi - \eta) \ge 0 \tag{2.4}$$

  for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \ne \eta$, and

$$a(x,\xi).\xi \ge \frac{1}{C}|\xi|^{p(x)} \tag{2.5}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where $C > 0$ is a constant independent of the variables x and ξ.

In the sequel C will denote a non-negative constant that may change from line to line.

Now, let us recall the notion of a $p(.)$-parabolic capacity, developed in [15]. Set $V = W_0^{1,p(.)}(\Omega) \cap L^2(\Omega)$ and endow V with the norm $\|.\|_{W_0^{1,p(.)}(\Omega)} + \|.\|_{L^2(\Omega)}$. Moreover, set

$$W_{p(.)}(0,T) = \left\{ u \in L^{p_-}(0,T;V) : \nabla u \in \big(L^{p(\cdot)}(Q)\big)^N,\ u_t \in L^{(p_-)'}(0,T;V') \right\}$$

and consider the norm $\|u\|_{W_{p(.)}(0,T)} = \|u\|_{L^{p_-}(0,T;V)} + \|\nabla u\|_{L^{p(\cdot)}(Q)} + \|u_t\|_{L^{(p_-)'}(0,T;V')}$. If $U \subset Q$ is an open set, we define the $p(.)$-parabolic capacity of U as

$$\mathrm{Cap}_{p(.)}(U) = \inf\left\{ \|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}(0,T),\, u > \chi_U \text{ almost everywhere in } Q \right\}$$

(we use the convention $\inf \emptyset = +\infty$). The above definition can be extended to any Borel subset $B \subset Q$ by setting $\mathrm{Cap}_{p(.)}(B) = \inf\left\{ \mathrm{Cap}_{p(.)}(U) : U \text{ is an open subset of } Q \text{ and } B \subset U \right\}$.

We denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the σ-algebra of Borel subsets of Q and by $\mathcal{M}_b^+(Q)$ its subspace of non-negative measures. $\mathcal{M}_0(Q)$ is defined as the set of bounded measures μ satisfying $\mu(E) = 0$ for every subset $E \subset Q$ such that $\mathrm{Cap}_{p(.)}(E) = 0$.

The main property of $\mathcal{M}_0(Q)$ is given by the following result.

Theorem 2.2 ([15]) *Let μ be a bounded measure on Q which does not charge sets of null capacity. Then, there exist $f \in L^1(Q)$, $F \in \left(L^{p'(\cdot)}(Q)\right)^N$, $g_1 \in L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)$ and $g_2 \in L^{p_-}\left(0,T;W^{1,p(\cdot)}(\Omega)\cap L^2(\Omega)\right)$ such that*

$$\int_Q \varphi\,\mathrm{d}\mu = \int_Q f\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q F.\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g_1,\varphi\rangle\,\mathrm{d}t - \int_0^T \langle \varphi_t, g_2\rangle\,\mathrm{d}t$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$.

We denote by $\langle\langle .,.\rangle\rangle$ the duality between $\left(W_{p(.)}(0,T)\right)'$ and $W_{p(.)}(0,T)$. Recall that $\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ is the set of all measures $\gamma\in\left(W_{p(.)}(0,T)\right)'$ such that there exists $C>0$ satisfying $|\langle\langle\gamma,\varphi\rangle\rangle|\le C\|\varphi\|_{L^\infty(Q)}$ for all $\varphi\in\mathcal{C}_c^\infty(Q)$.

Proposition 2.3 *Every $\gamma\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ can be identified by a unique linear transformation $\varphi\in\mathcal{C}_c^\infty(Q)\mapsto\int_Q\xi\,\mathrm{d}\gamma^{\text{meas}}$, where γ^{meas} belongs to $\mathcal{M}_b(Q)$.*

Proof. We define the positive linear functional F on $C_c(Q)$ by $F(\xi)=\langle\langle\gamma,\xi\rangle\rangle$. Since $|F(\xi)|=|\langle\langle\gamma,\xi\rangle\rangle|\le C\|\xi\|_{L^\infty(Q)}$ for all $\xi\in C_c(Q)$, by the Riesz representation theorem (see [10]) there exists a unique $\gamma^{\text{meas}}\in\mathcal{M}_b(Q)$ such that the relation $F(\xi)=\int_Q\xi\,\mathrm{d}\gamma^{\text{meas}}$ holds for all $\xi\in C_c(Q)$.
☐

Proposition 2.4 *Let $g\in L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)$. Then, there exists $G=(g_1,g_2,\cdots,g_N)\in\left(L^{p'(\cdot)}(Q)\right)^N$ such that*

$$\langle g,\varphi\rangle = \int_0^T\int_\Omega G.\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{2.6}$$

for any $\xi\in W_{p(.)}(0,T)$.

Proof. We start by introducing the following functional space

$$\mathcal{H} := \left\{ f\in L^{p_-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right) : |\nabla f|\in L^{p(\cdot)}(Q)\right\}$$

which, endowed with the norm

$$\|f\|_{\mathcal{H}} := \|\nabla f\|_{L^{p(\cdot)}(Q)},$$

or the equivalent norm

$$\|f\|_{\mathcal{H}} := \|f\|_{L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla f\|_{L^{p(\cdot)}(Q)},$$

is a separable and reflexive Banach space (see [20]). Moreover, we have

$$L^{(p_-)'}\left(0,T;W^{-1,p'(\cdot)}(\Omega)\right)\hookrightarrow\mathcal{H}',$$

where $\mathcal{H}'$ is the dual space of $\mathcal{H}$ and one can represent the elements of $\mathcal{H}'$ as follows: if $T\in\mathcal{H}'$, then there exists $F=(f_1,\cdots,f_n)\in\left(L^{p'(\cdot)}(Q)\right)^N$ such that $T=\operatorname{div}_x(F)$ in the sense that

$$\langle T,\xi\rangle = \int_Q F.\nabla\xi\,\mathrm{d}x\,\mathrm{d}t$$

for any $\xi \in \mathcal{H}$. Consequently, since $g \in L^{(p_-)'}\big(0,T;W^{-1,p'(.)}(\Omega)\big) \hookrightarrow \mathcal{H}'$ and $W_{p(.)}(0,T) \hookrightarrow \mathcal{H}$, it follows that

$$\langle g, \varphi \rangle = \int_0^T \int_\Omega G.\nabla \varphi \,\mathrm{d}x\,\mathrm{d}t$$

for any $\xi \in W_{p(.)}(0,T)$. □

Consequently, Theorem 2.2 can be reformulated as follows.

Lemma 2.5 *Let μ be a bounded measure on Q which does not charge sets of null $p(.)$-capacity. Then, there exist $f \in L^1(Q)$, $G_1 \in \big(L^{p'(.)}(Q)\big)^N$, and $g_2 \in L^{p_-}(0,T;V)$ such that*

$$\int_Q \varphi \,\mathrm{d}\mu = \int_Q f\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q G_1.\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \varphi_t, g_2\rangle \,\mathrm{d}t$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Moreover, from Proposition 2.4, Lemma 4.2 in [15] can be rewritten as follows.

Lemma 2.6 *Let $g \in \big(W_{p(.)}(0,T)\big)'$. Then, there exist $G_1 \in \big(L^{p'(.)}(Q)\big)^N$, $g_2 \in L^{p_-}(0,T;V)$ and $g_3 \in L^{(p_-)'}\big(0,T;L^2(\Omega)\big)$ such that*

$$\langle\langle g, u\rangle\rangle = \int_Q G_1.\nabla u \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle u_t, g_2\rangle \,\mathrm{d}t + \int_Q g_3 u \,\mathrm{d}x\,\mathrm{d}t \text{ for all } u \in W_{p(.)}(0,T).$$

Moreover, we can choose (G_1, g_2, g_3) such that

$$\|G_1\|_{(L^{p'(.)}(Q))^N} + \|g_2\|_{L^{p_-}(0,T;V)} + \|g_3\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \le C\|g\|_{\left(W_{p(.)}(0,T)\right)'}. \tag{2.7}$$

Definition 2.7 *For $L \in \big(W_{p(.)}(0,T)\big)'$, we say that $(G_1, g_2, g_3, h_1, h_2)$ is a pseudo-decomposition of L if $G_1 \in \big(L^{p'(.)}(Q)\big)^N$, $g_2 \in L^{p_-}(0,T;V)$, $g_3 \in L^{(p_-)'}(0,T;L^2(\Omega))$, $h_1 \in L^{p'(.)}(Q)$ and $h_2 \in L^2(0,T;L^2(\Omega))$, and*

$$\begin{aligned}\langle\langle L, \varphi\rangle\rangle = {}& \int_0^T \langle \operatorname{div}(G_1), \varphi\rangle \,\mathrm{d}t + \int_0^T \langle \varphi_t, g_2\rangle \,\mathrm{d}t \\ &+ \int_Q g_3\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q h_1\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q h_2\varphi \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.8}$$

for all $\varphi \in W_0^{1,p(.)}(\Omega)$.

Using Lemma 4.3 in [15] (see eq. (119) and eq. (120)), we get.

Proposition 2.8 *Let ρ_n be a sequence of symmetric regularizing kernels in $\mathbb{R}\times\mathbb{R}^N$ and let $\theta \in C_c^\infty([0,T]\times\Omega)$. If (G_1, g_2, g_3) is a decomposition of a measure ν according to Lemma 2.6, then $(\theta G_1, \theta g_2, \theta g_3, -\theta G_1\nabla\theta, \theta_t g_2)$ is a pseudo-decomposition of $\mu = \theta\nu$ and $\big((\theta G_1)*\rho_n, (\theta g_2)*\rho_n, (\theta g_3)*\rho_n, (-\theta G_1\nabla\theta)*\rho_n, (\theta_t g_2)*\rho_n\big)$ is a pseudo-decomposition of $\mu^{\text{meas}} * \rho_n$.*

For readers' convenience, we provide the proof of the following result, which can also be found in our previous article (cf. [15]).

Theorem 2.9 *Let $\mu \in \mathcal{M}_0(Q)$. Then, there exist $g \in \big(W_{p(.)}(0,T)\big)'$ and $h \in L^1(Q)$ such that $\mu = g + h$ in the sense that*

$$\int_Q \varphi \,\mathrm{d}\mu = \langle\langle g, \varphi\rangle\rangle + \int_Q h\varphi \,\mathrm{d}x\,\mathrm{d}t \tag{2.9}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Proof. Since μ belongs to $\mathcal{M}_0(Q)$, by the Hahn–Banach decomposition of μ we have $\mu^+, \mu^- \in \mathcal{M}_0(Q)$. So, we can assume that $\mu \in \mathcal{M}_0^+(Q)$. Hence, from Proposition 4.1 in [15], there exist $\gamma \in \big(W_{p(.)}(0,T)\big)' \cap \mathcal{M}_0^+(Q)$ and a non-negative Borel function $f \in L^1(Q, \mathrm{d}\gamma^{\text{meas}})$ such that

$$\mu(B) = \int_B f \,\mathrm{d}\gamma^{\text{meas}} \quad \text{for all Borel subsets } B \text{ of } Q.$$

Since γ^{meas} is a regular measure and $C_c^\infty(Q)$ is dense in $L^1(Q, \mathrm{d}\gamma^{\text{meas}})$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(Q)$ such that f_n converges strongly to f in $L^1(Q, \mathrm{d}\gamma^{\text{meas}})$. Moreover, we have $\sum_{n=0}^\infty \|f_n - f_{n-1}\|_{L^1(Q,\mathrm{d}\gamma^{\text{meas}})} < \infty$. Define $\nu_n = (f_n - f_{n-1})\gamma \in \big(W_{p(.)}(0,T)\big)'$. Then, thanks to [15, Lemma 4.3], we infer that $\nu \in \big(W_{p(.)}(0,T)\big)' \cap \mathcal{M}_b(Q)$ and $\sum_{n=0}^\infty \nu_n^{\text{meas}} = \sum_{n=0}^\infty (f_n - f_{n-1})\gamma^{\text{meas}}$ strongly converges to μ in $\mathcal{M}_b(Q)$. Therefore, we can consider μ as a compactly supported measure. Taking a standard sequence of mollifiers $\rho_l \in \mathcal{D}(\mathbb{R})$, from [15, Lemma 4.3], we deduce that $\rho_l * \nu_n^{\text{meas}}$ strongly converges to ν_n in $\big(W_{p(.)}(0,T)\big)'$. Hence, we can extract a subsequence l_n such that $\|\rho_{l_n} * \nu_n^{\text{meas}} - \nu_n\|_{(W_{p(.)}(0,T))'} \le \frac{1}{2^n}$.

Let us rewrite $\sum_{k=0}^n \nu_k^{\text{meas}}$ as follows:

$$\sum_{k=0}^n \nu_k^{\text{meas}} = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}} + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}}). \tag{2.10}$$

In the following we denote respectively by m_n and h_n the first and second term in (2.10) and we define the sequence g_n by $g_n = \sum_{k=0}^n (\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})$. So, m_n is a measure with compact support, h_n is a function in $C_c^\infty(Q)$ and g_n belongs to $\big(W_{p(.)}(0,T)\big)'$. Take θ_n in $C_c^\infty(Q)$ so that $\theta_n \equiv 1$ on a neighborhood of $\big(\operatorname{supp}(f_0) \cup \cdots \cup \operatorname{supp}(f_n)\big) \cap \operatorname{supp}(\sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}})$. Then, we can write $g_n = \theta_n g_n$.

Since all terms in (2.10) have compact support, we can use $\varphi \in C_c^\infty([0,T]\times\Omega)$ as a test function in (2.10) to obtain

$$\int_Q \varphi \,\mathrm{d}m_n = \int_Q h_n\varphi \,\mathrm{d}x\,\mathrm{d}t + \langle\langle g_n, \varphi\rangle\rangle. \tag{2.11}$$

Since

$$\int_Q \varphi \,\mathrm{d}g_n^{\text{meas}} = \int_Q \theta_n\varphi \,\mathrm{d}g_n^{\text{meas}} = \langle\langle g_n, \theta_n\varphi\rangle\rangle = \langle\langle g_n, \varphi\rangle\rangle,$$

we have

$$\|h\|_{L^1(Q)} \le \sum_{k=0}^\infty \|\rho_{l_k} * \nu_k^{\text{meas}}\|_{L^1(Q)} \le \sum_{k=0}^\infty \|\nu_k^{\text{meas}}\|_{\mathcal{M}_b(Q)} < \infty,$$

which implies the existence of a subsequence of $(h_n)_{n\in\mathbb{N}}$ converging to an element h in $L^1(Q)$. We also have

$$\|g_n\| \leq \sum_{k=0}^{\infty} \|\nu_k - \rho_{l_k} * \nu_n^{\text{meas}}\|_{(W_{p(.)}(0,T))'} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty,$$

hence $(h_n)_{n\in\mathbb{N}}$ converges strongly to an element g in $\big(W_{p(.)}(0,T)\big)'$. Then, it follows that

$$\langle\langle g_n, \varphi\rangle\rangle + \int_Q h_n\varphi \,\mathrm{d}x\,\mathrm{d}t \to \langle\langle g, \varphi\rangle\rangle + \int_Q h\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{2.12}$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Now, we prove that $\int_Q \varphi\,\mathrm{d}m_n$ converges to $\int_Q \varphi\,\mathrm{d}\mu$. For that we recall that the mapping $m \mapsto \widetilde{m}$, where $\widetilde{m}(f) = \int_Q f\,\mathrm{d}m$, is a continuous linear injection of $\mathcal{M}_b(Q)$ into $\big(C(\bar{Q})\big)'$. Thus, if m_n strongly converges to μ in $\mathcal{M}_b(Q)$, $\widetilde{m}_n$ strongly converges in $\big(C(\bar{Q})\big)'$ to $\widetilde{\mu}$, and so we have

$$\int_Q \varphi\,\mathrm{d}m_n = \widetilde{m}_n(\varphi) \to \widetilde{\mu}(\varphi) = \int_Q \varphi\,\mathrm{d}\mu. \tag{2.13}$$

Combining (2.11)–(2.13), we get (2.9). □

Using the previous theorem, we prove the following approximation result, which will play an important role in the proof of the existence of renormalized solutions of (P_μ).

Proposition 2.10 *Let $\mu \in \mathcal{M}_0(Q)$. Then, there exists a decomposition (f, G_1, g_2) of μ in the sense of Lemma* 2.5 *and an approximation μ_n of μ satisfying*

$$\mu_n \in C_c^\infty(Q), \qquad \|\mu_n\|_{\mathcal{M}_b(Q)} \leq C$$

and

$$\int_Q \varphi\,\mathrm{d}\mu_n = \int_Q f_n\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q G_1^n \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \varphi_t, g_2^n\rangle\,\mathrm{d}t$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, where

$$f_n \in C_c^\infty(Q) \text{ and } f_n \to f \text{ strongly in } L^1(Q),$$
$$G_1^n \in \big(C_c^\infty(Q)\big)^N \text{ and } G_1^n \to G_1 \text{ strongly in } \big(L^{p'(\cdot)}(Q)\big)^N,$$
$$g_2^n \in C_c^\infty(Q) \text{ and } g_2^n \to g_2 \text{ strongly in } L^{p_-}(0,T;V).$$

Proof. We will prove that there exists a decomposition (f, G_1, g_2) of μ such that for all $\epsilon > 0$ we can find $\mu_\epsilon \in C_c^\infty(Q)$ satisfying $\|\mu_\epsilon\|_{L^1(Q)} \leq C$ and

$$\int_Q \varphi\mu_\epsilon\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f_\epsilon\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q G_1^\epsilon\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle\varphi_t, g_2^\epsilon\rangle\,\mathrm{d}t$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$ with $f_\epsilon \in C_c^\infty(Q)$ such that $\|f_\epsilon - f\|_{L^1(Q)} \leq C\epsilon$, $G_1^\epsilon \in \big(C_c^\infty(Q)\big)^N$ such that $\|G_1^\epsilon - G_1\|_{(L^{p'(\cdot)}(Q))^N} \leq C\epsilon$, and $g_2^\epsilon \in C_c^\infty(Q)$ such that $\|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \leq C\epsilon$ (with C not depending on ϵ).

We use the notation introduced in the proof of Theorem 2.9. Recalling that $\nu_k = (f_k - f_{k-1})\gamma \in \big(W_{p(.)}(0,T)\big)'$, we choose $\xi_k \in C_c^\infty(Q)$ such that $\xi_k \equiv 1$ on a neighborhood of $\mathrm{supp}(f_k - f_{k-1})$. Then, there exists $C(\xi_k) > 0$, depending only on ξ_k, such that

- if $E\in\{(L^{p'(.)}(Q))^N,\ L^{p_-}(0,T;V),\ L^{(p_-)'}(0,T;L^2(\Omega))\}$ and $h\in E$, then $\|\xi_k h\|_E \le \|h\|_E C(\xi_k)$,
- if $H_1\in (L^{p'(.)}(Q))^N$, then $\|H_1\nabla\xi_k\|_{L^{p'(.)}(Q)} \le C(\xi_k)\|H_1\|_{(L^{p'(.)}(Q))^N}$,
- if $h\in L^{p_-}(0,T;L^2(\Omega))$, then $\|(\xi_k)_t h\|_{L^{p_-}(0,T;L^2(\Omega))} \le C(\xi_k)\|h\|_{L^{p_-}(0,T;L^2(\Omega))}$.

Instead of the index l chosen in the proof of Theorem 2.9, here we take l_k such that $\|\rho_{l_k}*\nu_n^{\text{meas}}-\nu_n\|_{(W_{p(.)}(0,T))'}\le 1/(2^k(C(\xi_k)+1))$ and $\xi_k\equiv 1$ on a neighborhood of $\operatorname{supp}(\rho_{l_k}*\nu_k^{\text{meas}})$. With this choice and taking (B_1^k,b_2^k,b_3^k) as a decomposition of $\nu_k-\rho_{l_k}*\nu_k^{\text{meas}}$ given by Lemma 2.6, satisfying moreover

$$\|B_1^k\|_{(L^{p'(.)}(Q))^N}+\|b_2^k\|_{L^{p_-}(0,T;V)}+\|b_3^k\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \le C\|\rho_{l_k}*\nu_k^{\text{meas}}-\nu_k^{\text{meas}}\|_{(W_{p(.)}(0,T))'}$$

with C not depending on k, we deduce that

$$\begin{aligned}
&\sum_{k\ge1}\xi_k B_1^k \text{ converges to an element } G_1 \text{ in } (L^{p'(.)}(Q))^N,\\
&\sum_{k\ge1}\xi_k b_2^k \text{ converges to an element } g_2 \text{ in } L^{p_-}(0,T;V),\\
&\sum_{k\ge1}\xi_k b_3^k \text{ converges to an element } f_1 \text{ in } L^{(p_-)'}(0,T;L^2(\Omega)),\\
&\sum_{k\ge1}B_1^k\nabla\xi_k \text{ converges to an element } f_2 \text{ in } L^{p'(.)}(Q),\\
&\sum_{k\ge1}(\xi_k)_t b_2^k \text{ converges to an element } f_3 \text{ in } L^{p_-}(0,T;L^2(\Omega)).
\end{aligned}\tag{2.14}$$

Notice also that the last three convergences imply in particular the convergence in $L^1(Q)$. We know that $\nu_k-\rho_{l_k}*\nu_k^{\text{meas}}=\xi_k(\nu_k-\rho_{l_k}*\nu_k^{\text{meas}})\in (W_{p(.)}(0,T))'$ and that (B_1^k,b_2^k,b_3^k) is a decomposition of $\nu_k-\rho_{l_k}*\nu_k^{\text{meas}}$. Then, by Remark 2.7 we deduce that $(\xi_k B_1^k,\xi_k b_2^k,\xi_k b_3^k,-B_1^k\nabla\xi_k,(\xi_k)_t b_2^k)$ is a pseudo-decomposition of $\nu_k-\rho_{l_k}*\nu_k^{\text{meas}}$.

Since from the definition of the sequence $(g_n)_n$ given in the proof of Theorem 2.9 we have

$$\langle\langle g_n,\varphi\rangle\rangle=\left\langle\!\!\left\langle\sum_{k=0}^n(\nu_k-\rho_{l_k}*\nu_k^{\text{meas}}),\varphi\right\rangle\!\!\right\rangle$$

for all $\varphi\in C_c^\infty([0,T]\times\Omega)$, using (2.11) it follow that

$$\begin{aligned}
\int_Q\varphi\,\mathrm{d}m_n=\int_Q\varphi h_n\,\mathrm{d}x\,\mathrm{d}t&+\int_0^T\left\langle\operatorname{div}\left(\sum_{k=0}^n\xi_k B_1^k\right),\varphi\right\rangle\mathrm{d}t\\
&+\int_0^T\left\langle\varphi_t,\sum_{k=0}^n\xi_k b_2^k\right\rangle\mathrm{d}t+\int_0^T\sum_{k=0}^n\xi_k b_3^k\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q\sum_{k=0}^n\left(-B_1^k\nabla\xi_k\right)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\sum_{k=0}^n(\xi_k)_t\,b_2^k\varphi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$. Hence, using (2.14) and the fact that m_n converges to μ and h_n converges to h, we obtain

$$\int_Q \varphi \,\mathrm{d}\mu = \int_Q (h+f_1-f_2+f_3)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \operatorname{div}(G_1),\varphi\rangle \,\mathrm{d}t - \int_0^T \langle \varphi_t, g_2\rangle \,\mathrm{d}t,$$

which is equivalent to saying that $(f=(h+f_1+F_2-f_2+f_3), \operatorname{div}(G_1), g_2)$ is a decomposition of μ in the sense of Lemma 2.5.

Let $\epsilon>0$ be fixed. Taking n large enough, we have

$$\left\|\sum_{k=0}^n \xi_k B_1^k - G_1\right\|_{(L^{p'(\cdot)}(Q))^N} \le \epsilon, \tag{2.15}$$

$$\left\|\sum_{k=0}^n \xi_k b_2^k - g_2\right\|_{L^{p-}(0,T;V)} \le \epsilon \tag{2.16}$$

and

$$\left\|h_n + \sum_{k=0}^n \xi_k b_3^k - \sum_{k=0}^n B_1^k \nabla\xi_k + \sum_{k=0}^n (\xi_k)_t b_2^k - f\right\|_{L^1(Q)} \le \epsilon. \tag{2.17}$$

Since $\nu_k - \rho_{l_k} * \nu_k^{\text{meas}} = \xi_k(\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})$ and (B_1^k, b_2^k, b_3^k) is its decomposition, thanks to Remark 2.7, $\big((\xi_k B_1^k)*\rho_j, (\xi_k b_2^k)*\rho_j, ((\xi_k)_t b_3^k)*\rho_j, (-B_1^k\nabla\xi_k)*\rho_j, ((\xi_k)_t b_2^k)*\rho_j\big)$ represents a pseudo-decomposition of $(\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})*\rho_j$ for j large enough. Now, we take j_n such that for all $k\in\{0,\cdots,n\}$ we have

$$\left\|(\xi_k B_1^k)*\rho_{j_n} - \xi_k B_1^k\right\|_{(L^{p'(\cdot)}(Q))^N} \le \frac{\epsilon}{n+1}, \tag{2.18}$$

$$\left\|(\xi_k b_2^k)*\rho_{j_n} - \xi_k b_2^k\right\|_{L^{p-}(0,T;V)} \le \frac{\epsilon}{n+1} \tag{2.19}$$

and

$$\begin{aligned}\left\|(\xi_k b_3^k)*\rho_{j_n} - \xi_k b_3^k\right\|_{L^1(Q)} &+ \left\|(B_1^k\nabla\xi_k)*\rho_{j_n} - B_1^k\nabla\xi_k\right\|_{L^1(Q)} \\ &+ \left\|\big((\xi_k)_t b_2^k\big)*\rho_{j_n} - (\xi_k)_t b_2^k\right\|_{L^1(Q)} \le \frac{\epsilon}{n+1}.\end{aligned} \tag{2.20}$$

Let us define $G_1^\epsilon = \sum_{k=0}^n (\xi_k B_1^k)*\rho_{j_n} \in \big(C_c^\infty(Q)\big)^N$. Then, by (2.15) and (2.18), we have $\|G_1^\epsilon - G_1\|_{(L^{p'(\cdot)}(Q))^N} \le 2\epsilon$. If we set $g_2^\epsilon = -\sum_{k=0}^n (\xi_k b_2^k)*\rho_{j_n} \in C_c^\infty(Q)$, then using (2.16) and (2.19) we obtain $\|g_2^\epsilon - g_2\|_{L^{p-}(0,T;V)} \le 2\epsilon$. Now, if we define $f_\epsilon = h_n + \sum_{k=0}^n (\xi_k b_3^k)*\rho_{j_n} - \sum_{k=0}^n (B_1^k\nabla\xi_k)*\rho_{j_n} + \sum_{k=0}^n ((\xi_k)_t b_2^k)*\rho_{j_n} \in C_c^\infty(Q)$, then by (2.17) and (2.20) we get $\|f_\epsilon - f\|_{L^1(Q)} \le \epsilon$.

We define $\mu_\epsilon = f_\epsilon + \operatorname{div}(G_1^\epsilon) + (g_2^\epsilon)_t \in C_c^\infty(Q)$. It remains to prove that $\|\mu_\epsilon\|_{L^1(Q)} \le C$ with C not depending on ϵ. To see this, we recall that $\big((\xi_k B_1^k)*\rho_{j_n}, (\xi_k b_2^k)*\rho_{j_n}, \big((\xi_k)_t b_3^k\big)*\rho_{j_n}, (-B_1^k\nabla\xi_k)*\rho_{j_n}, \big((\xi_k)_t b_2^k\big)*\rho_{j_n}\big)$ is a pseudo-decomposition of $(\nu_k - \rho_{l_k}*\nu_k^{\text{meas}})*\rho_{j_n}$ so that

$$\begin{aligned}\mu_\epsilon &= h_n + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k}*\nu_k^{\text{meas}})*\rho_{j_n} \\ &= h_n + \left(\sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k}*\nu_k^{\text{meas}})\right)*\rho_{j_n} = h_n + g_n^{\text{meas}}*\rho_{j_n}.\end{aligned}$$

Thanks to (2.10), we have $g_n^{\text{meas}} = m_n - h_n$. Then, it follows that $\|\mu_\epsilon\|_{L^1(Q)} \leq 2\|h_n\|_{L^1(Q)} + \|m_n\|_{L^1(Q)}$. Since h_n converges in $L^1(Q)$ and m_n converges in $\mathcal{M}_b(Q)$, we infer that $2\|h_n\|_{L^1(Q)} + \|m_n\|_{L^1(Q)}$ is bounded. This implies that $\|\mu_\epsilon\|_{L^1(Q)} \leq C$ with C not depending on ϵ. $\square$

3 The initial boundary value problem with data in $\mathcal{M}_0(Q)$

3.1 Definition and properties of renormalized solutions

In this part of the work we define a new concept of solution for the problem (P_μ) and prove the uniqueness result. We work with a measure μ which does not charge sets of null $p(.)$-parabolic capacity. Recall that such a measure belongs to $L^1(Q) + \big(W_{p(.)}(0,T)\big)'$. And if we deal with a data which belongs to $L^1(Q)$, the notion of a solution in the sense of distribution is not strong enough to guarantee uniqueness of solutions. To overcome this difficulty, we will work in a larger space than the Sobolev space in which the concept of a gradient is meaningful.

For any $k > 0$, we define the truncation function T_k at height k by the formula

$$T_k\,(s) = \max\{-k, \min\{k, s\}\} \text{ for } s \in \mathbb{R}.$$

Moreover, the primitive of the truncation function at height k is denoted by Θ_k, that is, $\Theta_k \colon \mathbb{R} \to \mathbb{R}$ is given by

$$\Theta_k(r) = \int_0^r T_k(s)\,\mathrm{d}s = \begin{cases} \frac{r^2}{2}, & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2}, & \text{if } |r| \geq k. \end{cases}$$

It is not difficult to see that for every $k > 0$ and all $s \in \mathbb{R}$ we have

$$\frac{T_k(s)^2}{2} \leq \Theta_k(s) \leq k|s|.$$

We set

$$\mathcal{T}_0^{1,p(.)}(Q) = \left\{ u \colon \Omega \times (0,T] \to \mathbb{R} \;:\; \begin{array}{l} u \text{ is measurable and } T_k(u) \in L^{p_-}\big(0,T; W_0^{1,p(.)}(\Omega)\big) \\ \text{with } \nabla T_k(u) \in \big(L^{p(.)}(Q)\big)^N \text{ for every } k > 0 \end{array} \right\}.$$

Next, we define the weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p(.)}(Q)$.

Proposition 3.1 (cf. [2]) *For every measurable function $u \in \mathcal{T}_0^{1,p(.)}(Q)$ there exists a unique measurable function $v \colon Q \to \mathbb{R}^N$, which we call the weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \text{ almost everywhere in } Q \text{ and for every } k > 0.$$

If u belongs to $L^1(0,T; W_0^{1,1}(\Omega))$, then this gradient coincides with the usual gradient in the distributional sense.

We introduce the following concept of a renormalized solution for the problem (P_μ).

Definition 3.2 *Let $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_0(Q)$ and let (f, G_1, g_2) be a decomposition of μ given in Theorem 2.2. A measurable function u is a renormalized solution of (P_μ) if*

$$u - g_2 \in L^\infty\big(0, T; L^1(\Omega)\big) \cap \mathcal{T}_0^{1,p(.)}(Q), \tag{3.1}$$

$$\lim_{n\to\infty} \int_{\{n \le |u-g_2| \le n+1\}} |\nabla u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.2}$$

and for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support,

$$\begin{aligned}(S(u-g_2))_t - \operatorname{div}(a(x,\nabla u)S'(u-g_2)) + S''(u-g_2)a(x,\nabla u)\nabla(u-g_2) \\ = S'(u-g_2)f + G_1 S''(u-g_2)\nabla(u-g_2) - \operatorname{div}(G_1 S'(u-g_2)) \text{ in } \big(C_c^\infty(Q)\big)'\end{aligned} \tag{3.3}$$

with

$$S(u-g_2)(0) = S(u_0) \text{ in } L^1(\Omega). \tag{3.4}$$

Remark 3.3 *Notice that the distributional meaning of each term in* (3.3) *is well-defined. Indeed, since S' has compact support, there exists $M > 0$ such that $\operatorname{supp}(S') \subset (-M, M)$. Then, it follows that $S'(u-g_2) = S''(u-g_2) = 0$ for $|u-g_2| \ge M$. Moreover, by* (3.1) *we have $\nabla T_M(u-g_2) \in \big(L^{p(.)}(Q)\big)^N$. Therefore, everywhere in* (3.3) *we can replace $\nabla(u-g_2)$ by $\nabla T_M(u-g_2) \in \big(L^{p(.)}(Q)\big)^N$ and ∇u by $\nabla T_M(u-g_2) + \nabla g_2 \in \big(L^{p(.)}(Q)\big)^N$. (Recall that $a(.,0) = 0$.) Since $S(u-g_2) = S(T_M(u-g_2)) \in L^{p_-}\big(0,T; W_0^{1,p(.)}(\Omega)\big)$, from* (3.3) *we deduce that $(S(u-g_2))_t \in L^{(p_-)'}\big(0,T; W^{-1,p'(.)}(\Omega)\big) + L^1(Q)$, which implies that $S(u-g_2)$ belongs to $C(0,T; L^1(\Omega))$ (see* [16]*). Thus,* (3.4) *is well-defined. Notice also that since $(S(u-g_2))_t \in L^{(p_-)'}\big(0,T; W^{-1,p'(.)}(\Omega)\big) + L^1(Q)$, we can use as test functions in* (3.3) *not only functions in $\big(C_c^\infty(Q)\big)'$ but also functions in $L^{p_-}\big(0,T; W_0^{1,p(.)}(\Omega)\big) \cap L^\infty(Q)$. Using the fact that $g_2 \in L^{p_-}\big(0,T; W_0^{1,p(.)}(\Omega)\big)$ and that $u-g_2$ is almost everywhere finite, one can show that* (3.2) *is equivalent to*

$$\lim_{n\to\infty} \int_{\{n \le |u-g_2| \le n+1\}} |\nabla(u-g_2)|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.5}$$

In what follows, we need the following auxiliary functions of real variable.

Definition 3.4 *We define*:

$$\theta_n(s) = T_1(s - T_n(s)), \quad h_n(s) = 1 - |\theta_n(s)|, \quad S_n(s) = \int_0^s h_n(r)\,\mathrm{d}r \quad \text{for } s \in \mathbb{R}.$$

3.2 Proofs of the existence and uniqueness theorems

First, we introduce the approximate problems. Let $\mu_n \in C_c^\infty(\Omega)$ be an approximation of μ given by Proposition 2.10 and let $u_{0n} \in C_c^\infty(\Omega)$ be strongly convergent to u_0 in $L^1(\Omega)$ such that $\|u_{0n}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}$. Then, we consider the approximate problem

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,\nabla u_n)) = \mu_n & \text{in } (0,T)\times\Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0) = u_{0n} & \text{in } \Omega. \end{cases} \tag{3.6}$$

Let

$$\mathcal{H} := \Big\{ f \in L^{p_-}\big(0,T; W_0^{1,p(.)}(\Omega)\big) : |\nabla f| \in L^{p(.)}(Q) \Big\}.$$

Thanks to [14, Proposition 4.7], there exists a unique weak solution u_n for the Cauchy–Dirichlet problem (3.6) in the sense that $u_n \in \mathcal{H}$ and

$$(u_n)_t - \operatorname{div}(a(x,\nabla u_n)) = \mu_n \quad \text{in} \quad \big(C_c^\infty(Q)\big)' \tag{3.7}$$

and

$$u_n(0) = u_{0n}.$$

Then, it follows that

$$\int_0^t \langle (u_n)_t, \varphi\rangle \,\mathrm{d}s + \int_0^t\int_\Omega a(x,\nabla u_n)\varphi \,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega \varphi\mu_n \,\mathrm{d}x\,\mathrm{d}s \tag{3.8}$$

for any $\varphi \in C_c^\infty(Q)$ and any $t \in (0,T)$. Moreover, from Proposition 2.10, there exist $g_1^n, g_2^n, f_n \in C_c^\infty(Q)$ such that

$$\begin{aligned}\int_0^t \langle (u_n - g_2^n)_t, \varphi\rangle \,\mathrm{d}s + \int_0^t\int_\Omega a(x,\nabla u_n)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s &= \int_0^t\int_\Omega f_n\varphi \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t\int_\Omega G_1^n\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{3.9}$$

for any $\varphi \in C_c^\infty(Q)$ and any $t \in (0,T)$. Since the space $C_c^\infty(Q)$ is dense in $\mathcal{H}$ (see [1]), the equality (3.9) remains true for all $\varphi \in \mathcal{H}$.

Now, let us give some a priori estimates on u_n.

Proposition 3.5 *Let u_n be the solution of* (3.6). *Then, we have*

(i) $\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C,$

(ii) $\displaystyle\int_0^T \|\nabla T_k(u_n)\|_{p(.)}^{p_-} \,\mathrm{d}t \le C,$

(iii) $\|u_n - g_2^n\|_{L^\infty(0,T;L^1(\Omega))} \le C,$

(iv) $\displaystyle\int_0^T \|\nabla T_k(u_n - g_2^n)\|_{p(.)}^{p_-} \,\mathrm{d}t \le C,$

(v) $\displaystyle\lim_{h\to\infty}\left(\sup_n \int_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u_n|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t\right) = 0$ *for every* $k > 0$.

Moreover, there exists a measurable function u such that $u - g_2$ belong to $L^\infty\big(0,T;L^1(\Omega)\big) \cap \mathcal{T}_0^{1,p(.)}(Q)$ and, up to a subsequence, for any $k > 0$:

$$\begin{gathered} u_n \to u \text{ a.e. in } Q, \\ T_k(u_n - g_2^n) \to T_k(u - g_2) \text{ weakly in } L^{p_-}\big(0,T; W_0^{1,p(.)}(\Omega)\big) \text{ a.e. in } Q. \end{gathered} \tag{3.10}$$

Finally, for every $k > 0$ we have

$$\lim_{h\to\infty}\int_{\{h\le|u_n-g_2^n|\le h+k\}} |\nabla u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.11}$$

Proof of Proposition 3.5. We take $T_k(u_n)$ as a test function in (3.6) and we integrate over $(0,t)$ to obtain

$$\begin{aligned}\int_\Omega \Theta_k(u_n)(t)\,\mathrm{d}x + \int_0^t\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}s\\ = \int_0^t\int_\Omega \mu_n T_k(u_n)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \Theta_k(u_{0n})\,\mathrm{d}x.\end{aligned} \tag{3.12}$$

Now, we use the hypothesis (2.5) and the properties of Θ_k to get

$$\int_\Omega \Theta_k(u_n)(t)\,\mathrm{d}x + \frac{1}{C}\int_0^t\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}s = k\big(\|\mu_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(Q)}\big). \tag{3.13}$$

Since $\Theta_k(s)\geq 0$ and $|s|-1\leq|\Theta_1(s)|\leq\Theta_k(s)$, we have

$$\begin{aligned}\int_\Omega |u_n(t)|\,\mathrm{d}x + \frac{1}{C}\int_0^t\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}s\\ = k\big(\|\mu_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(Q)}\big) + \operatorname{meas}(\Omega).\end{aligned} \tag{3.14}$$

Finally, taking the supremum over $[0,T]$ and using the fact that $\|\mu_n\|_{L^1(Q)}$, $\|u_{0n}\|_{L^1(Q)}$ and Ω are bounded we deduce that

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))}\leq C(k+1) \quad \text{and} \quad \int_0^T |\nabla T_k(u_n)|^{p_-}\,\mathrm{d}t \leq C(k+1). \tag{3.15}$$

Combining (3.15) with Proposition 2.1 and the Hölder inequality type, we deduce that

$$\begin{aligned}&\int_0^T \|\nabla T_k(u_n)\|_{p(.)}^{p_-}\,\mathrm{d}t\\ &\leq \int_0^T \max\Big\{\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x,\ \Big(\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x\Big)^{\frac{p_-}{p_+}}\Big\}\,\mathrm{d}t\\ &\leq \int_0^T\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t + T^{1-p_-/p_+}\Big(\int_0^T\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p_-}{p_+}}\\ &\leq C(k+1) + T^{1-p_-/p_+}(C(k+1))^{\frac{p_-}{p_+}}.\end{aligned} \tag{3.16}$$

For the estimates on $u_n-g_2^n$, we take $T_k(u_n-g_2^n)$ as a test function in (3.8) and we use the integration by parts formula, while having in mind that $u_n(0)-g_2^n(0)=u_n(0)=u_{0n}$. Because g_2^n has compact support, this gives

$$\begin{aligned}&\int_\Omega \Theta_k(u_n-g_2^n)(t)\,\mathrm{d}x + \int_0^t\int_\Omega a(x,\nabla u_n)\nabla u_n\chi_{\{|u_n-g_2^n|\leq k\}}\,\mathrm{d}x\,\mathrm{d}s\\ &\leq \int_\Omega \Theta_k(u_{0n})\,\mathrm{d}x + \int_0^t\int_\Omega f_n T_k(u_n-g_2^n)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega a(x,\nabla u_n)\nabla g_2^n\chi_{\{|u_n-g_2^n|\leq k\}}\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega G_1^n\nabla u_n\chi_{\{|u_n-g_2^n|\leq k\}}\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega G_1^n\nabla g_2^n\chi_{\{|u_n-g_2^n|\leq k\}}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.17}$$

Therefore, using the hypothesis (2.3), (2.5) and the Young inequality, we obtain

$$\begin{aligned}&\int_\Omega \Theta_k(u_n - g_2^n)(t)\,\mathrm{d}x + \frac{1}{2C}\int_0^t\int_\Omega |\nabla u_n|^{p(x)}\chi_{\{|u_n-g_2^n|\le k\}}\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\le k\big(\|u_{0n}\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\big)\\ &\qquad + C_1\int_0^T\int_\Omega\big(j^{p'(x)} + |G_1^n|^{p'(x)} + |F_n|^{p'(x)} + |\nabla g_2^n|^{p(x)}\big).\end{aligned}\tag{3.18}$$

As G_1^n is bounded in $\big(L^{p'(\cdot)}(Q)\big)^N$, ∇g_2^n is bounded in $\big(L^{p(\cdot)}(Q)\big)^N$, f_n is bounded in $L^1(Q)$ and u_{0n} is bounded in $L^1(Q)$, we deduce that

$$\int_\Omega \Theta_k(u_n - g_2^n)(t)\,\mathrm{d}x + \frac{1}{2C_1}\int_0^t\int_\Omega |\nabla u_n|^{p(x)}\chi_{\{|u_n-g_2^n|\le k\}}\,\mathrm{d}x\,\mathrm{d}s \le C(k+1).\tag{3.19}$$

This implies that

$$\int_\Omega |u_n - g_2^n|(t)\,\mathrm{d}x + \frac{1}{2C_1}\int_0^t\int_\Omega |\nabla u_n|^{p(x)}\chi_{\{|u_n-g_2^n|\le k\}}\,\mathrm{d}x\,\mathrm{d}s \le C(k+1) + \mathrm{meas}(\Omega).\tag{3.20}$$

Consequently, we have

$$\|u_n - g_2^n\|_{L^\infty(0,T;L^1(\Omega))} \le C(k+2)\tag{3.21}$$

and

$$\int_0^t\int_\Omega |\nabla u_n|^{p(x)}\chi_{\{|u_n-g_2^n|\le k\}}\,\mathrm{d}x\,\mathrm{d}s \le C(k+2),\tag{3.22}$$

which implies that

$$\int_0^t\int_\Omega |\nabla T_k(u_n - g_2)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}s \le C(k+2).\tag{3.23}$$

Similarly as in (3.16), we obtain

$$\int_0^T \|\nabla T_k(u_n - g_2)\|_{p(.)}^{p_-}\,\mathrm{d}t \le C(k+2) + T^{1-p_-/p_+}(C(k+2))^{\frac{p_-}{p_+}}.\tag{3.24}$$

For the condition (v), we consider the function $\psi(s) = T_k(s - T_h(s))$ since

$$\begin{aligned}&\int_0^t \langle (u_n - g_2^n)_t, \psi(u_n - g_2^n)\rangle\,\mathrm{d}s\\ &= \int_0^t \langle (u_n - g_2^n - T_h(u_n - g_2^n))_t, \Theta_k'(u_n - g_2^n - T_h(u_n - g_2^n))\rangle\,\mathrm{d}s\\ &\quad + \int_0^t \langle (T_h(u_n - g_2^n))_t, \Theta_k'(u_n - g_2^n - T_h(u_n - g_2^n))\rangle\,\mathrm{d}s\\ &= \int_0^t \langle (u_n - g_2^n - T_h(u_n - g_2^n))_t, \Theta_k'(u_n - g_2^n - T_h(u_n - g_2^n))\rangle\,\mathrm{d}s.\end{aligned}\tag{3.25}$$

Then, taking $\psi(u_n - g_2^n)$ as a test function in (3.9) we get

$$\begin{aligned}
&\int_\Omega \Theta_k(u_n - g_2^n - T_h(u_n - g_2^n))(t)\,\mathrm{d}x + \int_0^t\int_\Omega a(x,\nabla u_n)\nabla u_n \chi_{\{h\le |u_n-g_2^n|\le k+h\}}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad \le \int_\Omega \Theta_k(u_{0n} - T_h(u_{0n}))\,\mathrm{d}x + \int_0^t\int_\Omega f_n\psi(u_n - g_2^n)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_0^t\int_\Omega a(x,\nabla u_n)\nabla g_2^n \chi_{\{h\le |u_n-g_2^n|\le k+h\}}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_0^t\int_\Omega G_1^n.\nabla u_n \chi_{\{h\le |u_n-g_2^n|\le k+h\}}\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega G_1^n.\nabla g_2^n \chi_{\{h\le |u_n-g_2^n|\le h+k\}}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.26}$$

Using the Young inequality and the assumptions (2.3)–(2.5), we deduce that

$$\begin{aligned}
&\int_\Omega \Theta_k(u_n - g_2^n - T_h(u_n - g_2^n))(t)\,\mathrm{d}x + \frac{1}{2C_1}\int_0^t\int_\Omega |\nabla u_n|^{p(x)} \chi_{\{h\le |u_n-g_2^n|\le k+h\}}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad \le k\int_{\{|u_{0n}|>h\}} |u_{0n}|\,\mathrm{d}x + k\int_{\{|u_n-g_2^n|>h\}} |f_n|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C\int_{\{|u_n-g_2^n|>h\}} \Big(j^{p'(x)} + |G_1^n|^{p'(x)} + |\nabla g_2^n|^{p(x)}\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.27}$$

We know that the sequence $u_n - g_2^n$ is bounded in $L^\infty\big(0,T;L^1(\Omega)\big)$. Then,

$$\lim_{h\to\infty}\sup_n \operatorname{meas}\{|u_n - g_2^n| > h\} = 0. \tag{3.28}$$

Consequently, as the sequences $f_n, j^{p'(.)}, |G_1^n|^{p'(.)}, |\nabla g_2^n|^{p(.)}$ and u_{0n} are equi-integrable, from (3.27) we deduce that

$$\lim_{h\to\infty}\left(\sup_n \int_{\{h\le|u_n-g_2^n|\le k+h\}} |\nabla u_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\right) = 0. \tag{3.29}$$

For the proof of the last part of Proposition 3.5, we will show that u_n converges (up to subsequences) almost everywhere in Q to a measurable function u. We take a non-decreasing function τ_k in $C^2(\mathbb{R})$ such that $\tau_k(s) = s$ for $|s| \le \frac{k}{2}$ and $\tau_k(s) = \operatorname{sign}(s)k$ for $|s| > k$. If we multiply pointwise equation (3.6) by $\tau_k'(u_n - g_2^n)$ or if we take $\tau_k'(u_n - g_2^n)\psi$ as a test function in (3.9) with $\psi \in C_c^\infty(Q)$, we get

$$\begin{aligned}
&(\tau_k(u_n - g_2^n))_t - \operatorname{div}(a(x,\nabla u_n))\tau_k'(u_n - g_2^n) + a(x,\nabla u_n)\nabla(u_n - g_2^n)\tau_k''(u_n - g_2^n)\\
&\quad = \tau_k'(u_n - g_2^n)f_n - \operatorname{div}(G_1^n\tau_k'(u_n - g_2^n)) + \tau_k''(u_n - g_2^n)G_1^n.\nabla(u_n - g_2^n).
\end{aligned} \tag{3.30}$$

Since τ_k has compact support and $|\nabla u_n|^{p(.)}\chi_{\{|u_n-g_2^n|\le k\}}$ is bounded in $L^1(Q)$, we can use the hypothesis (2.4) to prove that the sequence $a(x,\nabla u_n)\nabla(u_n-g_2^n)\tau_k''(u_n-g_2^n)$ is bounded in $L^1(Q)$ and so is $\tau_k''(u_n-g_2^n)(G_1^n)\nabla(u_n-g_2^n)$ (since G_1^n is bounded in $L^{p'(.)}(Q)$). By the same arguments we can prove that the sequences $(a(x,\nabla u_n))\tau_k'(u_n - g_2^n)$ and $G_1^n\tau_k'(u_n - g_2^n)$ are bounded in $\big(L^{p'(.)}(Q)\big)^N$. Hence, by (3.27) it follows that $(\tau_k(u_n - g_2^n))_t$ is bounded in $L^{(p_-)'}\big(0,T;W^{1,p'(.)}(\Omega)\big) + L^1(Q)$. Since we have just proven that $\tau_k(u_n - g_2^n)$ is bounded in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$, a classical compactness result (see [17]) allows us to deduce that $\tau_k(u_n - g_2^n)$ is compact in $L^1(Q)$. Thus, up to a subsequence, it also converges in measure. Let then $\sigma > 0$, and given $\epsilon > 0$ let us fix h such that $\operatorname{meas}\{|u_n - g_2^n| \le \frac{h}{2}\} \le \epsilon$ for every n. Since $\tau_h(u_n - g_2^n)$ converges in measure, for n and m sufficiently large we have

$$\operatorname{meas}\big\{|\tau_h(u_n - g_2^n) - \tau_h(u_m - g_2^m)| > \sigma\big\} \le \epsilon.$$

This implies that

$$\begin{aligned}
&\operatorname{meas}\big\{|(u_n - g_2^n) - (u_m - g_2^m)| > \sigma\big\}\\
&\quad \le \operatorname{meas}\big\{|u_n - g_2^n| \le \tfrac{h}{2}\big\}\\
&\qquad + \operatorname{meas}\big\{|u_m - g_2^m| \le \tfrac{h}{2}\big\} + \operatorname{meas}\big\{|\tau_h(u_n - g_2^n) - \tau_k(u_m - g_2^m)| > \sigma\big\}.
\end{aligned}$$

Hence, the choice of h implies for n and m sufficiently large that

$$\operatorname{meas}\big\{|(u_n - g_2^n) - (u_m - g_2^m)| > \sigma\big\} \le 3\epsilon,$$

which proves that the sequence $u_n - g_2^n$ is a Cauchy sequence in measure and thus, up to a subsequence, converges almost everywhere to some measurable function in Q. Moreover, we know that g_2^n strongly converges to g_2 in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$. Therefore, $u_n = (u_n - g_2^n) + g_2^n$ almost everywhere converges to some function u in Q. Consequently, the sequence $T_k(u_n - g_2^n)$ weakly converges to $T_k(u - g_2)$ in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$. Thanks to (3.14) $u \in L^\infty\big(0,T;L^1(\Omega)\big)$ (indeed, apply Fatou's lemma to the first term on the left-hand side of (3.14)), and $T_k(u_n)$ weakly converges to $T_k(u)$ in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$.

Now, we prove the last part of Proposition 3.5. For that we use the function $\psi(s) = T_k(s - T_h(s))$ to obtain the following estimate

$$\begin{aligned}
\int_Q |\nabla\psi(u_n - g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t &= \int_{\{k \le |u_n - g_2^n| k+h\}} |\nabla\psi(u_n - g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_Q |\nabla T_{k+h}(u_n - g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \le C,
\end{aligned}$$

which implies that $(\psi(u_n - g_2^n))_n$ is bounded in $W_0^{1,p(.)}(Q)$. Then, up to a subsequence, we infer that $(\psi(u_n - g_2^n))_n$ weakly converges to v_k in $W_0^{1,p(.)}(Q)$ and, by compact embedding it converges strongly to v_k in $L^{p_-}(Q)$ and a.e. in Q. Using the continuity of ψ and the fact that $u_n - g_2^n \to u - g_2$ a.e. in Q, we deduce that $\psi(u_n - g_2^n) \to \psi(u - g_2)$ a.e. in Q, and $v_k = \psi(u - g_2)$ a.e. in Q.

In Proposition 2.1, we replace Ω by Q to obtain

$$\begin{aligned}
&\int_Q |\nabla\psi(u - g_2)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \le \|\nabla\psi(u - g_2)\|^{p_+}_{L^{p(.)}(Q)} + \|\nabla\psi(u - g_2)\|^{p_-}_{L^{p(.)}(Q)}\\
&\quad \le C\Big(\|\psi(u - g_2)\|^{p_+}_{W_0^{1,p(.)}(Q)} + \|\psi(u - g_2)\|^{p_-}_{W_0^{1,p(.)}(Q)}\Big).
\end{aligned} \tag{3.31}$$

Since $(\psi(u_n - g_2^n))_n$ weakly converges to v_k in $W_0^{1,p(.)}(Q)$, we have

$$\|\nabla\psi(u - g_2)\|_{W_0^{1,p(.)}(Q)} \le \liminf_{n\to\infty} \|\nabla\psi(u_n - g_2^n)\|_{W_0^{1,p(.)}(Q)}. \tag{3.32}$$

From (3.31)–(3.32) and Proposition 2.1, we have

$$\begin{aligned}
&\int_Q |\nabla\psi(u-g_2)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\liminf_{n\to\infty}\Big(\|\psi(u_n-g_2^n)\|^{p_+}_{W_0^{1,p(.)}(Q)}+\|\psi(u_n-g_2^n)\|^{p_-}_{W_0^{1,p(.)}(Q)}\Big)\\
&\leq C\liminf_{n\to\infty}\Big(\|\nabla\psi(u_n-g_2^n)\|^{p_+}_{L^{p(.)}(Q)}+\|\nabla\psi(u_n-g_2^n)\|^{p_-}_{L^{p(.)}(Q)}\Big)\\
&\leq C\liminf_{n\to\infty}\Bigg(\max\Bigg\{\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t,\ \Big(\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p_+}{p_-}}\Bigg\}\\
&\quad+\max\Bigg\{\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t,\ \Big(\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p_-}{p_+}}\Bigg\}\Bigg)\\
&\leq C\liminf_{n\to\infty}\Bigg(\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t+\Big(\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p_+}{p_-}}\\
&\quad+\Big(\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p_-}{p_+}}\Bigg).
\end{aligned}\tag{3.33}$$

Also, we have

$$\int_Q|\nabla\psi(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\leq C\int_{\{h\leq|u_n-g_2^n|\leq h+k\}}\Big(|\nabla u_n|^{p(x)}+|\nabla g_n^2|^{p(x)}\Big)\,\mathrm{d}x\,\mathrm{d}t.\tag{3.34}$$

Combining (3.29) and (3.33)–(3.34), we obtain

$$\lim_{h\to\infty}\int_{\{h\leq|u_n-g_2^n|\leq h+k\}}|\nabla(u_n-g_2^n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t=0.\tag{3.35}$$

This ends the proof of Proposition 3.5. □

In order to prove the strong convergence of $T_k(u_n-g_2^n)$ in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$ we follow the method introduced in [16]. So, need to recall the following definition of a time-regularization of $T_k(u)$, which was first introduced in [11]. Let z_ν be a sequence of functions such that:

- $z_\nu\in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ and $\|z_\nu\|_{L^\infty(\Omega)}\leq k$,
- $z_\nu\to T_k(u_0)$ a.e. in Ω as ν tends to infinity,
- $\frac{1}{\nu}\|z_\nu\|_{W_0^{1,p(.)}(\Omega)}\to 0$ as $\nu\to\infty$.

For $k>0$ fixed and $\nu>0$, $(T_k(u))_\nu$ is the unique solution of the problem

$$\begin{cases}\dfrac{\partial T_k(u)_\nu}{\partial t}=\nu(T_k(u)-T_k(u)_\nu) & \text{in } \mathcal{D}'(Q),\\ T_k(u)_\nu(0)=z_\nu & \text{in } \Omega.\end{cases}\tag{3.36}$$

Then, $(T_k(u))_\nu\in L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)\cap L^\infty(Q)$ and following [11] we can prove that

$$\frac{\partial T_k(u)_\nu}{\partial t}\in L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)\cap L^\infty(Q),\quad \|(T_k(u))_\nu\|_{L^\infty(Q)}\leq k.\tag{3.37}$$

Moreover, $T_k(u)_\nu\to T_k(u)$ and a.e. in Q, weak* in $L^\infty(Q)$ and strongly in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$.

Proposition 3.6 *Let u_n be the solution of* (3.6)*, where μ_n is given by Proposition* 2.10*, and let u be given by Proposition* 3.5*. Then, there exists a subsequence not relabelled such that*

$$T_k(u_n - g_2^n) \to T_k(u - g_2) \text{ strongly in } L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big) \text{ for any } k > 0. \tag{3.38}$$

Proof. We consider a subsequence u_n such that u_n converges almost everywhere to u in Q, where u is given by Proposition 3.5. We set $v_n = u_n - g_2^n$ and $v = u - g_2$. Thanks to Proposition 3.5 $v \in L^\infty\big(0,T;L^1(\Omega)\big)$ and it is almost everywhere finite. Moreover, $T_k(v) \in L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$ for every $k > 0$ and

$$T_k(v_n) \to T_k(v) \text{ weakly in } L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big) \text{ a.e. in } Q. \tag{3.39}$$

Let $(T_k(v))_\nu$ be the approximation of $T_k(v)$ given by (3.36). We define the function w_n by

$$w_n = T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu) \text{ with } h > 2k.$$

We have

$$\nabla w_n = \nabla(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\chi_{E_n},$$

where $E_n = \big\{|v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu| \leq 2k\big\}$. Notice that $\nabla w_n = 0$ if $|v_n| > h + 2k$. From Proposition 3.5, w_n is bounded in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$, and since v_n converges almost every where to v, we deduce that

$$w_n \to T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k v)_\nu) \text{ weakly in } L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big) \text{ a.e. in } Q. \tag{3.40}$$

In the following, we set $M = h + 4k$ and we denote by $\omega\,(n,\nu,h)$ all quantities (possibility different) such that

$$\lim_{h\to\infty}\lim_{\nu\to\infty}\limsup_{n\to\infty}|\omega(n,\nu,h)| = 0.$$

Similarly, we will write $\omega(n)$ or $\omega(n,\nu)$ to mean that the limits are taken only with respect to the specified parameters. Taking w_n as a test function in (3.8), we obtain

$$\int_0^T \langle (v_n)_t, w_n\rangle\,\mathrm{d}t + \int_Q a(x,\nabla u_n)\nabla w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q f_n w_n\,\mathrm{d}x\,\mathrm{d}t + \int_Q G_1^n.\nabla w_n\,\mathrm{d}x\,\mathrm{d}t. \tag{3.41}$$

Therefore, from (3.40) we get

$$\lim_{n\to\infty}\int_Q f_n w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q f T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\,\mathrm{d}x\,\mathrm{d}t,$$

$$\lim_{n\to\infty}\int_Q G_1^n \nabla w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q G_1 \nabla T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu)\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, since $T_k(v)_\nu$ converges to $T_k(v)$ strongly in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$ and almost everywhere in Q as ν tends to infinity, we have

$$\lim_{n\to\infty}\int_Q f_n w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q f T_{2k}(v - T_h(v))\,\mathrm{d}x\,\mathrm{d}t,$$

$$\lim_{n\to\infty}\int_Q G_1^n \nabla w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q G_1 \nabla(T_{2k}(v - T_h(v)))\,\mathrm{d}x\,\mathrm{d}t.$$

From the Lebesgue theorem we obtain $\lim_{n\to\infty}\int_Q f_n w_n = 0$. We have

$$\int_Q G_1\nabla(T_{2k}(v-T_h(v)))\,\mathrm{d}x\,\mathrm{d}t = \int_Q \nabla v\chi_{\{h\leq|v|\leq h+2k\}}\,\mathrm{d}x\,\mathrm{d}t.$$

For simplicity, let us also set

$$\begin{aligned}
J_1 &:= \left(\int_{\{h\leq|v|\leq h+2k\}} |\nabla v|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_+}},\\
J_2 &:= \left(\int_{\{h\leq|v|\leq h+2k\}} |\nabla v|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_-}},\\
J_3 &:= \left(\int_{\{h\leq|v|\leq h+2k\}} |\nabla (u-g_2)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_+}},\\
J_4 &:= \left(\int_{\{h\leq|v|\leq h+2k\}} |\nabla (u-g_2)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_-}}.
\end{aligned}$$

Then, using the Hölder inequality and Proposition 2.1 we obtain

$$\begin{aligned}
&\int_Q G_1\nabla v\chi_{\{h\leq|v|\leq k+h\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq 2\|G_1\|_{(L^{p'(\cdot)}(Q))^N}\|\nabla v\chi_{\{h\leq|v|\leq h+2k\}}\|_{L^{p(\cdot)}(Q)}\\
&\quad\leq 2\|G_1\|_{(L^{p'(\cdot)}(Q))^N}\max\{J_1,J_2\}\\
&\quad\leq 2\|G_1\|_{(L^{p'(\cdot)}(Q))^N}\max\{J_3,J_4\}.
\end{aligned}\tag{3.42}$$

Therefore, by (3.42) we deduce that

$$\lim_{h\to\infty}\int_Q G_1\nabla(T_{2k}(v-T_h(v)))\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Hence, we can write

$$\int_Q f_n w_n\,\mathrm{d}x\,\mathrm{d}t + \int_Q G_1^n\nabla w_n\,\mathrm{d}x\,\mathrm{d}t = \omega(n,\nu,h).\tag{3.43}$$

For the second term in (3.41), we have

$$\int_Q a(x,\nabla u_n)\nabla w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q a(x,\nabla u_n\chi_{\{|v_n|\leq M\}})\nabla w_n\,\mathrm{d}x\,\mathrm{d}t,$$

because $\nabla w_n = 0$ if $|v_n|\geq M = h+2k$. Now, we use the fact that

$$\nabla w_n = \nabla(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\chi_{E_n} \text{ with } h > 2k$$

to obtain

$$
\begin{aligned}
&\int_Q a(x,\nabla u_n)\nabla w_n \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_Q a(x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla(v_n - T_k(v)_\nu)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)_\nu\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t := I_1 + I_2 + I_3.
\end{aligned} \tag{3.44}
$$

Let us estimate I_2. Since $v_n - T_h(v_n) = 0$ if $|v_n| \le h$, we have

$$
\begin{aligned}
&\left|\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \le \int_Q |a(x,\nabla u_n\chi_{\{|v_n|\le M\}})||\nabla(v_n - T_h(v_n))|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_{\{h\le|v_n|\le h+2k\}} |a(x,\nabla u_n)||\nabla(u_n - g_2^n)|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.45}
$$

From (2.3) and the Young inequality we deduce that

$$
\begin{aligned}
&\int_{\{h\le|v_n|\le h+2k\}} |a(x,\nabla u_n)||\nabla(u_n - g_2^n)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\Bigg(\int_{\{h\le|v_n|\le h+2k\}} |\nabla u_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\{h\le|v_n|\le h+2k\}} |\nabla g_2^n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\{h\le|v_n|\le h+2k\}} |j(x)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Bigg).
\end{aligned} \tag{3.46}
$$

Hence, from (3.46) and the fact that $|\nabla g_2^n|^{p(\cdot)}$ is integrable and $\operatorname{meas}\{h \le |v_n| \le h+2k\}$ converges to zero as h tends to infinity uniformly with respect to n, we obtain

$$
\lim_{h\to\infty}\limsup_{n\to\infty}\left|\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(v_n - T_h(v_n))\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t\right| = 0,
$$

which means that $I_2 = \omega(n,h)$. To estimate the third term we rewrite I_3 as

$$
\begin{aligned}
I_3 = &-\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(T_k(v)_\nu - T_k(v))\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using (2.3) and the fact that $\nabla u_n\chi_{\{|v_n|\le M\}}$ is bounded in $\left(L^{p(\cdot)}(Q)\right)^N$, we infer that $|a(x,\nabla u_n\chi_{\{|v_n|\le M\}})|$ is bounded in $L^{p'(\cdot)}(Q)$. Since v_n converges to v almost everywhere,

$|\nabla T_k(v)|\chi_{\{|v_n|>k\}}$ strongly converges to zero in $L^{p(.)}(Q)$. Consequently,

$$\lim_{n\to\infty}\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla T_k(v)\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Using the fact that $|a(x,\nabla u_n\chi_{\{|v_n|\le M\}})|$ is bounded in $L^{p'(.)}(Q)$ and thanks to (3.38), we can apply the Lebesgue dominated convergence theorem to obtain

$$\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(T_k(v)_\nu - T_k(v))\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t = \omega(n,\nu),$$

which is equivalent to saying

$$I_3 = -\int_{\{|v_n|>k\}} a(x,\nabla u_n\chi_{\{|v_n|\le M\}})\nabla(T_k(v)_\nu - T_k(v))\chi_{E_n}\,\mathrm{d}x\,\mathrm{d}t = \omega(n,\nu).$$

Since I_2 and I_3 converge to zero, by (3.40) we deduce that

$$\int_Q a(x,\nabla u_n)\nabla w_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q a(x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla(v_n - T_k(v)_\nu)\,\mathrm{d}x\,\mathrm{d}t + \omega(n,\nu,h). \quad (3.47)$$

Combining (3.41), (3.43) and (3.47), we can write

$$\int_0^T \langle (v_n)_t, w_n\rangle\,\mathrm{d}t + \int_Q a(x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla(v_n - T_k(v)_\nu)\,\mathrm{d}x\,\mathrm{d}t = \omega(n,\nu,h). \quad (3.48)$$

As far as the first term is concerned, that is

$$\int_0^T \langle (v_n)_t, T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)\rangle\,\mathrm{d}t,$$

we can apply [16, Lemma 2.1] to the function v_n, using the fact that u_{0n} and z_ν strongly converge in $L^1(\Omega)$ to u_0 and $T_k(u)$, respectively. The above-mentioned lemma, together with the monotonicity properties of the time-regularization $T_k(v)_\nu$, gives

$$\int_0^T \langle (v_n)_t, w_n\rangle\,\mathrm{d}t \ge \omega(n,\nu,h).$$

Hence, (3.48) becomes

$$\int_Q a(x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla(v_n - T_k(v_n)_\nu)\,\mathrm{d}x\,\mathrm{d}t \le \omega(n,\nu,h). \quad (3.49)$$

Since $\chi_{\{|v_n|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ (see [4, Lemma 3.2]), from the strong convergence of g_2^n in $L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$ we deduce that $a(x,\nabla(g_2^n + T_k(v)\chi_{\{|v_n|\le k\}}))$ strongly converges to $a(x,\nabla(g_2 + T_k(v)\chi_{\{|v|\le k\}}))$ in $\left(L^{p(.)}(Q)\right)^N$. Hence, since $T_k(v_n)$ weakly converges to $T_k(v)$ in $L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$ we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_Q a(x,\nabla(g_2^n + T_k(v)\chi_{\{|v_n|\le k\}}))\nabla(v_n - T_k(v))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \lim_{n\to\infty}\int_Q a(x,\nabla(g_2^n + T_k(v)\chi_{\{|v_n|\le k\}}))\nabla(T_k(v_n) - T_k(v))\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

We know that $(T_k(v))_\nu$ strongly converges to $T_k(v)$ in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$. Then, from (3.49) we deduce that

$$\begin{aligned}\lim_{n\to\infty}\int_Q \big[a(x,\nabla u_n\chi_{\{|v_n|\le k\}})-a(x,\nabla(g_2^n+T_k(v)\chi_{\{|v_n|\le k\}}))\big]\times\\ \times(\nabla u_n-\nabla(g_2^n-T_k(v)))\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}\tag{3.50}$$

Using the fact that $\chi_{\{|v_n|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ and that g_2^n strongly converges to g_2 in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$, by the standard monotonicity argument which relies on (2.4) (see [5, Lemma 5]), from (3.50) we can deduce that

$$\nabla u_n\chi_{\{|v_n|\le k\}}\to\nabla(g_2^n+T_k(v))\chi_{\{|v_n|\le k\}}=\nabla u\chi_{\{|v|\le k\}} \text{ a.e. in } Q,$$

and then that $a(x,\nabla u_n\chi_{\{|v_n|\le k\}})\nabla u_n$ strongly converges to $a(x,\nabla u\chi_{\{|v|\le k\}})\nabla u$ in $L^1(Q)$. Finally, together with (2.3), this proves that the sequence $|\nabla u_n|^{p(.)}\chi_{\{|v|\le k\}}$ is equi-integrable in Q, which, as a consequence of Vitali's theorem and since g_2^n strongly converges in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$, yields

$$T_k(u_n-g_2^n)\to T_k(u-g_2) \text{ strongly in } L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big).$$

Since we have proved it for almost every k, the result holds true for any k as well. □

Now, we have at our disposal all the necessary tools for the proof of the existence of renormalized solutions for the problem (P_μ).

Theorem 3.7 *Assume that* (2.3)*,* (2.4)*,* (2.5) *hold true, and let $\mu\in\mathcal{M}_0(Q)$, $u_0\in L^1(\Omega)$. Then, there exists a renormalized solution u of the problem (P_μ) in the sense of Definition* 3.2*. Moreover, u belongs to $L^\infty\big(0,T;L^1(\Omega)\big)$ and $T_k(u)\in L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$ for every $k>0$.*

Proof. We consider the sequence u_n of solutions of (3.6) and the function $u\in L^\infty\big(0,T;L^1(\Omega)\big)$ given by Proposition 3.5. Let us recall that from Proposition 3.5 and Proposition 3.6 we know that

$$\begin{aligned}&u_n\to u \text{ a.e. in } Q,\\ &T_k(u_n-g_2^n)\to T_k(u-g_2) \text{ strongly in } L^{p_-}\big(0,T;W^{1,p}(\Omega)\big) \text{ for any } k>0 \text{ and a.e. in } Q.\end{aligned}\tag{3.51}$$

We consider $\varphi\in C_c^\infty(Q)$ and $S\in W^{2,\infty}(\mathbb{R})$ such that S' has compact support. Taking $S'(u_n-g_2^n)\varphi$ as a test function in (3.9), we get

$$\begin{aligned}&-\int_Q\varphi_t S(u_n-g_2^n)\,\mathrm{d}x\,\mathrm{d}t+\int_Q a(x,\nabla u_n)\nabla\varphi S'(u_n-g_2^n)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_Q S''(u_n-g_2^n)a(x,\nabla u_n)\nabla(u_n-g_2^n)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_Q f_nS'(u_n-g_2^n)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q G_1^n\nabla\varphi S'(u_n-g_2^n)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_Q S''(u_n-g_2^n)G_1^n\nabla(u_n-g_2^n)\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.52}$$

Using the compactness of $\operatorname{supp}(S)$, we can write

$$a(x,\nabla u_n)S'(u_n-g_2^n)=a(x,\nabla T_M(u_n-g_2^n)+\nabla g_2^n)S'(u_n-g_2^n),$$

where $M > 0$. And, as g_2^n converges strongly in $L^{p_-}\big(0,T;W^{1,p(.)}(\Omega)\big)$, from hypothesis (2.3) we deduce that

$$a(x,\nabla u_n)S'(u_n-g_2^n) \to a(x,\nabla u)S'(u-g_2) \text{ strongly in } \big(L^{p'(.)}(Q)\big)^N.$$

Similarly, we have that

$$\begin{aligned}&S''(u_n-g_2^n)a(x,\nabla u_n)\nabla(u_n-g_2^n)\\ &\qquad \to S''(u-g_2)a(x,\nabla u)\nabla(u-g_2) \text{ strongly in } L^1(Q)\end{aligned}$$

and

$$S''(u_n-g_2^n)\nabla(u_n-g_2^n) \to S''(u-g_2)\nabla(u-g_2) \text{ strongly in } \big(L^{p(.)}(Q)\big)^N.$$

Consequently, using (3.51) and the Lebesgue dominated convergence theorem, we can pass to the limit in (3.52) as n tends to infinity and obtain

$$\begin{aligned}&-\int_Q \varphi_t S(u-g_2)\,\mathrm{d}x\,\mathrm{d}t+\int_Q a(x,\nabla u)\nabla\varphi S'(u-g_2)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_Q S''(u-g_2)a(x,\nabla u)\nabla\varphi\nabla(u-g_2)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\int_Q fS'(u-g_2)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q G_1\nabla\varphi S'(u-g_2)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_Q S''(u-g_2)G_1\nabla(u-g_2)\varphi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.53}$$

which is equivalent to saying that u satisfies (3.3). As regards (3.2) it suffices to take $k=1$ in (3.11) given by Proposition 3.5. Finally, passing to the limit written in distributional sense (thanks to (3.51), (3.52)) we have

$$(S(u_n-g_2^n))_t \text{ is strongly convergent in } L^{(p_-)'}\big(0,T;W^{-1,p'(.)}(\Omega)\big)+L^1(Q).$$

Since $S(u_n-g_2^n)$ strongly converges in $L^{p_-}\big(0,T;W_0^{1,p(.)}(\Omega)\big)$, we deduce that

$$S(u_n-g_2^n) \to S(u-g_2) \text{ strongly in } C\big(0,T;L^1(\Omega)\big)$$

(see [16, Theorem 1.1]). In particular, we have $S(u_n-g_2^n)(0)=S(u_{0n})$. This concludes the proof that u is a renormalized solution of the problem (P_μ). $\square$

Now, we prove the uniqueness of the renormalized solution of the problem (P_μ).

Theorem 3.8 *Assume that* (2.3)*,* (2.4)*,* (2.5) *hold true. Let $\mu\in\mathcal{M}(Q)$. Then, there exists a unique renormalized solution of (P_μ).*

Remark 3.9 *One can remark that in the proof of this theorem we do not use the fact that $u-g_2\in L^\infty\big(0,T;L^1(Q)\big)$ but only the fact that the renormalized solution is almost everywhere finite.*

Proof of Theorem 3.8. Assume on the contrary that there are two renormalized solutions u_1, u_2 of problem (P_μ). Let (f, G_1, g_2) be a decomposition of μ, so that u_1 and u_2 both satisfy (3.3).

We consider the sequence S_n given by Definition 3.4. Then, we have that $S_n(u_1 - g_2)$ and $S_n(u_2 - g_2)$ belong to $L^{p_-}\big(0, T; W^{1,p(\cdot)}(\Omega)\big)$. We take $T_k(S_n(u_1 - g_2) - S_n(u_2 - g_2))$ as a test function in both the equations solved by u_1 and u_2; subtracting the equations we then have

$$\begin{aligned}
&\int_0^T \langle (S_n(v_1) - S_n(v_2))_t, T_k(S_n(v_1) - S_n(v_2))\rangle \,\mathrm{d}t \\
&\qquad + \int_Q [S_n'(v_1)a(x, \nabla u_1) - S_n'(v_2)a(x, \nabla u_2)]\nabla T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_Q f(S_n'(v_1) - S_n'(v_2))T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_Q G_1(S_n'(v_1) - S_n'(v_2))\nabla T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_Q (S_n''(v_1)G_1\nabla v_1 - S_n''(v_2)G_1\nabla v_2)T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_Q (S_n''(v_2)a(x, \nabla u_2)\nabla v_2 - S_n''(v_1)a(x, \nabla u_1)\nabla v_1)T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned} \tag{3.54}$$

where $v_1 = u_1 - g_2$ and $v_2 = u_2 - g_2$. We set (A)–(F) for the integrals above. We will study the limit of each of those integrals as n tends to infinity. Since $S_n(s)$ converges to 1 for every s in $\mathbb{R}$, then using the Lebesgue theorem we obtain

$$\lim_{n\to\infty} (C) = 0.$$

For the study of the limit of (E), we set $(E) = (E_1) + (E_2)$, where

$$(E_1) = \int_Q S_n''(v_1)G_1\nabla v_1 T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t.$$

Recalling that $S_n''(s) = -\operatorname{sign}(s)\chi_{n\le|s|\le n+1}$, we have

$$|(E_1)| \le k \int_{\{n\le|v_1|\le n+1\}} |G_1||\nabla v_1| \,\mathrm{d}x \,\mathrm{d}t.$$

So, using Hölder's inequality and Proposition 2.1 we get

$$\begin{aligned}
|(E_1)| \le k\|G_1\|_{L^{p'(\cdot)}(Q)} \max\Bigg\{ &\int_{\{n\le|u_1-g_2|\le n+1\}} |\nabla v_1 - \nabla g_2|^{p(x)} \,\mathrm{d}x \,\mathrm{d}t \\
&+ \left(\int_{\{n\le|u_1-g_2|\le n+1\}} |\nabla v_1 - \nabla g_2|^{p(x)} \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{p_-}} \Bigg\}.
\end{aligned}$$

Thus, by (3.2) written for u_1, we get that (E_1) converges to zero as n tends to infinity. The same is true for (E_2), hence we deduce that $\lim_{n\to\infty}(E) = 0$. Similarly, for (F) we write $(F) = (F_1) + (F_2)$ with

$$(F_1) = \int_Q S_n''(v_2)a(x, \nabla u_2)\nabla v_2 T_k(S_n(v_1) - S_n(v_2)) \,\mathrm{d}x \,\mathrm{d}t.$$

Because of the symmetry between (F_1) and (F_2), it is enough to prove that (F_1) tends to zero. Using again the properties of S_n'' and (2.3) we have

$$|(F_1)| = C_1 k \int_{\{n \le |v_2| \le n+1\}} |\nabla v_2| \Big(|j(x)| + |\nabla u_2|^{p(x)-1} \Big) \,\mathrm{d}x\,\mathrm{d}t,$$

which by Young's inequality yields

$$\begin{aligned}(F_1) \le C \Bigg(&\int_{\{n \le |u_2-g_2| \le n+1\}} \Big(|j(x)|^{p'(x)} + |\nabla g_2|^{p(x)} \Big) \,\mathrm{d}x\,\mathrm{d}t \\ &\int_{\{n \le |u_2-g_2| \le n+1\}} |\nabla u_2|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t \Bigg).\end{aligned}$$

Since $u_2 - g_2$ is almost everywhere finite and thanks to (3.2) written for u_2, we conclude that (F_1) – and (F_2) as well – converges to zero. So,

$$\lim_{n\to\infty} (F) = 0.$$

Now we focus our attention on (D). Since $S_n'(v_1) - S_n'(v_2) = 0$ in $\{|v_1| \le n,\ |v_2| \le n\} \cup \{|u_1| > n,\ |v_2| > n+1\}$, we split the integral (D) as follows

$$\begin{aligned} &\int_{\{|S_n(v_1)-S_n(v_2)| \le k\}} G_1(S_n'(v_1) - S_n'(v_2)) \nabla (S_n'(v_1) - S_n'(v_2)) \chi_{\{|v_1| \le n\}} \chi_{\{|v_2\}| > n} \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\{|S_n(v_1)-S_n(v_2)| \le k\}} G_1(S_n'(v_1) - S_n'(v_2)) \nabla (S_n'(v_1) - S_n'(v_2)) \chi_{\{n \le |v_1| < n+1\}} \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\{|S_n(v_1)-S_n(v_2)| \le k\}} G_1(S_n'(v_1) - S_n'(v_2)) \nabla (S_n'(v_1) - S_n'(v_2)) \chi_{\{|v_2| \le n+1\}} \chi_{\{|v_1| > n+1\}} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.55}$$

By (D_1)–(D_3) we denote the three terms of (3.55). Using the properties of S_n and S_n' (recall that $S_n(t) = |t|$ if $|t| \le n$, S_n is non-decreasing and $\mathrm{supp}(S_n') \subset [-n-1, n+1]$) we have

$$\begin{aligned}(D_1) \le &\int_{\{n-k \le |u_1-g_2| \le n\}} |G_1| |\nabla (u_1 - g_2)| \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\{n \le |u_2-g_2| \le n+1\}} |G_1| \nabla (u_2 - g_2) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Applying Hölder's inequality, Proposition 2.1 and using property (3.2) we get that (D_1) converges to zero as n tends to infinity. As $|S_n(t)| > n - k$ implies $|t| > n - k$, we have

$$\begin{aligned}(D_2) \le &\int_{\{n \le |u_1-g_2| \le n+1\}} |G_1| \nabla (u_1 - g_2) \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\{n-k \le |u_2-g_2| \le n+1\}} |G_1| \nabla (u_2 - g_2) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then, as in the case of (D_1), we deduce that (D_2) converges to zero as well. Using the same method (recall that $S_n'(t) = 0$ if $|t| > n + 1$), one can show that (D_3) also tends to zero. Consequently,

$$\lim_{n\to\infty} (D) = 0.$$

As regards the term (B), we decompose it as follows

$$\begin{aligned}(B) = &\int_{\mathcal{B}_1} [a(x,\nabla u_1) - a(x,\nabla u_2)](\nabla u_1 - \nabla u_2)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\mathcal{B}_2} [S_n'(v_1)a(x,\nabla u_1) - S_n'(v_2)a(x,\nabla u_2)]\nabla(S_n'(v_1) - S_n'(v_2))\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\mathcal{B}_3} [S_n'(v_1)a(x,\nabla u_1) - S_n'(v_2)a(x,\nabla u_2)]\nabla(S_n'(v_1) - S_n'(v_2))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}\mathcal{B}_1 &= \big\{|v_1 - v_2| \le k,\ |v_1| \le n,\ |v_2| \le n\big\},\\ \mathcal{B}_2 &= \big\{|S_n(v_1) - S_n(v_2)| \le k,\ |v_1| \le n,\ |v_2| > n\big\},\\ \mathcal{B}_3 &= \big\{|S_n(v_1) - S_n(v_2)| \le k,\ |v_1| > n\big\}.\end{aligned}$$

By (B_1)–(B_3) we denote the three integrals above. Since $\big\{|S_n(v_1) - S_n(v_2)| \le k,\ |v_1| > n\big\} \subset \big\{|v_1| > n,\ |v_2| > n-k\big\}$, using the fact that $S_n'(t) = 0$, if $|t| > n+1$, we have

$$\begin{aligned}|(B_3)| \le &\int_{\{n \le |u_1 - g_2| \le n+1\}} |a(x,\nabla u_1)||(\nabla u_2 - \nabla u_2)|\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\{n \le |u_1 - g_2| \le n+1\}} |a(x,\nabla u_1)||(\nabla u_2 - \nabla g_2)|\chi_{\{n-k \le |u_2 - g_2| \le n+1\}}\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\{n \le |u_1 - g_2| \le n+1\}} |a(x,\nabla u_2)||(\nabla u_1 - \nabla g_2)|\chi_{\{n-k \le |u_2 - g_2| \le n+1\}}\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\{n-k \le |u_2 - g_2| \le n+1\}} |a(x,\nabla u_2)||(\nabla u_2 - \nabla g_2)|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.56}$$

Using assumption (2.3), Young's inequality and the condition (3.2) for renormalized solutions, we can conclude, as we did before, that all the four terms on the right-hand side of (3.56) converge to zero. Thus we get that (B_3) converges to zero. Changing the roles of u_1 and u_2, the same arguments prove that (B_2) also converges to zero as n tends to infinity. Thus, using Fatou's lemma in (B_1), we conclude that

$$\liminf_{n\to\infty}(B) \ge \int_{|u_1 - u_2| \le k} [a(x,\nabla u_1) - a(x,\nabla u_2)](\nabla u_1 - \nabla u_2)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $S_n(v_1),\ S_n(v_2) \in C\big([0,T]; L^1(\Omega)\big)$ and $S_n(v_1)(0) = S_n(v_2)(0) = S_n(v_0)$, we can integrate the term (A) to obtain

$$(A) = \int_\Omega \Theta_k(S_n(v_1) - S_n(v_2))(T)\,\mathrm{d}x,$$

where $\Theta_k(s) = \int_0^s T_k(t)\,\mathrm{d}t$. Since Θ_k is non-negative, we conclude that $(A) \ge 0$.

Using the above estimates concerning the integrals (A)–(F) and passing to the limit in (3.54), we get

$$\int_{\{|u_1 - u_2| \le k\}} [a(x,\nabla u_1) - a(x,\nabla u_2)](\nabla u_1 - \nabla u_2)\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Then, letting k tend to infinity (recall that u_1 and u_2 are finite a.e. on Q) we obtain

$$\int_Q [a(x,\nabla u_1) - a(x,\nabla u_2)](\nabla u_1 - \nabla u_2)\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

The strict monotonicity assumption (2.4) then implies that $\nabla u_1 = \nabla u_2$ almost everywhere in Q. Let $\xi_n = T_1(T_{n+1}(v_1) - T_{n+1}(v_2))$. We have $\xi_n \in L^{p-}\big(0, T; W_0^{1p(.)}(\Omega)\big)$ and, since $\nabla v_1 = \nabla v_2$ almost everywhere,

$$\xi_n = \begin{cases} 0 & \text{on } \big\{|v_1| \leq n+1,\, |v_2| \leq n+1\big\}, \\ \nabla v_1 \chi_{\{|v_1 - T_{n+1}(v_2)| \leq 1\}} & \text{on } \big\{|v_1| \leq n+1,\, |v_2| > n+1\big\}, \\ \nabla v_1 \chi_{\{|T_{n+1}(v_1) - v_2| \leq 1\}} & \text{on } \big\{|v_1| > n+1,\, |v_2| \leq n+1\big\}, \end{cases}$$

so that

$$\int_Q |\xi_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \leq \int_{\{n \leq v_1 \leq n+1\}} |\nabla v_1|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t + \int_{\{n \leq v_2 \leq n+1\}} |\nabla v_2|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t.$$

Thanks to (3.2), we deduce that ξ_n tends to 0 in $L^{p-}\big(0, T; W_0^{1,p(.)}(\Omega)\big)$ and, since ξ_n tends to $T_1(v_1 - v_2)$ almost everywhere, $T_1(u_1 - u_2) = T_1(v_1 - v_2) = 0$. Hence, $u_1 = u_2$. □

References

[1] M. Bendahmane, P. Wittbold, A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 data*, Journal of Differential Equation **249** (2010), no. 6, 1483–1515.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy , M. Pierre, J. L. Vasquez, *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Annali Della Scuola Normale Superiore Di Pisa, Classe di Scienze 4e Série **22** (1995), no. 2, 240–273.

[3] L. Boccardo, T. Gallouët, L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Annales de l'Institut Henri Poincarè (C) Non Linear Analysis **13** (1996), no. 5, 539–551.

[4] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, *Nonlinear parabolic equations with measure data*, Journal of Functional Analysis **147** (1997), no. 1, 237–258.

[5] L. Boccardo, F. Murat, J. P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problem*, Annali di Matematica Pura ed Applicata **152** (1998), no. 1, 183–196.

[6] O. M. Buhrii, R. A. Mashiyev, *Uniqueness of solutions of the parabolic variational inequality with variable exponent of nonlinearity*, Nonlinear Analysis: Theory, Methods & Applications **70** (2009), no. 6, 2325–2331.

[7] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2017.

[8] J. Droniou, A. Porretta, A. Prignet, *Parabolic capacity and soft measures for nonlinear equations*, Potential Analysis **19** (2003), no. 2, 99–161.

[9] X. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, Journal of Mathematical Analysis and Application **263** (2001), 424–446.

[10] F. Harpain, M. Sion, *A representation theorem for measure on infinite dimensional spaces*, Pacific Journal of Mathematics **30** (1969), no. 1, 47–58.

[11] R. Landes, *On the existence of weak solutions for quasilinear parabolic boundary value problems*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **89** (1981), 217–237.

[12] R. A. Mashiyev, O. M. Buhrii, *Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity*, Journal of Mathematical Analysis and Application **377** (2011), no. 2, 450–463.

[13] I. Nyanquini, S. Ouaro, S. Soma, *Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data*, Annals of the University of Craiova, Mathematics and Computer Science Series **40** (2013), no. 2, 174–198.

[14] S. Ouaro, A. Ouédraogo, *Nonlinear parabolic problems with variable exponent and L^1-data*, Electronic Journal of Differential Equations **2017** (2017), no. 32, 1–32.

[15] S. Ouaro, U.Traoré, *$p(.)$-parabolic capacity and decomposition of measures*, Annals of the University of Craiova, Mathematics and Computer Science Series **44** (2017), no. 1, 30–63.

[16] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations,* Annali di Matematica Pura ed Applicata **177** (1999), no. 1, 143–172.

[17] J. Simon, *Compact sets in $L^p(0,T;B)$*, Annali di Matematica Pura ed Applicata **146** (1986), no. 1, 65–96.

[18] C. Zhang, S. Zhou, *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data*, Journal of Differential Equations **248** (2010), 1376–1400.

[19] D. Zhao, W. J. Qiang, X. L. Fan, *On generalized Orlicz spaces $L^{p(x)}$*, Journal of Gansu Education College **9** (1997), no. 2, 1–7.

[20] A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 data*, Diplom-Mathematikerin, Berlin, 2010.