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CONTROL OF A MIGRATION PROBLEM OF A POPULATION BY THE SENTINEL METHOD

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Abstract. In this paper we study the dynamics of a single species population subjected to a migratory phenomenon and whose initial distribution is unknown. The aim of this paper is to use the sentinel method to control the migration.

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1 Introduction

We consider the model describing the dynamics of population with age dependence, spatial structure and incomplete data. More precisely, let Ω be an open and bounded domain of $\mathbb{R}^N$, $N \in \{1, 2, 3\}$, with boundary Γ of class $\mathcal{C}^\infty$. For the time $T > 0$ and the life expectancy $A > 0$ of an individual, set $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$, $\Sigma = U \times \Gamma$, $\Sigma_1 = U \times \Gamma_1$, where Γ_1 is a non-empty open subset of Γ. Then, consider the following two time scales varying

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equation:

$$
\begin{cases}
\dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y + \mu y = 0 & \text{in } Q, \\
y(0,a,x) = y^0 + \tau \widehat{y}^0 & \text{in } Q_A, \\
y(t,0,x) = \displaystyle\int_0^A \beta(t,a,x) y(t,a,x)\,\mathrm{d}a & \text{in } Q_T, \\
\dfrac{\partial y}{\partial \nu} = \xi + \displaystyle\sum_{i=1}^{M} \lambda_i \widehat{\xi}_i & \text{on } \Sigma_1, \\
\dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma \setminus \Sigma_1,
\end{cases}
\tag{1.1}
$$

where:

- $y(t,a,x)$ is the distribution of a-year-old individuals at time t at the point $x \in \Omega$,
- $\frac{\partial y}{\partial \nu}$ is the derivative of y with respect to ν; here, ν is the unit exterior normal to Γ,
- $\beta(t,a,x)$ and $\mu(t,a,x)$ are, respectively, the natural fertility and the natural death rates of age a at time t and position $x \in \Omega$,
- the boundary condition is unknown on the part Σ_1 of the boundary and represents a pollution with the structure of the form $\xi + \sum_{i=1}^{M} \lambda_i \widehat{\xi}_i$. In this structure, the functions ξ and ξ_i, $i = 1, \dots, M$, are known whereas the real numbers λ_i, $i = 1, \dots, M$, are unknown,
- the initial distribution of individuals is unknown and its structure is of the form $y^0 + \tau \widehat{y}^0$, where the function y^0 is known and the term $\tau \widehat{y}^0$ is unknown.

System (1.1) describes the migratory phenomenon of single species population with age dependence and spatial structure. We say that it is a system with incomplete data because the information on the boundary condition as well as on the initial condition are partially or completely unknown. Here, the pollution is isolated on the boundary Γ_1 and we do not know with certainty the number of individuals leaving the boundary Γ_1. The missing term in the initial conditions expresses the fact that we do not know when the migratory phenomenon begins. In what follows, we assume as in [6] that:

$$
(H_1)\quad
\begin{cases}
\beta \in L^\infty(Q) \text{ and } \beta(t,a,x) \geq 0 \text{ a.e. in } Q, \\
\displaystyle\sup_{(t,x)\in(0,T)\times\Omega} \int_{(0,A)} \left(|\beta^2(t,a,x)| + |\nabla \beta|^2(t,a,x)\right)\,\mathrm{d}a < \infty, \\
\text{there exists } \delta \in (0,A) \text{ such that } \beta(a,.,.) = 0 \text{ for } a \in (\delta, A),
\end{cases}
$$

(H_2) $\mu \in \mathcal{C}([0,T] \times [0,A] \times \overline{\Omega})$ and $\mu(t,a,x) \geq 0$ a.e. in Q,

$$
(H_3)\quad
\begin{cases}
\displaystyle\lim_{a\to A} \int_0^t \mu(\tau, a-t+\tau, x)\,\mathrm{d}\tau = +\infty \text{ for each } 0 < t < A \text{ and } x \in \Omega \\
\displaystyle\lim_{a\to A} \int_0^a \mu(t-a+\alpha, \alpha, x)\,\mathrm{d}\alpha = +\infty \text{ for each } A < t < T \text{ and } x \in \Omega \\
\nabla \mu \in [L^\infty(Q)]^n.
\end{cases}
$$

We also assume that:

- y^0 and $\widehat{y}^0$ belong to $L^2(Q_A)$, ξ and $\widehat{\xi}_i$ belong to $L^2(\Sigma)$,
- the real numbers τ, λ_i, $1 \leq i \leq M$, are sufficiently small and $\|\widehat{y}^0\|_{L^2(Q_A)} \leq 1$, and we set $\lambda = (\lambda_1, \ldots, \lambda_M)$.

Under the above assumptions on the data, one can prove as in [12] that problem (1.1) has a unique solution $y = y(\lambda, \tau) \in L^2(U; H^1(\Omega))$ which satisfies

(i) for all $\phi \in L^2(U; H^1(\Omega))$,

$$\int_U \left\langle \phi, \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} \right\rangle_{H^1(\Omega),(H^1(\Omega))'} \mathrm{d}t\mathrm{d}a + \int_Q (\nabla y \nabla \phi + \mu y \phi)\, \mathrm{d}t\mathrm{d}a\mathrm{d}x = \int_{\Sigma_1} \left(\xi + \sum_{i=1}^{M} \lambda_i \widehat{\xi}_i \right) \phi \, \mathrm{d}t\mathrm{d}a\mathrm{d}x,$$

(ii) $y(0, a, x) = y^0 + \tau \widehat{y}^0$ a.e. in Q_A,

(iii) $y(t, 0, x) = \int_0^A \beta(t, a, x) y(t, a, x)\, \mathrm{d}a$ a.e. in Q_T.

Moreover, if we denote by $I \subset \mathbb{R}$ a neighbourhood of zero, the maps

$$\tau \longmapsto y(\lambda, \tau) \text{ and } \lambda_i \longmapsto y(\lambda, \tau) \ \ (1 \leq i \leq M)$$

are in $\mathcal{C}^1(I, L^2(U; H^1(\Omega)))$. From now on, we denote by $W(U)$ the space

$$W(U) = \left\{ \rho \in L^2(U; H^1(\Omega)) : \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2(U; (H^1(\Omega))') \right\}, \tag{1.2}$$

where $(H^1(\Omega))'$ is the dual of $H^1(\Omega)$.

Remark 1 *Notice that if μ satisfies (H_2) and if $\rho \in L^2(U; H^1(\Omega))$ is such that $\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta \rho + \mu \rho \in L^2(Q)$, then we have*

$$\Delta \rho \in L^2(U; H^{-1}(\Omega)), \quad \mu \rho \in L^2(Q) \subset L^2(U; H^{-1}(\Omega))$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2(U; H^{-1}(\Omega)) \subset L^2(U; (H^1(\Omega))').$$

This implies that $\rho \in W(U)$.

Remark 2 *Note that if a function ρ belongs to $W(U)$, then $(\rho(0, ., .), \rho(., 0, .))$ and $(\rho(T, ., .), \rho(., A, .))$ exist and belong to $L^2((0, A); L^2(\Omega)) \times L^2((0, T); L^2(\Omega))$ (see* [6]).

For more literature on the model describing the dynamics of population with age dependence and spatial structure as well as for some existence results on such problems, we refer for instance to [3, 4, 6, 12] and the references therein. For the model (1.1), we are interested in identifying the parameters λ_i without any attempt at computing $\tau\widehat{y}^0$. To identify these parameters, we use the theory of sentinel in a general framework. More precisely, let O be a non-empty open subset of Ω and let $y = y(t, a, x; \lambda, \tau) = y(\lambda, \tau)$ be the solution of (1.1). Then, for any non-empty open subset ω of Ω such that $O \cap \omega \neq 0$, we build a function $S(\lambda, \tau)$ which depends on the solution of the following problem: *Given $h_0 \in L^2(U \times O)$, find $w \in L^2(U \times \omega)$ such that*

(i) *the function S defined by*

$$S(\lambda,\tau) = \int_U\int_O h_0 y(t,a,x;\lambda,\tau)\,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega w y(t,a,x;\lambda,\tau)\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \tag{1.3}$$

satisfies

- *S is stationary to the first order with respect to the missing term $\tau\widehat{y}^0$:*

$$\frac{\partial S}{\partial \tau}(0,0) = 0 \textit{ for all } \widehat{y}^0, \tag{1.4}$$

- *S is sensitive to the first order with respect to the pollution terms $\lambda_i\widehat{\xi_i}$:*

$$\frac{\partial S}{\partial \lambda_i}(0,0) = c_i,\ 1 \le i \le M, \tag{1.5}$$

where c_i, $1 \le i \le M$, are given constants not all identically zero,

(ii) *the control w is of minimum norm in $L^2(U \times \omega)$ among "the admissible controls,"* i.e.,

$$\|w\|_{L^2(U\times\omega)} = \min_{\overline{w}\in E} \|\overline{w}\|_{L^2(U\times\omega)}, \tag{1.6}$$

where $E = \{\overline{w} \in L^2(U \times \omega) : (\overline{w}, S(\overline{w}))$ satisfies (1.3), (1.4), (1.5)$\}$.

Remark 3 *To estimate the parameters λ_i, one proceeds as in* [14]: *Assume that the solution of* (1.1) *when $\lambda = 0$ and $\tau = 0$ is known. Then, one has the following information*

$$S(\lambda,\tau) - S(0,0) \approx \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0).$$

Therefore, fixing $i \in \{1, \dots, M\}$ and choosing

$$\frac{\partial S}{\partial \lambda_j}(0,0) = 0 \textit{ for } j \neq i, \textit{ and } \frac{\partial S}{\partial \lambda_i}(0,0) = c_i,$$

one obtains the following estimate of the parameter λ_i:

$$\lambda_i \approx \frac{1}{c_i}(S(\lambda,\tau) - S(0,0)).$$

Remark 4 *J. L. Lions* [7] *refers to the function S as a sentinel with given sensitivity c_i. In* (1.5)*, the numbers c_i are chosen according to the importance which is conferred to the component ξ_i of the pollution.*

Remark 5 *Notice that for the J. L. Lions' sentinels theory, the observatory $O \subset \Omega$ is also the support of the control function w.*

For more information on the theory of sentinel, we refer to [7, 8, 9, 10, 11, 13] and the references therein. By $y_0 = y(0,0) \in W(U)$ we denote the solution of (1.1) when $\lambda = 0$ and $\tau = 0$, and moreover by y_τ and y_{λ_i} we denote the derivatives of y at $(0,0)$ with respect to τ and λ_i, respectively, *i.e.*,

$$y_\tau = \frac{\partial}{\partial \tau} y(\lambda, \tau)|\tau = 0, \lambda_i = 0$$

and

$$y_{\lambda_i} = \frac{\partial}{\partial \lambda_i} y(\lambda, \tau)|\tau = 0, \lambda_i = 0.$$

Then, y_τ and y_{λ_i} are, respectively, solutions of

$$\begin{cases} \dfrac{\partial y_\tau}{\partial t} + \dfrac{\partial y_\tau}{\partial a} - \Delta y_\tau + \mu y_\tau = 0 & \text{in } Q, \\ y_\tau(0,a,x) = \widehat{y}^0 & \text{in } Q_A, \\ y_\tau(t,0,x) = \displaystyle\int_0^A \beta(t,a,x) y_\tau(t,a,x)\,\mathrm{d}a & \text{in } Q_T, \\ \dfrac{\partial y_\tau}{\partial \nu} = 0 & \text{on } \Sigma, \end{cases} \tag{1.7}$$

and

$$\begin{cases} \dfrac{\partial y_{\lambda_i}}{\partial t} + \dfrac{\partial y_{\lambda_i}}{\partial a} - \Delta y_{\lambda_i} + \mu y_{\lambda_i} = 0 & \text{in } Q, \\ y_{\lambda_i}(0,a,x) = 0 & \text{in } Q_A, \\ y_{\lambda_i}(t,0,x) = \displaystyle\int_0^A \beta(t,a,x) y_{\lambda_i}(t,a,x)\,\mathrm{d}a & \text{in } Q_T, \\ \dfrac{\partial y_{\lambda_i}}{\partial \nu} = \widehat{\xi}_i \chi_{\Sigma_1} & \text{on } \Sigma, \end{cases} \tag{1.8}$$

where by χ_X we denote (here and in the sequel) the characteristic function of the set X. Under the assumptions (H_1)–(H_3), we have on the one hand that (1.7) has a unique solution $y_\tau \in L^2(U; H^1(\Omega))$ because $\widehat{y}^0 \in L^2(Q_A)$ and, on the other hand, because $\widehat{\xi}_i \in L^2(\Sigma_1)$, that (1.8) admits a unique solution $y_{\lambda_i} \in L^2(U; H^1(\Omega))$ (see [6, 12, 3]). From now on, we assume that

$$\text{the functions } \widehat{\xi}_i \chi_{\Sigma_1},\ 1 \le i \le M, \text{ are linearly independent} \tag{1.9}$$

and we set

$$Y = \mathrm{Span}\{y_{\lambda_1}\chi_\omega, \dots, y_{\lambda_M}\chi_\omega\}, \tag{1.10}$$

that is, Y is the vector subspace of $L^2(U \times \omega)$ generated by the M functions $\{y_{\lambda_i}\chi_\omega\}_{i=1}^M$. Further, $Y_\theta = \frac{1}{\theta} Y$ is the vector subspace of $L^2(U \times \omega)$ generated by the M functions $\left\{\frac{1}{\theta} y_{\lambda_i}\chi_\omega\right\}_{i=1}^M$, where θ is a positive function precisely defined later on (see (3.6)).

Remark 6 *We will prove in Lemma 1 that the functions $\{y_{\lambda_i}\chi_\omega\}_{i=1}^M$ and $\left\{\frac{1}{\theta} y_{\lambda_i}\chi_\omega\right\}_{i=1}^M$ are linearly independent.*

We denote by $Y^\perp$ the orthogonal of Y in $L^2(U \times \omega)$. Assume that

$$\begin{cases} \text{any function } k \in Y \text{ such that} \\ \frac{\partial k}{\partial t} + \frac{\partial k}{\partial a} - \Delta k + \mu k = 0 \text{ in } U \times \Omega \text{ and } k = 0 \text{ in } U \times \omega \\ \text{is identically zero in } U \times \Omega. \end{cases} \tag{1.11}$$

We now consider the following controllability problem: *Given* $h_0 \in L^2(U \times O)$, $w_0 \in Y_\theta$, *find* $v \in L^2(U \times \omega)$ *such that*

$$v \in Y^\perp, \tag{1.12}$$

and if $q = q(t, a, x, v)$ *is a solution of*

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t, 0, x) + h_0 \chi_O + (w_0 - v)\chi_\omega & \text{in } Q, \\ \dfrac{\partial q}{\partial \nu} = 0 & \text{on } \Sigma, \\ q(T, a, x) = 0 & \text{in } Q_A, \\ q(t, A, x) = 0 & \text{in } Q_T, \end{cases} \tag{1.13}$$

q satisfies

$$q(0, a, x, v) = 0 \text{ in } Q_A. \tag{1.14}$$

Remark 7 *Let us notice that if* v *exists, the set*

$$\mathcal{E} = \Big\{ \overline{v} \in Y^\perp : (\overline{v}, \overline{q} = q(t, a, x, \overline{v})) \textit{ satisfies } (1.13)\text{–}(1.14) \Big\}$$

is a non-empty, closed, and convex subset of $L^2(U \times \omega)$. *Therefore, there exists* $v \in \mathcal{E}$ *of minimal norm.*

The problem (1.12)–(1.14) is a null-controllability problem with constraint on the control. When $Y^\perp = L^2(U \times \omega)$, this problem becomes a null-controllability problem without constraint on the control. This kind of problems has been studied by many authors with various methods [1, 2]. In this paper we solve the null internal controllability problem with constraint on the control (1.12)–(1.14). This allows us to prove the existence of the sentinel for given sensitivity (1.3)–(1.6). More precisely, we have the following result.

Theorem 1 *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ *with boundary* Γ *of class* $\mathcal{C}^\infty$. *Let* Γ_1 *be a non-empty open subset of* Γ. *Let also* O *and* ω *be two non-empty subsets of* Ω *such that* $O \cap \omega \neq \emptyset$. *Assume that the assumptions of the data of the system* (1.1) *are satisfied. Assume also that* (1.9) *and* (1.11) *hold. Then, the existence of sentinel* (1.3)–(1.6) *holds if and only if the internal null-controllability problem with constraints on the control* (1.12)–(1.14) *has a solution.*

To prove the internal null-controllability problem with constraints on the control (1.12)–(1.14), we use an inequality of Carleman adapted to the constraints that we establish by means of a global Carleman inequality. More precisely we prove the following result.

Theorem 2 *Assume that the hypotheses of Theorem* 1 *are satisfied. Then, there exists a positive real weight function* θ (*a precise definition of* θ *will be given later on*) *such that for any function* $h_0 \in L^2(U \times O)$ *with* $\theta h_0 \in L^2(U \times O)$ *there exists a unique control* $\widehat{v} \in L^2(U \times \omega)$ *such that* $(\widehat{v}, \widehat{q})$ *with* $\widehat{q} = q(\widehat{v})$ *is a solution of the null internal controllability problem with constraint on the control* (1.12)–(1.14) *and provides a control* $\widehat{w} = w_0\chi_\omega - \widehat{v}$ *of the sentinel problem satisfying* (1.6). *Moreover, the control* $\widehat{w}$ *is given by*

$$\widehat{w} = P(w_0) + (I - P)(\widehat{\rho}\chi_\omega), \tag{1.15}$$

where P *is the orthogonal projection operator from* $L^2(U \times \omega)$ *into* Y, $w_0 \in Y_\theta$ *depends on* h_0 *and* c_i, $i \in \{1, \dots, M\}$, *and will be precisely determined in* (2.7), *and* $\widehat{\rho}$ *satisfies*

$$\begin{cases} \dfrac{\partial \widehat{\rho}}{\partial t} + \dfrac{\partial \widehat{\rho}}{\partial a} - \Delta \widehat{\rho} + \mu \widehat{\rho} = 0 & \text{in } Q, \\ \dfrac{\partial \widehat{\rho}}{\partial \nu} = 0 & \text{on } \Sigma, \\ \widehat{\rho}(t,0,x) = \displaystyle\int_0^A \beta(t,a,x)\widehat{\rho}(t,a,x)\,\mathrm{d}a & \text{in } Q_T. \end{cases} \tag{1.16}$$

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. In Section 3, we study the internal null-controllability problem with constraint on the control (1.12)–(1.14) and prove Theorem 2.

2 Equivalence between the sentinel problem and the controllability problem with constraint on the control

In this section we prove Theorem 1. But before going further, we need the following result.

Lemma 1 *Assume that* (1.9) *and* (1.11) *hold. Then, the functions* $y_{\lambda_i}\chi_\omega$, $1 \leq i \leq M$, *are linearly independent. Moreover, the functions* $\frac{1}{\theta} y_{\lambda_i}\chi_\omega$, $1 \leq i \leq M$, *are also linearly independent.*

Proof. Let $\alpha_i \in \mathbb{R}, 1 \leq i \leq M$, be such that $\sum_{i=1}^M \alpha_i y_{\lambda_i}\chi_\omega = 0$. Set $k = \sum_{i=1}^M \alpha_i y_{\lambda_i}\chi_\omega$. Then, using (1.8), k is such that

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial k}{\partial a} - \Delta k + \mu k = 0 & \text{in } Q, \\ k(0,a,x) = 0 & \text{in } Q_A, \\ k(t,0,x) = \displaystyle\int_0^A \beta(t,a,x)k(t,a,x)\,\mathrm{d}a & \text{in } Q_T, \\ \dfrac{\partial k}{\partial \nu} = \displaystyle\sum_{i=1}^M \alpha_i \widehat{\xi}_i \chi_{\Sigma_1} & \text{on } \Sigma, \\ k = 0 & \text{in } U \times \omega. \end{cases} \tag{2.1}$$

Assumption (1.11) allows us to say that $k = 0$ in Q. Therefore, we deduce that $\sum_{i=1}^M \alpha_i \widehat{\xi}_i \chi_{\Sigma_1} = 0$ on Σ, and it follows from (1.9) that $\alpha_i = 0$, $1 \leq i \leq M$. The second assertion of the lemma follows immediately. $\square$

Now, let us prove Theorem 1. To this end, we interpret (1.4) and (1.5). Actually, in view of (1.3), the stationary condition (1.4) and, respectively, the sensitivity conditions (1.5) hold if and only if

$$\int_U\int_O h_0 y_\tau \,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega w y_\tau \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = 0 \text{ for all } \widehat{y}^0 \text{ with } \left\|\widehat{y}^0\right\|_{L^2(Q_A)} \leq 1. \tag{2.2}$$

and

$$\int_U\int_O h_0 y_{\lambda_i} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega w y_{\lambda_i} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = c_i,\ 1 \leq i \leq M. \tag{2.3}$$

Therefore, in order to transform equation (2.2), we consider the following linear adjoint problem:

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x) + h_0\chi_O + w\chi_\omega & \text{in } Q, \\ \dfrac{\partial q}{\partial \nu} = 0 & \text{on } \Sigma, \\ q(T,a,x) = 0 & \text{in } Q_A, \\ q(t,A,x) = 0 & \text{in } Q_T. \end{cases} \tag{2.4}$$

Since $h_0\chi_O + w\chi_\omega \in L^2(Q)$, and the assumptions (H_1)–(H_3) hold, we can prove that the problem (2.4) has a unique solution $q \in L^2(U, H^1(\Omega))$. Therefore, $q \in W(U)$ and, in view of Remark 2, $(q(0,.,.), q(.,0,.))$ and $(q(T,.,.), q(.,A,.))$ exist and belong to $L^2(Q_A) \times L^2(Q_T)$.

Now multiplying both sides of the differential equation in (2.4) by y_τ, that is, the solution of (1.7), and integrating by parts in Q, we get for all $\widehat{y}^0 \in L^2(Q_A)$ that

$$\int_U\int_O h_0 y_\tau \,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega w y_\tau \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = \int_0^A\int_\Omega q(0,a,x)\widehat{y}^0 \,\mathrm{d}a\mathrm{d}x.$$

Thus, the condition (1.4) (or (2.2)) holds if and only if

$$q(0,a,x) = 0 \text{ in } Q_A. \tag{2.5}$$

Then, multiplying both sides of the differential equation in (2.4) by y_{λ_i}, that is, the solution of (1.8), and integrating by parts in Q, we have

$$\int_U\int_O h_0 y_{\lambda_i} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega w y_{\lambda_i} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = \int_{\Sigma_1} q\widehat{\xi}_i \,\mathrm{d}t\mathrm{d}a\mathrm{d}x,\ 1 \leq i \leq M.$$

Thus, the condition (1.5) (or (2.3)) is equivalent to

$$\int_{\Sigma_1} q\widehat{\xi}_i \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = c_i,\ 1 \leq i \leq M. \tag{2.6}$$

Now, consider the matrix

$$\left(\int_U\int_\omega \frac{1}{\theta} y_{\lambda_i} y_{\lambda_j} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x\right)_{1\leq i,j\leq M}.$$

Since this matrix is symmetric positive definite, there exists a unique $w_0 \in Y_\theta$ such that

$$c_i - \int_U\int_O h_0 y_{\lambda_i} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = \int_U\int_\omega w_0 y_{\lambda_i} \,\mathrm{d}t\mathrm{d}a\mathrm{d}x,\ 1 \leq i \leq M. \tag{2.7}$$

Consequently, combining (2.3) with (2.7), we observe that condition (1.5) (or the constraints (2.6)) holds if and only if

$$w - w_0 = -v \in Y^\perp,$$

where Y is given by (1.10). Replacing w by $w_0 - v$ in the second expression of (2.4), we obtain (1.13). We have just proved that the sentinel problem (1.3)–(1.6) holds if and only if null-controllability problem with constraints on the control (1.12)–(1.14) has a solution.

Remark 8 *If $\mathcal{E}$ is the set of admissible controls $v \in L^2(U \times \omega)$ such that* (1.12)–(1.14) *are satisfied, then $\mathcal{E}$ is a closed convex subset of $L^2(U \times \omega)$. Since $w_0 - \mathcal{E}$ is also a closed convex subset of $L^2(U \times \omega)$, we can obtain w to be of minimum norm in $L^2(U \times \omega)$ by minimizing the norm of $w_0 - v$ when $v \in \mathcal{E}$. Then, the pair $(v, q(v))$ satisfying* (1.12)–(1.14) *necessarily provides a control w satisfying* (1.6).

3 Study of the internal null-controllability problem with constraints on the control

3.1 An adapted observability inequality

The observability inequality we are looking for is a consequence of Carleman's inequality. We consider an auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) > 0 \text{ for every } x \in \Omega,\ \psi(x) = 0 \text{ for every } x \in \Gamma,\ |\nabla\psi(x)| \neq 0 \text{ for every } x \in \overline{\Omega \setminus \omega_0}, \quad (3.1)$$

where ω_0 denotes any open set such that $\omega_0 \Subset \omega$. Such a function ψ exists according to A. Fursikov and O. Yu. Imanuvilov [5].

For any positive parameter λ we define the following weight functions:

$$\varphi(t,a,x) = \frac{e^{\lambda\psi(x)}}{at(T-t)}, \quad (3.2)$$

$$\widetilde{\varphi}(t,a,x) = \frac{e^{-\lambda\psi(x)}}{at(T-t)}, \quad (3.3)$$

$$\eta(t,a,x) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{at(T-t)}, \quad (3.4)$$

$$\widetilde{\eta}(t,a,x) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{-\lambda\psi(x)}}{at(T-t)}. \quad (3.5)$$

Since φ does not vanish on Q, we set

$$\frac{1}{\theta^2} = \min\left[\left(\frac{e^{-2s\eta}}{\varphi} + e^{-2s\widetilde{\eta}}\widetilde{\varphi}\right), (\varphi e^{-2s\eta} + \widetilde{\varphi}e^{-2s\widetilde{\eta}}), (\varphi^3 e^{-2s\eta} + \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}})\right] \quad (3.6)$$

and we adopt the following notation

$$\begin{aligned} L &= \frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \Delta + \mu I, \\ L^* &= -\frac{\partial}{\partial t} - \frac{\partial}{\partial a} - \Delta + \mu I, \\ \mathcal{V} &= \left\{\rho \in C^\infty(\overline{Q}) : \frac{\partial\rho}{\partial\nu} = 0 \text{ on } \Sigma\right\}. \end{aligned} \quad (3.7)$$

Using the introduced notations and the definition of θ given by (3.6), we obtain the following Carleman inequality.

Proposition 1 (Global Carleman inequality) *Let ψ, φ, $\widetilde{\varphi}$, η and $\widetilde{\eta}$ be defined respectively by* (3.1), (3.2)–(3.5). *Then, there exist $\lambda_o > 1$, $s_o > 1$ and $C > 0$ such that for any $\lambda \geq \lambda_o$, for any $s \geq s_o$ and for any $\rho \in \mathcal{V}$ the following inequality holds*:

$$\begin{aligned}
&\int_Q \left(\frac{e^{-2s\eta}}{s\varphi} + \frac{e^{-2s\widetilde{\eta}}}{s\widetilde{\varphi}} \right) \left(\left| \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \right|^2 + |\Delta \rho|^2 \right) \mathrm{d}t\mathrm{d}a\mathrm{d}x \\
&\quad + s\lambda^2 \int_Q (\varphi e^{-2s\eta} + \widetilde{\varphi} e^{-2s\widetilde{\eta}}) \, |\nabla \rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \\
&\quad + s^3\lambda^4 \int_Q (\varphi^3 e^{-2s\eta} + \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}}) \, |\rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \\
&\leq C \left(\int_Q (e^{-2s\eta} + e^{-2s\widetilde{\eta}}) \, |L\rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \right) \\
&\quad + C \left(s^3\lambda^4 \int_U \int_\omega (\varphi^3 e^{-2s\eta} + \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}}) \, |\rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \right).
\end{aligned} \tag{3.8}$$

Proof. See [15]. □

Since $e^{-2s\eta}$, $e^{-2s\widetilde{\eta}}$, $\varphi^k e^{-2s\eta}$ and $\widetilde{\varphi}^k e^{-2s\widetilde{\eta}}$ with $k > 0$ are bounded, it is immediate that $\frac{1}{\theta^2}$, $e^{-2s\eta} + e^{-2s\widetilde{\eta}}$ and $\varphi^3 e^{-2s\eta} + \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}}$ are also bounded in Q. Hence, from Proposition 1, we obtain the following result.

Proposition 2 *Let θ be defined by* (3.6). *Then, there exist $\lambda_o > 1$, $s_o > 1$ and $C > 0$ such that for any $\lambda \geq \lambda_o$, for any $s \geq s_o$ and for any $\rho \in \mathcal{V}$,*

$$\begin{aligned}
&\int_Q \frac{1}{\theta^2} \left(\left| \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \right|^2 + |\Delta \rho|^2 \right) \mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_Q \frac{1}{\theta^2} \, |\nabla \rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_Q \frac{1}{\theta^2} \, |\rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \\
&\leq C \left(\int_Q |L\rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U \int_\omega |\rho|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \right).
\end{aligned} \tag{3.9}$$

Lemma 2 *Assume that* (1.9) *holds. Let Y be the real vector subspace of $L^2(U \times \omega)$ of finite dimensions defined in* (1.10). *Then, any function ρ such that*

$$\left\{ \begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta \rho + \mu \rho &= 0 \quad \text{in } Q, \\
\frac{\partial \rho}{\partial \nu} &= 0 \quad \text{on } \Sigma, \\
\rho \chi_\omega &\in Y,
\end{aligned} \right. \tag{3.10}$$

is identically zero.

Proof. For any ρ satysfying (3.10), there exist α_i, $1 \leq i \leq M$, such that $\rho = \sum_{i=1}^M \alpha_i y_{\lambda_i}$ in $U \times \omega$. We set $z = \rho - \sum_{i=1}^M y_{\lambda_i}$. Then, in view of (1.8), we have

$$\left\{ \begin{aligned}
\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - \Delta z + \mu z &= 0 \quad \text{in } Q, \\
\frac{\partial z}{\partial \nu} &= 0 \quad \text{on } \Sigma, \\
z &= 0 \quad \text{in } U \times \omega.
\end{aligned} \right. \tag{3.11}$$

Then, it follows from (1.11) that $z = 0$ in Q. Consequently, $\rho = \sum_{i=1}^{M} \alpha_i y_{\lambda_i}$. Since $\frac{\partial z}{\partial \nu} = 0$ on Σ, we deduce from (1.8) that $\sum_{i=1}^{M} \alpha_i \widehat{\xi_i} = 0$ in Σ_1. Therefore, assumption (1.9) allows us to conclude that $\alpha_i = 0$ for $1 \leq i \leq M$. This means that $\rho = 0$ in Q. □

Proposition 3 (Adapted Carleman inequality) *Assume that* (1.9) *holds. Let* Y *be the real vector subspace of* $L^2(U \times \omega)$ *of finite dimensions defined in* (1.10) *and let* P *be the orthogonal projection operator from* $L^2(U \times \omega)$ *into* Y. *Let also* θ *be the function defined by* (3.6). *Then, there exist numbers* $\lambda_0 > 1$, $s_0 > 1$, $C > 0$ *such that for fixed* $\lambda \geq \lambda_0$ *and* $s \geq s_0$ *and for any* $\rho \in \mathcal{V}$,

$$\int_U \int_\Omega \frac{1}{\theta^2} |\rho|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq C\bigg(\int_U \int_\Omega |L\rho|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U \int_\omega |\rho\chi_\omega - P\rho|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x\bigg). \quad (3.12)$$

Proof. As in [8], we use a well-known compactness-uniqueness argument and the inequality (3.9). Indeed, suppose that (3.12) does not hold. Then, for any $j \in \mathbb{N}^*$ there exists $\rho_j \in \mathcal{V}$ such that

$$\int_U \int_\Omega \frac{1}{\theta^2} |\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x = 1, \quad (3.13)$$

$$\int_U \int_\Omega |L\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq \frac{1}{j}, \quad (3.14)$$

$$\int_U \int_\omega |\rho_j - P\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq \frac{1}{j}. \quad (3.15)$$

In what follows, we prove in three steps that (3.13)–(3.15) yield a contradiction.

Step 1. We have

$$\int_U \int_\omega \frac{1}{\theta^2} |P\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq 2\int_U \int_\omega \frac{1}{\theta^2} |\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x + 2\int_U \int_\omega \frac{1}{\theta^2} |\rho_j - P\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x.$$

Since $\frac{1}{\theta^2}$ is bounded, using (3.13) and (3.15), it follows that there exists a positive constant C such that

$$\int_U \int_\omega \frac{1}{\theta^2} |P\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq C. \quad (3.16)$$

Therefore, because Y is a finite dimensional vector subspace of $L^2(U \times \omega)$, we deduce that

$$\int_U \int_\omega |P\rho_j|^2 \,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq C. \quad (3.17)$$

As $\rho\chi_\omega = P\rho\chi_\omega + (\rho\chi_\omega - P\rho\chi_\omega)$, using (3.15) and (3.17), we obtain

$$\|\rho_j\|^2_{L^2(U\times\omega)} \leq C. \quad (3.18)$$

Step 2. Let

$$L^2\left(\frac{1}{\theta^2}, U\times\omega\right) = \left\{\rho\in L^2(U\times\omega) : \int_U\int_\omega \frac{1}{\theta^2}\,|\rho|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x < \infty\right\}.$$

Then, in view of (3.13) and (3.18), we deduce from (3.9) that $\left(\frac{\partial\rho_j}{\partial t}+\frac{\partial\rho_j}{\partial a}\right)$, (ρ_j), $(\nabla\rho_j)$ and $(\Delta\rho_j)$ are bounded in $L^2\left(\frac{1}{\theta^2}, U\times\omega\right)$. Let us take a subsequence still denoted by (ρ_j) such that

$$\rho_j \rightharpoonup \rho \text{ weakly in } L^2\left(\frac{1}{\theta^2}, U\times\omega\right). \tag{3.19}$$

Then, it follows from (3.2)–(3.5) and the definition of $\frac{1}{\theta}$ given by (3.6) that (ρ_j) is bounded in $L^2((\beta, T-\beta)\times(\gamma, A-\gamma); H^2(\Omega))$ for any $\beta>0$ and any $\gamma>0$. In particular, for all $\beta>0$ and $\gamma>0$, we have $\rho_j\rightharpoonup\rho$ weakly in $L^2((\beta,T-\beta)\times(\gamma,A-\gamma)\times\Omega)$, which implies that $\rho_j\rightharpoonup\rho$ weakly in $\mathcal{D}'(Q)$. Therefore, we get from (3.14) and (3.18) that

$$L\rho_j \to L\rho = 0 \text{ strongly in } L^2(U\times\Omega), \tag{3.20}$$

$$\rho_j \rightharpoonup \rho \text{ weakly in } L^2(U\times\omega). \tag{3.21}$$

And, since P is a compact operator, it follows from (3.21) that

$$P\rho_j \to P\rho \text{ strongly in } L^2(U\times\omega). \tag{3.22}$$

In view of (3.15), we also have

$$\rho_j\chi_\omega - P\rho_j \to 0 \text{ strongly in } L^2(U\times\omega). \tag{3.23}$$

Thus, combining (3.22) and (3.23), we get

$$\rho_j \to P\rho \text{ strongly in } L^2(U\times\omega). \tag{3.24}$$

Thanks to the uniqueness of the limit in $L^2(U\times\omega)$, the convergence relations (3.21) and (3.24) imply that $P\rho=\rho\chi_\omega$. This means that $\rho\chi_\omega\in Y$. We thus have proved that ρ satisfies (3.10). Hence, thanks to Lemma 2, ρ is identically zero. Therefore, (3.24) becomes

$$\rho_j \to 0 \text{ strongly in } L^2(U\times\omega). \tag{3.25}$$

Step 3. Since $\rho_j\in\mathcal{V}$, it follows from the observability inequality (3.9) that

$$\int_U\int_\Omega \frac{1}{\theta^2}\,|\rho_j|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \le C\left(\int_U\int_\Omega |L\rho_j|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega |\rho_j|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x\right).$$

Therefore, passing in this latter inequality to the limit while using (3.20)–(3.25), we obtain

$$\lim_{j\to\infty}\int_U\int_\omega \frac{1}{\theta^2}\,|\rho_j|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x = 0.$$

This contradicts (3.13). □

3.2 Proof of Theorem 2

In this subsection, we are concerned with the proof of Theorem 2, that is, the optimality system for the control $\widehat{v}$ such that the pair $(\widehat{v};\widehat{q})$ satisfies (1.12)–(1.14). A classical way to derive this optimality system is the method of penalization due to J. L. Lions [7]. The proof of Theorem 2 will be divided in three steps.

Step 1. Let w_0 be defined by (2.7). If $v \in Y^\perp$ and q is solution of (1.13), then $q(0,.,.) \in L^2(Q_A)$ and we can define the functional

$$J_\epsilon(v) = \frac{1}{2}\left\|w0 - v\right\|^2_{L^2(U\times\omega)} + \frac{1}{2\epsilon}\left\|q(0,.,.)\right\|^2_{L^2(Q_A)}. \tag{3.26}$$

We consider the optimal control problem: *Find $v_\epsilon \in Y^\perp$ such that*

$$J_\epsilon(v_\epsilon) = \min_{v\in Y^\perp} J_\epsilon(v). \tag{3.27}$$

Since $Y^\perp$ is a closed and convex subset of $L^2(U\times\omega)$, it is classical to prove that there exists a unique solution to (3.27). If by q_ϵ we denote the solution of (1.13) corresponding to v_ϵ, using an adjoint state ρ_ϵ, we have that the triplet $(q_\epsilon,\rho_\epsilon,v_\epsilon)$ is a solution of the first order optimality system:

$$\begin{cases} L^* q_\epsilon &= \beta q_\epsilon(t,0,x) + h_0\chi_O + (w_0 - v_\epsilon)\chi_\omega \quad \text{in } Q,\\ \dfrac{\partial q_\epsilon}{\partial\nu} &= 0 \quad \text{on } \Sigma,\\ q_\epsilon(T,a,x) &= 0 \quad \text{in } Q_A,\\ q_\epsilon(t,A,x) &= 0 \quad \text{in } Q_T, \end{cases} \tag{3.28}$$

$$\begin{cases} L\rho_\epsilon &= 0 \quad \text{in } Q,\\ \dfrac{\partial \rho_\epsilon}{\partial\nu} &= 0 \quad \text{on } \Sigma,\\ \rho_\epsilon(0,a,x) &= \dfrac{1}{\epsilon} q_\epsilon(0,a,x) \quad \text{in } Q_A,\\ \rho_\epsilon(t,0,x) &= \displaystyle\int_0^A \beta(t,a,x)\rho_\epsilon(t,a,x)\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \quad \text{in } Q_T, \end{cases} \tag{3.29}$$

$$v_\epsilon = (w_0\chi_\omega + \rho_\epsilon\chi_\omega) - P(w_0\chi_\omega + \rho_\epsilon\chi_\omega) \in Y^\perp. \tag{3.30}$$

Step 2. Multiplying the state equation (3.28) by ρ_ϵ and integrating by parts over Q, we get

$$\frac{1}{\epsilon}\left\|q_\epsilon(0,.,.)\right\|^2_{L^2(Q_A)} = \int_U\int_O h_0\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega (w_0 - v_\epsilon)\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x,$$

which in view of (3.30) and the fact that $v_\epsilon \in Y^\perp$ gives

$$\begin{aligned}\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} &= \int_U\int_O h_0\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \\ &\quad + \int_U\int_\omega (w_0 - v_\epsilon)(v_\epsilon - w_0 + P(w_0\chi_\omega + \rho_\epsilon\chi_\omega))\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \\ &= \int_U\int_O h_0\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \\ &\quad - \|w_0 - v_\epsilon\|_{L^2(U\times\omega)} + \|Pw_0\chi_\omega\|_{L^2(U\times\omega)} \\ &\quad + \int_U\int_\omega w_0\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x.\end{aligned}$$

As on $U\times\omega$,

$$v_\epsilon - w_0 = (I-P)\rho_\epsilon\chi_\epsilon - P(w_0\chi_\omega),$$

we have that

$$\|v_\epsilon - w_0\|^2_{L^2(U\times\omega)} = \|(I-P)\rho_\epsilon\chi_\omega\|^2_{L^2(U\times\omega)} + \|P(w_0\chi_\omega)\|^2_{L^2(U\times\omega)},$$

so that

$$\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \|(I-P)\rho_\epsilon\chi_\omega\|^2_{L^2(U\times\omega)} = \int_U\int_O h_0\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_U\int_\omega w_0\rho_\epsilon\,\mathrm{d}t\mathrm{d}a\mathrm{d}x.$$

This implies that

$$\begin{aligned}&\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \|(I-P)\rho_\epsilon\chi_\omega\|^2_{L^2(U\times\omega)} \\ &\leq \left(\int_U\int_O(\theta h_0)^2\right)^{1/2}\left(\int_U\int_\omega\frac{1}{\theta^2}\rho_\epsilon^2\right)^{1/2} + \left(\int_U\int_\omega(\theta w_0)^2\right)^{1/2}\left(\int_U\int_\omega\frac{1}{\theta^2}\rho_\epsilon^2\right)^{1/2}.\end{aligned} \tag{3.31}$$

If we apply the adapted Carleman inequality (3.12) to ρ_ϵ, we obtain

$$\int_U\int_\Omega\frac{1}{\theta^2}|\rho_\epsilon|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x \leq C\int_U\int_\omega|\rho_\epsilon\chi_\omega - P\rho_\epsilon|^2\,\mathrm{d}t\mathrm{d}a\mathrm{d}x, \tag{3.32}$$

where $C>0$ is independent of ϵ. From (3.31), the choice of $w_0 \in Y_\theta$ and hypothesis on h_0, we deduce that

$$\|(I-P)\rho_\epsilon\chi_\omega\|_{L^2(U\times\omega)} \leq C\left[\left(\int_U\int_O(\theta h_0)^2\right)^{1/2} + \left(\int_U\int_\omega(\theta w_0)^2\right)^{1/2}\right], \tag{3.33}$$

and then

$$\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|_{L^2(Q_A)} \leq C\left[\left(\int_U\int_O(\theta h_0)^2\right)^{1/2} + \left(\int_U\int_\omega(\theta w_0)^2\right)^{1/2}\right], \tag{3.34}$$

$$\|v_\epsilon\|^2_{L^2(U\times\omega)} \leq C\left[\left(\int_U\int_O(\theta h_0)^2\right)^{1/2} + \left(\int_U\int_\omega(\theta w_0)^2\right)^{1/2}\right]. \tag{3.35}$$

Since q_ϵ satisfies (3.28), we can prove that

$$\|q_\epsilon \chi_\omega\|^2_{L^2(U;H^1(\Omega))} \leq C \left[\left(\int_U \int_O (\theta h_0)^2 \right)^{1/2} + \left(\int_U \int_\omega (\theta w_0)^2 \right)^{1/2} \right]. \tag{3.36}$$

In view of (3.32) and (3.33), we get

$$\left\| \frac{1}{\theta} \rho_\epsilon \right\|_{L^2(U\times\Omega)} \leq C \left[\left(\int_U \int_O (\theta h_0)^2 \right)^{1/2} + \left(\int_U \int_\omega (\theta w_0)^2 \right)^{1/2} \right]. \tag{3.37}$$

Using again (3.33) and the fact that $\frac{1}{\theta}$ is bounded, we obtain

$$\left\| \frac{1}{\theta} P \rho_\epsilon \chi_\omega \right\|_{L^2(U\times\omega)} \leq C \left[\left(\int_U \int_O (\theta h_0)^2 \right)^{1/2} + \left(\int_U \int_\omega (\theta w_0)^2 \right)^{1/2} \right]. \tag{3.38}$$

Therefore, since Y is a finite dimensional vector subspace of $L^2(U \times \omega)$, we deduce that

$$\|P \rho_\epsilon \chi_\omega\|_{L^2(U\times\omega)} \leq C \left[\left(\int_U \int_O (\theta h_0)^2 \right)^{1/2} + \left(\int_U \int_\omega (\theta w_0)^2 \right)^{1/2} \right], \tag{3.39}$$

from which we deduce by using (3.33) that

$$\|\rho_\epsilon\|_{L^2(U\times\omega)} \leq C \left[\left(\int_U \int_O (\theta h_0)^2 \right)^{1/2} + \left(\int_U \int_\omega (\theta w_0)^2 \right)^{1/2} \right]. \tag{3.40}$$

Using (3.9), we have that

$$\begin{aligned} \int_Q \frac{1}{\theta^2} \left(\left| \frac{\partial \rho_\epsilon}{\partial t} + \frac{\partial \rho_\epsilon}{\partial a} \right|^2 + |\Delta \rho_\epsilon|^2 \right) \mathrm{d}t\mathrm{d}a\mathrm{d}x + \int_Q \frac{1}{\theta^2} |\nabla \rho_\epsilon|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \\ + \int_Q \frac{1}{\theta^2} |\rho_\epsilon|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x \leq C \int_U \int_\omega |\rho_\epsilon|^2 \, \mathrm{d}t\mathrm{d}a\mathrm{d}x. \end{aligned} \tag{3.41}$$

Step 3. We prove the convergence of $(v_\epsilon, q_\epsilon)_\epsilon$ and $(\rho_\epsilon)_\epsilon$ towards $(\widehat{v}, \widehat{q})$ and $\widehat{\rho}$ as $\epsilon \to 0$. According to (3.35), (3.36) and (3.37) we can extract subsequences of $(v_\epsilon, q_\epsilon)_\epsilon$ (still called $(v_\epsilon, q_\epsilon)_\epsilon$) such that

$$v_\epsilon \rightharpoonup \widetilde{v} \text{ weakly in } L^2(U \times \omega), \tag{3.42}$$

$$q_\epsilon \rightharpoonup \widetilde{q} \text{ weakly in } L^2(U; H^1(\Omega)), \tag{3.43}$$

$$\frac{1}{\theta} \rho_\epsilon \rightharpoonup \text{ weakly in } L^2 \left(\frac{1}{\theta}, Q \right). \tag{3.44}$$

As v_ϵ belongs to $Y^\perp$, which is a closed vector subspace of $L^2(U \times \omega)$, we have

$$\widetilde{v} \in Y^\perp. \tag{3.45}$$

From (3.43) and Remark 1 we have that $\widetilde{q} \in W(U)$. Hence, in view of Remark 2, we know that the traces $(\widetilde{q}(0,.,.), \widetilde{q}(.,0,.))$ and $(\widetilde{q}(T,.,.), \widetilde{q}(.,A,.))$ exist and belong to $L^2(Q_A) \times L^2(Q_T)$. So, using (3.43) and (3.42) while passing (3.28) to the limit as $\epsilon \to 0$, we can prove that $\widetilde{q}$ is a solution of

$$\begin{cases} -\dfrac{\partial \widetilde{q}}{\partial t} - \dfrac{\partial \widetilde{q}}{\partial a} - \Delta \widetilde{q} + \mu \widetilde{q} = \beta \widetilde{q}(t,0,x) + h_0 \chi_O + (w_0 - \widetilde{v})\chi_\omega & \text{in } Q, \\ \dfrac{\partial \widetilde{q}}{\partial \nu} = 0 & \text{on } \Sigma, \\ \widetilde{q}(T,a,x) = 0 & \text{in } Q_A, \\ \widetilde{q}(t,A,x) = 0 & \text{in } Q_T, \end{cases} \tag{3.46}$$

and it follows from (3.34) that

$$q_\epsilon(0,.,.) \rightharpoonup \widetilde{q}(0,.,.) = 0 \text{ weakly in } L^2(Q_A). \tag{3.47}$$

In view of (3.45), (3.46) and (3.47), $(\widetilde{v}, \widetilde{q})$ satisfies the null-controllability (1.12)–(1.14). From (3.44),

$$\rho_\epsilon \rightharpoonup \widetilde{\rho} \text{ weakly in } \mathcal{D}'(Q).$$

Consequently, using (3.29) and (3.41), we can prove that $\widetilde{\rho}$ satisfies

$$\begin{cases} L\widetilde{\rho} = 0 & \text{in } Q, \\ \dfrac{\partial \widetilde{\rho}}{\partial \nu} = 0 & \text{on } \Sigma, \\ \widetilde{\rho}(t,0,x) = \displaystyle\int_0^A \beta(t,a,x)\widetilde{\rho}(t,a,x)\, \mathrm{d}t\mathrm{d}a\mathrm{d}x & \text{in } Q_T. \end{cases} \tag{3.48}$$

From (3.40), we see that

$$\rho_\epsilon \rightharpoonup \widetilde{\rho} \text{ weakly in } L^2(U \times \omega), \tag{3.49}$$

and therefore

$$v_\epsilon \rightharpoonup \widetilde{v} = (I - P)(w_0\chi_\omega + \widetilde{(\rho)}\chi_\omega) \text{ weakly in } L^2(U \times \omega). \tag{3.50}$$

We know on the one hand that $(\widetilde{v}, \widetilde{q})$ is a solution to null-controllability (1.12)–(1.14), and on the other hand that there exists a unique $\widehat{v} \in \mathcal{E}$ such that $w_0 - v$ is of minimal norm in $L^2(U \times \omega)$. If we denote by $\widehat{q}$ the corresponding solution to (1.13), we have $\widehat{q}(0,.,.) = 0$ and, as $\widetilde{v} \in \mathcal{E}$,

$$\frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\omega)} \le J_\epsilon(v_\epsilon) \le J_\epsilon(\widehat{v}) = \frac{1}{2}\|w_0 - \widehat{v}\|^2_{L^2(U\times\omega)}$$

and

$$\frac{1}{2}\|w_0 - \widehat{v}\|_{L^2(U\times\omega)} \le \frac{1}{2}\|w_0 - v_\epsilon\|_{L^2(U\times\omega)}.$$

Using (3.42),

$$\frac{1}{2}\|w_0 - \widetilde{v}\|_{L^2(U\times\omega)} \le \liminf_{\epsilon\to 0} \frac{1}{2}\|w_0 - v_\epsilon\|_{L^2(U\times\omega)}.$$

Hence,

$$\widetilde{v} = \widehat{v}$$

and

$$v_\epsilon \to \widehat{v} \text{ strongly in } L^2(U \times \omega).$$

Writing $\widetilde{\rho} = \widehat{\rho}$, we have

$$\widehat{v} = (I - P)(w_0\chi_\omega + \widehat{\rho}\chi_\omega).$$

This end the proof of Theorem 2.

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