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A CAUCHY PROBLEM FOR SOME FRACTIONAL DIFFERENTIAL EQUATION VIA DEFORMABLE DERIVATIVES

M. Mebrat*, **G. M. N'Guérékata**†
Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, M.D. 21251, USA

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Abstract. In this paper we investigate further properties of the new concept of deformable derivative and use the results to study the existence (and uniqueness) of mild solutions to the Cauchy problem for the nonlinear differential equation with non-local conditions $D^\alpha x(t) = f(t, x(t))$, $t \in (0, T]$, $x(0) + g(x) = x_0$, where $D^\alpha x(t)$ is the deformable derivative of x, $0 < \alpha < 1$. We use the Krasnoselskii's theorem to achieve our main result.

Keywords: Deformable derivative, Krasnoselskii's theorem, mild solution.

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1 Introduction

In their paper [8], F. Zulfeqarr, A. Ujlayan and P. Ahuja introduced the new concept of deformable derivative using limit approach as in the usual derivative. They called it ,,deformable" because of its intrinsic property of continuously deforming function to derivative. This derivative is linearly related to the usual derivative. Deformable derivatives can be viewed as derivatives of the fractional order. Recently, A. Meraj, D. N. Pandey [5] used this concept to study the existence and uniqueness of solutions to the Cauchy problem $D^\alpha x(t) = Ax(t) + f(t, x(t))$, $t \in (0, T]$, $x(0) = x_0$, where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X (using the Banach fixed point theorem) and the approximate controllability to the related problem (using the Schauder fixed point theorem).

*e-mail address: mameb1@morgan.edu
†e-mail address: Gaston.N'Guerekata@morgan.edu

In the present paper, we first review some elementary properties of deformable derivatives and then study the existence of mild solutions to the Cauchy problem for the differential equation with non-local conditions $D^\alpha x(t) = f(t, x(t))$, $t \in (0, T]$, $x(0) + g(x) = x_0$, where $D^\alpha x(t)$ is the deformable derivative of x, $0 < \alpha < 1$. This problem has been studied by the second author in the context of fractional derivative in the sense of Caputo (cf. [6]).

Throughout the paper we denote by $\mathcal{C} := C([0, T], \mathbb{R})$ the Banach space of all real-valued continuous functions defined on $[0, T]$, endowed with the topology of uniform convergence. The norm in this space will be denoted by $\|f\|_{\mathcal{C}} = \sup_{t\in[0,T]} |f(t)|$.

2 Deformable derivative

Definition 2.1 ([8]) *Let f be a real valued-function on $[a, b]$, $\alpha \in [0, 1]$. The* deformable derivative *of f of order α at $t \in (a, b)$ is defined as*

$$D^\alpha f(t) = \lim_{\epsilon \to 0} \frac{(1 + \epsilon\beta)f(t + \epsilon\alpha) - f(t)}{\epsilon},$$

*where $\alpha + \beta = 1$. If the limit exists, we say that f is α-*differentiable *at t.*

Remark 2.2 *If $\alpha = 1$, then $\beta = 0$ and we recover the usual derivative.*

Definition 2.3 ([8]) *For a continuous function f defined on $[a, b]$, the α-*integral *of f is given by*

$$I_a^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_a^t e^{\frac{\beta}{\alpha} x} f(x)\, \mathrm{d}x, \quad t \in [a, b],$$

where $\alpha + \beta = 1$ and $\alpha \in (0, 1]$. When $a = 0$, we use the notation

$$I^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} x} f(x)\, \mathrm{d}x.$$

Remark 2.4 *If $\alpha = 1$, then $\beta = 0$ and we recover the usual Riemann integral.*

Theorem 2.5 ([1]) *A differentiable function f at a point $t \in (a, b)$ is always α-differentiable at that point for any α. Moreover, we have $D^\alpha f(t) = \beta f(t) + \alpha D f(t)$, where $\alpha + \beta = 1$ and $D := \frac{\mathrm{d}}{\mathrm{d}t}$ is the usual derivative.*

Theorem 2.6 ([1]) *Let f be differentiable at a point t for some α. Then, it is continuous there.*

Theorem 2.7 ([1]) *Let f be defined in (a, b). For any α, f is α-differentiable if and only if it is differentiable.*

The operators D^α and I_a^α possess the following properties.

Theorem 2.8 ([4]) *Let $\alpha, \alpha_1, \alpha_2 \in (0, 1]$ be such that $\alpha + \beta = 1$ and $\alpha_i + \beta_i = 1$ for $i = 1, 2$. Then,*

(a) $D^\alpha(af + bg) = aD^\alpha f + bD^\alpha g$ *(linearity of* D^α*),*

(b) $D^{\alpha_1} \cdot D^{\alpha_2} = D^{\alpha_2} \cdot D^{\alpha_1}$ *(commutativity of* D^α*),*

(c) $D^\alpha(c) = \beta c$ *for any constant c,*

(d) $D^\alpha(fg) = (D^\alpha f)g + \alpha f Dg$*,*

(e) $I_a^\alpha(bf + cg) = bI_a^\alpha f + cI_a^\alpha g$ *(linearity of* I_a^α*),*

(f) $I_a^{\alpha_1} I_a^{\alpha_2} = I_a^{\alpha_2} I_a^{\alpha_1}$ *(commutativity of* I_a^α*).*

Theorem 2.9 (Taylor's theorem, see [8]) *Suppose* f *is* n*-times* α*-differentiable and such that all* α*-derivatives are continuous on* $[a, a+h]$*. Then,*

$$f(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\alpha^k}\left[D_k^\alpha f(a) - \beta \frac{(1-\theta)^{(k-n+1)}h}{\alpha n} D_k^\alpha f(a+\theta h)\right] + \frac{h^n}{n!\alpha} D_n^\alpha f(a+\theta h),$$

where $D_k^\alpha = D^\alpha D^\alpha \ldots\ D^\alpha$ *(*k *times) and* $0 < \theta < 1$*.*

We state and prove the following result.

Theorem 2.10 *The operator* D^α *possesses also the following property*

$$D^\alpha(fg^{-1}) = \frac{gD^\alpha(f) - \alpha f}{g^2}.$$

Proof. We have

$$\begin{aligned} D^\alpha(fg^{-1}) &= \beta(fg^{-1}) + \alpha D(fg^{-1}) \\ &= \beta(fg^{-1}) + \alpha[Dfg^{-1} + fDg^{-1}] \\ &= \beta fg^{-1} + \alpha Dfg^{-1} + \alpha fDg^{-1} \\ &= [\beta f + \alpha Df]g^{-1} + \alpha fDg^{-1} \\ &= \frac{gD^\alpha(f) - \alpha f}{g^2}. \end{aligned}$$

The proof is complete. □

Theorem 2.11 *Suppose f and g are* α*-differentiable. Then,*

$$D^\alpha(f\circ g)(t) = \beta(f\circ g)(t) + \alpha D(f\circ g)(t) = \beta(f\circ g)(t) + \alpha f'(g(t))g'(t).$$

Proof. Since

$$D^\alpha f(t) = \lim_{\epsilon\to 0} \frac{(1+\epsilon\beta)f(t+\epsilon\alpha) - f(t)}{\epsilon} = \lim_{\epsilon\to 0}\left[\frac{f(t+\epsilon\,\alpha) - f(t)}{\epsilon} + \beta f(t+\epsilon\alpha)\right],$$

we have

$$\begin{aligned} D^\alpha f(g(t)) =& \lim_{\epsilon\to 0}\left[\frac{f(g(t+\epsilon\alpha))-f(g(t))}{\epsilon}+\beta f(g(t+\epsilon\alpha))\right] \\ =& \lim_{\epsilon\to 0}\left[\frac{f(g(t+\epsilon\alpha))-f(g(t))}{g(t+\epsilon\alpha)-g(t)}\cdot\frac{g(t+\epsilon\alpha)-g(t)}{\epsilon}+\beta f(g(t+\epsilon\alpha))\right] \\ =& \lim_{\epsilon\to 0}\left[\frac{f(g(t)+\epsilon_0)-f(g(t))}{\epsilon_0}\cdot\frac{g(t+\epsilon\alpha)-g(t)}{\epsilon}+\beta f(g(t+\epsilon\alpha))\right], \end{aligned}$$

where $\epsilon_0 \to 0$ as $\epsilon \to 0$. We obtain

$$\begin{aligned} D^\alpha f(g(t)) = & \lim_{\epsilon_0\to 0}\frac{f(g(t)+\epsilon_0)-f(g(t))}{\epsilon_0}\cdot\lim_{\epsilon\to 0}\frac{g(t+\epsilon\alpha)-g(t)}{\epsilon}+\lim_{\epsilon\to 0}\beta f(g(t+\epsilon\alpha)) \\ =& f'(g(t))\alpha g'(t)+\beta f(g(t)) \\ =& \alpha D[f(g(t))]+\beta f(g(t)) \\ =& \beta f(g(t))+\alpha D[f(g(t))]. \end{aligned}$$

The proof is now complete. □

Theorem 2.12 ([8]) *Let f be continuous on $[a,b]$. Then, $I_a^\alpha f$ is α-differentiable in (a,b), and we have*

$$D^\alpha(I_a^\alpha f)(t) = f(t) \quad \textit{and} \quad I_a^\alpha(D^\alpha f)(t) = f(t) - e^{\frac{\beta}{\alpha}(a-t)}f(a).$$

3 Application to evolution equations

We consider the following Cauchy problem with non-local condition

$$D^\alpha x(t) = f(t,x(t)), \;\; t\in(0,T], \tag{3.1}$$

$$x(0)+g(x) = x_0, \tag{3.2}$$

where D^α is the deformable derivative of order $\alpha\in(0,1)$, and $g\colon \mathcal{C}\to\mathbb{R}$ is a continuous function.

Theorem 3.1 *The system* (3.1)–(3.2) *is equivalent to the following integral equation*

$$x(t) = [x_0-g(x)]e^{\frac{-\beta}{\alpha}t}+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,x(s))\,\mathrm{d}s. \tag{3.3}$$

Proof. Assume (3.1)–(3.2). Then, $I^\alpha(D^\alpha x)(t) = I^\alpha f(t,x(t))$. Using Theorem 2.12 and Definition 2.3, we get

$$x(t)-e^{\frac{-\beta}{\alpha}t}x(0) = \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,x(s))\,\mathrm{d}s.$$

Using (3.2), we get

$$x(t)-[x_0-g(x)]e^{\frac{-\beta}{\alpha}t} = \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,x(s))\,\mathrm{d}s.$$

Therefore,

$$x(t) = [x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s.$$

Conversely, assuming (3.3) and applying D^α to both sides of the equation, we get

$$\begin{aligned} D^\alpha x(t) &= \beta\bigg([x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s\bigg) \\ &\qquad + \alpha D\bigg([x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s\bigg) \\ &= \beta[x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} + \frac{\beta}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s - \beta[x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} \\ &\qquad + f(t, x(t)) - \frac{\beta}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s \\ &= f(t, x(t)). \end{aligned}$$

The proof is complete. □

Definition 3.2 *A function $x \in \mathcal{C}$ is said to be a* mild solution *to* (3.1)–(3.2) *if*

$$x(t) = [x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s, \quad t \in (0, T],$$

provided the integral exists.

Now, let us investigate the Cauchy problem (3.1)–(3.2) with the following assumptions:

(H1) $f\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ is jointly continuous,

(H2) $|f(t, x) - f(t, y)| \le L|x - y|$ for all $t \in [0, T]$ and $x, y \in \mathbb{R}$,

(H3) $g\colon \mathcal{C} \to \mathbb{R}$ is continuous and $G = \sup_{x\in\mathcal{C}} |g(x)| < \infty$; moreover, there exists $b > 0$ such that $|g(x) - g(y)| \le b\|x - y\|_{\mathcal{C}}$ for all $x, y \in \mathcal{C}$.

Theorem 3.3 ([3]) *Under assumptions* **(H1)–(H3)***, if $b < \frac{1}{2}$ and $L \le \frac{\beta}{2}$, then the Cauchy problem* (3.1)–(3.2) *has a unique mild solution.*

Proof. Define $F\colon \mathcal{C} \to \mathcal{C}$ by

$$(Fx)(t) := [x_0 - g(x)]e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, x(s))\,\mathrm{d}s.$$

Let $M = \sup_{t\in[0,T]} |f(t, 0)|$. Clearly $M < \infty$. Choose $r > 2(|x_0| + G + \frac{M}{\beta})$. Then, we can show that $F(B_r) \subset B_r$, where $B_r := \{x \in \mathcal{C} : \|x\| \leq r\}$. Indeed, let $x \in B_r$; then, we get

$$\begin{aligned}
|Fx(t)| &\leq |x_0| + G + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, x(s))| \, \mathrm{d}s \\
&\leq |x_0| + G + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t \Big(e^{\frac{\beta}{\alpha} s} |f(s, x(s)) - f(s, 0)| + |f(s, 0)| \Big) \, \mathrm{d}s \\
&\leq |x_0| + G + (Lr + M) \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} \, \mathrm{d}s \\
&\leq |x_0| + G + (Lr + M) \frac{1}{\beta} \Big(1 - e^{\frac{-\beta}{\alpha} t} \Big) \\
&\leq |x_0| + G + (Lr + M) \frac{1}{\beta} \\
&< r,
\end{aligned}$$

by the choice of L and r. Now, take $x, y \in B_r$. Then, we get

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq |g(x) - g(y)| + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, x(s)) - f(s, y(s))| \, \mathrm{d}s \\
&\leq b \|x - y\|_{\mathcal{C}} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} L \|x - y\|_C \frac{\alpha}{\beta} \Big(e^{\frac{\beta}{\alpha} t} - 1 \Big) \\
&\leq b \|x - y\|_{\mathcal{C}} + \frac{L}{\beta} \|x - y\|_{\mathcal{C}} \\
&= \left(b + \frac{L}{\beta} \right) \|x - y\|_{\mathcal{C}} \\
&\leq \Omega \|x - y\|_{\mathcal{C}},
\end{aligned}$$

where $\Omega = \Omega_{b,L,\beta} := (b + \frac{L}{\beta})$ depends on the parameters of the problem.

Since $\Omega < 1$ and $\|Fx - Fy\|_{\mathcal{C}} \leq \Omega \|x - y\|_{\mathcal{C}}$, the mapping F is a contraction. Therefore, F has a unique fixed point, which is the solution of the Cauchy problem. The proof is complete. □

Let us recall our tool.

Theorem 3.4 (Krasnoselskii) *Let M be a closed convex and non-empty subset of a Banach space X. Let A, B be two operators such that*

(1) *$Ax + By \in M$, whenever $x, y \in M$,*

(2) *A is compact and continuous,*

(3) *B is a contraction mapping.*

Then, there exists $z \in M$ such that $z = Az + Bz$.

Now, let

(H4) $|f(t, x)| \leq \mu(t)$ for all $(t, x) \in [0, T] \times \mathbb{R}$, where $\mu \in L^1([0, T], \mathbb{R}^+)$.

Then, we can state and prove our main result.

Theorem 3.5 *Assume* **(H1)**, **(H3)** *with* $b < 1$, *and* **(H4)**. *Then the Cauchy problem* (3.1)–(3.2) *has at least one mild solution on* $[0, T]$.

Proof. Choose $r \geq |x_0| + G + \frac{\|\mu\|_{L^1}}{\alpha}$ and consider $B_r := \{x \in \mathcal{C} : \|x\| \leq r\}$. Now, define on B_r the operators A and B by

$$\begin{aligned} A(x)(t) &:= \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, x(s))\, \mathrm{d}s, \\ B(x)(t) &:= [x_0 - g(x)] e^{\frac{-\beta}{\alpha} t}. \end{aligned}$$

Let us observe that if $x, y \in B_r$, then $Ax + By \in B_r$. Indeed, we have

$$\begin{aligned} |A(x)(t) + B(y)(t)| &= \left| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, x(s))\, \mathrm{d}s + [x_0 - g(x)] e^{\frac{-\beta}{\alpha} t} \right| \\ &\leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, x(s))|\, \mathrm{d}s + |x_0 - g(x)| e^{\frac{-\beta}{\alpha} t} \\ &\leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} e^{\frac{\beta}{\alpha} t} \int_0^t |f(s, x(s))|\, \mathrm{d}s + |x_0| + G \\ &\leq \frac{1}{\alpha} \int_0^t \mu(s)\, \mathrm{d}s + |x_0| + G \\ &\leq \frac{1}{\alpha} \int_0^T \mu(s)\, \mathrm{d}s + |x_0| + G \\ &\leq \frac{1}{\alpha} \|\mu\|_{L^1} + |x_0| + G \\ &\leq r. \end{aligned}$$

By **(H3)**, it is also clear that B is a contraction mapping for $b < 1$.

Since x is continuous, then $A(x)(t)$ is continuous in view of **(H1)**. Suppose now that $x_n \to x$ in B_r. Then, given $\varepsilon > 0$ there exists N large enough such that $\|x_n - x\|_{\mathcal{C}} < \varepsilon$ whenever $n > N$. Now, for such n we get

$$\begin{aligned} |A(x_n)(t) - A(x)(t)| &\leq \frac{1}{\alpha} e^{-\frac{\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, x_n(s)) - f(s, x(s))|\, \mathrm{d}s \\ &\leq \frac{L}{\alpha} e^{-\frac{\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |x_n(s) - x(s)|\, \mathrm{d}s. \end{aligned}$$

Continuing the calculation we obtain

$$||A(x_n) - A(x)||_{\mathcal{C}} \leq \frac{\varepsilon L}{\beta},$$

which shows that A is a continuous operator from B_r to $\mathcal{C}$.

Let us now note that A is uniformly bounded on B_r. This follows from the inequality $\|A(x)\|_{\mathcal{C}} \leq \frac{\|\mu\|_{L^1}}{\alpha}$, because

$$|A(x)(t)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, x(s))|\, \mathrm{d}s$$

for any $x \in B_r$.

Now let's prove that $A(B_r)$ is equicontinuous. Let $t_1, t_2 \in [0,T]$, $t_2 \leq t_1$ and $x \in B_r$. Using the fact that f is bounded on the compact set $[0,T] \times [-r,r]$ (and thus $K := \sup_{(t,x)\in[0,T]\times[-r,r]} |f(t,x)| < \infty$), we will get

$$
\begin{aligned}
&|Ax(t_1) - Ax(t_2)| \\
&\quad = \frac{1}{\alpha}\left| e^{\frac{-\beta}{\alpha}t_1} \int_0^{t_1} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s - e^{\frac{-\beta}{\alpha}t_2} \int_0^{t_2} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s \right| \\
&\quad = \frac{1}{\alpha}\left| \int_0^{t_1} e^{\frac{-\beta}{\alpha}t_1} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s - \int_0^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s \right| \\
&\quad = \frac{1}{\alpha}\left| \int_{t_2}^{t_1} e^{\frac{-\beta}{\alpha}t_1} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s - \int_0^{t_2} \left[e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}\right] e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s \right| \\
&\quad = \frac{1}{\alpha}\left| e^{\frac{-\beta}{\alpha}t_1} \int_{t_2}^{t_1} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s - \left[e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}\right] \int_0^{t_2} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s \right| \\
&\quad \leq \frac{1}{\alpha}\left| e^{\frac{-\beta}{\alpha}t_1} \int_{t_2}^{t_1} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s \right| + \frac{1}{\alpha}\left| \left[e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}\right] \int_0^{t_2} e^{\frac{\beta}{\alpha}s} f(s,x(s))\,\mathrm{d}s \right| \\
&\quad \leq \frac{K}{\beta}\left| e^{\frac{-\beta}{\alpha}t_1}\left(e^{\frac{\beta}{\alpha}t_1} - e^{\frac{\beta}{\alpha}t_2}\right) + \left(e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}\right)\left(e^{\frac{\beta}{\alpha}t_2} - 1\right) \right| \\
&\quad \leq \frac{K}{\beta}\left| \left(1 - e^{\frac{\beta}{\alpha}(t_2 - t_1)}\right) + \left(1 - e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{\beta}{\alpha}(t_2-t_1)} + e^{\frac{-\beta}{\alpha}t_1}\right) \right| \\
&\quad = \frac{K}{\beta}\left| 2 - 2e^{\frac{\beta}{\alpha}(t_2-t_1)} - e^{\frac{-\beta}{\alpha}t_2} + e^{\frac{-\beta}{\alpha}t_1} \right| \\
&\quad \leq \frac{K}{\beta}\left| 2 - 2e^{\frac{-\beta}{\alpha}(t_1-t_2)} \right| \\
&\quad = \frac{2K}{\beta}\left| e^{\frac{-\beta}{\alpha}(t_1-t_2)} - 1 \right|,
\end{aligned}
$$

which is independent of x and $|Ax(t_1) - Ax(t_2)| \to 0$ as $t_2 \to t_1$. Therefore, $Ax(t)$ is equicontinuous. So, by Arzela–Ascoli's theorem, $A(B_r)$ is a relatively compact subset of $\mathcal{C}$. This, in turn, implies that the operator A is compact. We finally complete the proof using Krasnoselskii's theorem. □

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