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SIMULTANEOUS NULL CONTROLLABILITY FOR A SYSTEM OF TWO STROKE EQUATIONS

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Abstract. In this paper, we study the null controllability for a system of two stroke equations arising from coupled population dynamics models. First, we transform the system of two stroke equations with the same control to a system of coupled two stroke equations with a control function acting only in one equation. Then, we establish a global Carleman inequality and we deduce an observability inequality that we use to solve the problem thanks to appropriate estimates adapted to the system.

Keywords: Carleman inequalities, null controllability, observability inequality, population dynamics equation.

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1 Introduction and main result

We consider a linear model describing the dynamics of population with age dependence and spatial structure. More precisely, let Ω be an open and bounded domain of $\mathbb{R}^N$, where $N \in \{1, 2, 3\}$, with boundary Γ of class C^∞. For the time $T > 0$ and the life expectancy of an individual $A > 0$, we set $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$ and $\Sigma = U \times \Gamma$. Let $y := y(t, a, x)$ be the distribution of individuals of age a at time t and location $x \in \Omega$. We denote by $\mu := \mu(t, a, x) \geq 0$ and $\beta := \beta(t, a, x) \geq 0$, respectively, the natural death and birth rates of individuals of age a at time t and location x. We mention that if the flux of individuals takes the form

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$-\nabla y(t,a,x)$, where ∇ is the gradient vector with respect to the spatial variable x, then y solves the following evolution system:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y + \mu y = f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,a,x) = y^0(a,x) & \text{in } Q_A, \\ y(t,0,x) = \displaystyle\int_0^A \beta(t,a,x) y(t,a,x)\,\mathrm{d}a & \text{in } Q_T, \end{cases} \tag{1.1}$$

where Δ is the Laplacian with respect to the spatial variable and $y^0(a,x)$ is the initial distribution of individuals of age a at location x. The function f is an external input. The formula $\int_0^A \beta(t,a,x)y(t,a,x)\,\mathrm{d}a$ denotes the distribution of newborn at time t and position x. We assume as in [1] that:

$$(H_1) \begin{cases} \beta \in C^2([0,T]\times[0,A]\times\overline{\Omega}), \\ \beta(t,a,x) \geq 0 \text{ in } [0,T]\times[0,A]\times\overline{\Omega}, \\ \text{there exist } 0 < a_1 < a_0 < A \text{ such that } \beta(t,a,x) = 0 \text{ in } [0,T]\times[(0,a_1)\cup(a_0,A)]\times\Omega, \end{cases}$$

$$(H_2) \begin{cases} \mu(t,a,x) = \mu_0(a) + \mu_1(t,a,x) \text{ a.e. in } U\times\Omega, \\ \mu_1 \in L^\infty(U\times\Omega) \text{ and } \mu_1(t,a,x) \geq 0 \text{ for a.e. } (t,a,x)\in U\times\Omega, \\ \mu(t,a,x) \geq 0 \text{ for a.e. } (t,a,x)\in U\times\Omega, \\ \mu_0 \in L^1_{\text{loc}}(0,A) \text{ and } \displaystyle\lim_{a\to A}\int_0^a \mu_0(s)\,\mathrm{d}s = +\infty. \end{cases}$$

The third assumption in (H_1) means that younger and older individuals are not fertile. The fourth assumption in (H_2) means that all individuals die before the age A. For more literature on the significance of assumptions (H_1) and (H_2), we refer to [24, 3] and the reference therein.

Since $f \in L^2(Q)$ and $y^0 \in L^2(Q_A)$, we can prove as in [20] using assumptions (H_1)–(H_2) that the problem (1.1) has a unique solution $y \in L^2(U; H_0^1(\Omega))$ and $\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} \in L^2(U; H^{-1}(\Omega))$.

Let $\omega \Subset \Omega$ be a subdomain compactly embedded in Ω. In the sequel, we set $G = U \times \omega$ and $W(U) = \left\{\rho \in L^2(U; H_0^1(\Omega)) : \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2(U; H^{-1}(\Omega))\right\}$. The space $W(U)$ is continuously injected into $C((0,T); L^2(Q_A)) \cap C((0,A); L^2(Q_T))$. So, the solution of (1.1) belongs to $C([0,T]; L^2(Q_A)) \cap C([0,A]; L^2(Q_T)) \cap L^2(U; H_0^1(\Omega))$. The adjoint system associated to (1.1) is the following:

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x) + \xi & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T,a,x) = 0 & \text{in } Q_A, \\ q(t,A,x) = 0 & \text{in } Q_T, \end{cases} \tag{1.2}$$

for some $\xi \in L^2(Q)$. For $i = 1, 2$ we consider the following two stroke equations

$$\begin{cases} -\dfrac{\partial q_i}{\partial t} - \dfrac{\partial q_i}{\partial a} - \Delta q_i + \mu_i q_i = \beta_i q_i(t,0,x) + h + v\chi_\omega & \text{in } Q, \\ q_i = 0 & \text{on } \Sigma, \\ q_i(T,a,x) = 0 & \text{in } Q_A, \\ q_i(t,A,x) = 0 & \text{in } Q_T, \end{cases} \tag{1.3}$$

where $h \in L^2(Q)$ is some function, $v \in L^2(G)$ represents the control function and χ_ω is the characteristic function of ω, the set where the controls are chosen. In the system (1.3), we have the same control function v and the same function h on the right-hand side of the evolution equations. Using assumptions (H_1)–(H_2), we can also prove as in [20] that the problem (1.3) has a unique solution $q_i \in W(U)$. Sometimes, to underline that the solution q_i, where $i = 1, 2$, depends on v, we will write $q_i(t, a, x, v)$ instead of q_i.

The null controllability of the system (1.3) can be formulated as follows: given $h \in L^2(Q)$ find

$$v \in L^2(G) \tag{1.4}$$

such that the solutions q_i, $i = 1, 2$, of (1.3) satisfy

$$q_1(0, a, x) = q_2(0, a, x) = 0 \text{ in } Q_A. \tag{1.5}$$

Controllability problems for an age and space structured population dynamics model have been studied by several authors. For instance, B. Ainseba and M. Langlais proved that the set of profiles is approximatively reachable [4]. It has been shown in [5] that if the initial distribution is small enough, we can find a control which leads to extinction of the population. The result was achieved by means of Carleman's inequality for parabolic equations. Exact and approximate controllability results are obtained for a linear population dynamics problem structured in age and space by Ainseba (see [1]). Concerning the nonlinear population dynamics model, a null controllability result was established by B. Ainseba and M. Iannelli by means of the Kakutani fixed point theorem [2]. Using an extension of the Leray-Schauder fixed point theorem and Carleman's inequality for the adjoint system, O. Traoré showed that for all given initial densities, there exists an internal control acting on a small open set of the domain and leading the population to extinction [25]. S. Sawadogo and G. Mophou [23] gave a null controllability result for a population dynamics model with constraints on the state when the age of the population belongs to (γ, A) for any $\gamma > 0$. Following this work, M. Mercan and G. Mophou [16] proved a null controllability problem with constraints on the state for the adjoint system of a population dynamics model. The result was achieved by means of Carleman's adapted to the constraints. C. L. Rose solved a problem of simultaneous null controllability with constraint on the control for a system of coupled linear heat equations by means of an appropriate Carleman's estimate adapted to the constraint [15]. M. Kéré and O. Nakoulima proved a simultaneous null controllability for coupled backward heat equations and used the result to build a simultaneous sentinel [12]. However, the problem of simultaneous null controllability is little known in population dynamics problems. This is what motivates us to extend the results from [15, 12] to the population dynamics problem. More precisely, we study a simultaneous null controllability for a system of two stroke equations. Assume that

$$\mu_1 \neq \mu_2 \text{ and } \beta_1 \neq \beta_2 \text{ in } G. \tag{1.6}$$

By setting

$$\begin{aligned} &p_1 = q_1 + q_2, && p_2 = q_1 - q_2, && a_\mu = \tfrac{1}{2}(\mu_1 + \mu_2), && b_\mu = \tfrac{1}{2}(\mu_1 - \mu_2), \\ &a_\beta = \tfrac{1}{2}(\beta_1 + \beta_2), && b_\beta = \tfrac{1}{2}(\beta_1 - \beta_2), && f = 2h, && k = 2v, \end{aligned} \tag{1.7}$$

one gets:

$$
\begin{cases}
-\dfrac{\partial p_1}{\partial t} - \dfrac{\partial p_1}{\partial a} - \Delta p_1 + a_\mu p_1 + b_\mu p_2 = a_\beta p_1(t,0,x) \\
\qquad + b_\beta p_2(t,0,x) \\
\qquad + f + k\chi_\omega & \text{in } Q, \\
-\dfrac{\partial p_2}{\partial t} - \dfrac{\partial p_2}{\partial a} - \Delta p_2 + a_\mu p_2 + b_\mu p_1 = b_\beta p_1(t,0,x) \\
\qquad + a_\beta p_2(t,0,x) & \text{in } Q, \\
p_1 = p_2 = 0 & \text{on } \Sigma, \\
p_1(T,a,x) = p_2(T,a,x) = 0 & \text{in } Q_A, \\
p_1(t,A,x) = p_2(t,A,x) = 0 & \text{in } Q_T.
\end{cases}
\tag{1.8}
$$

Consequently, our aim is: for any a_μ, b_μ, a_β, $b_\beta \in L^\infty(Q)$, $f \in L^2(Q)$ find a control

$$k \in L^2(G) \tag{1.9}$$

such that the solution (p_1, p_2) of (1.8) satisfies

$$p_1(0,a,x) = p_2(0,a,x) = 0 \ \text{ in } \ Q_A. \tag{1.10}$$

The main result of this paper reads as follows.

Theorem 1 *Let $\omega \subset \Omega$. Assume that (H_1), (H_2) and* (1.6) *hold true. Then, there exists a positive real function θ (a precise definition of θ will be give later on) such that for any function $f \in L^2(Q)$ and a_μ, b_μ, a_β, $b_\beta \in L^\infty(Q)$ with $\theta f \in L^2(Q)$, θa_μ, θb_μ, θa_β, $\theta b_\beta \in L^\infty(Q)$ there exists a unique control $\tilde{k}$, of minimal norm in $L^2(G)$, such that $(\tilde{k}, \tilde{p}_1, \tilde{p}_2)$ is a solution of the simultaneous null controllability problem* (1.8)–(1.10)*. Moreover, the control $\tilde{k}$ is given by*

$$\tilde{k} = \tilde{\rho}_1 \chi_\omega, \tag{1.11}$$

where $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$ satisfies

$$
\begin{cases}
\dfrac{\partial \tilde{\rho}_1}{\partial t} + \dfrac{\partial \tilde{\rho}_1}{\partial a} - \Delta \tilde{\rho}_1 + a_\mu \tilde{\rho}_1 + b_\mu \tilde{\rho}_2 = 0 & \textit{in } Q, \\
\dfrac{\partial \tilde{\rho}_2}{\partial t} + \dfrac{\partial \tilde{\rho}_2}{\partial a} - \Delta \tilde{\rho}_2 + a_\mu \tilde{\rho}_2 + b_\mu \tilde{\rho}_1 = 0 & \textit{in } Q, \\
\tilde{\rho}_1 = \tilde{\rho}_2 = 0 & \textit{on } \Sigma, \\
\tilde{\rho}_1(t,0,x) = \displaystyle\int_0^A (a_\beta \tilde{\rho}_1 + b_\beta \tilde{\rho}_2)\,\mathrm{d}a & \textit{in } Q_T, \\
\tilde{\rho}_2(t,0,x) = \displaystyle\int_0^A (b_\beta \tilde{\rho}_1 + a_\beta \tilde{\rho}_2)\,\mathrm{d}a & \textit{in } Q_T.
\end{cases}
\tag{1.12}
$$

The rest of this paper is organized as follows. In Section 2, we establish a global Carleman's inequality from which we deduce an observability inequality adapted to our problem. In Section 3, we prove the existence of the solution of the problem (1.8)–(1.10). The Section 4 is devoted to proving Theorem 1.

2 Carleman's inequalities

Let us recall the following lemma due to A. V. Fursikov and O. Yu Imanuvilov.

Lemma 1 ([10]) *Let $\omega_0 \Subset \omega$ be a subset of ω. Then, there exists a function $\psi \in C^2(\Omega)$ which satisfies the following conditions*:

(a) $\left|\nabla\psi(x)\right| > 0$ *for all* $x \in \overline{\Omega \setminus \omega_0}$,

(b) $\psi(x) > 0$ *for all* $x \in \Omega$,

(c) $\psi(x) = 0$ *for all* $x \in \Gamma$ *and* $\frac{\partial\psi}{\partial\nu} \leq 0$ *on* Γ,

(d) $\min\{\psi(x) : x \in \Omega\} \geq \max\{\frac{3}{4}\|\psi\|_\infty, \ln 3\}$, *where* $\| \ \|_\infty = \| \ \|_{L^\infty(\Omega)}$.

For any positive parameters λ and τ, let us consider the weight functions which for all $(t, a, x) \in Q$ are given by

$$\alpha(t,a,x) = \tau\frac{e^{\frac{4}{3}\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{at\,(T-t)} > 0 \text{ and } \varphi(t,a,x) = \frac{e^{\lambda\psi(x)}}{at\,(T-t)} > 0.$$

Remark 1 *Note, in particular, that* $\varphi > \frac{4}{AT^2}$.

In the sequel, C represents different positive constants. We adopt the following notations:

$$\mathcal{V} = \left\{\rho \in C^\infty\left(\overline{Q}\right) : \rho_{|\Sigma} = 0\right\}, \quad \mathcal{W} = \mathcal{V}\times\mathcal{V}, \quad Q_{\omega_0} = (0,T)\times(0,A)\times\omega_0$$

and

$$\begin{aligned}
L\rho &= -\frac{\partial\rho}{\partial t} - \frac{\partial\rho}{\partial a} - \Delta\rho, & L^*\rho &= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial a} - \Delta\rho,\\
M(\rho_1,\rho_2) &= L^*\rho_1 + a_\mu\rho_1 + b_\mu\rho_2, & N(\rho_1,\rho_2) &= L^*\rho_2 + b_\mu\rho_1 + a_\mu\rho_2,\\
M(\rho_1,\rho_2) &= L\rho_1 + a_\mu\rho_1 + b_\mu\rho_2, & N(\rho_1,\rho_2) &= L\rho_2 + b_\mu\rho_1 + a_\mu\rho_2,\\
\|(a_\mu,b_\mu)\|_\infty^2 &= \|a_\mu\|_\infty^2 + \|b_\mu\|_\infty^2, & \mathrm{d}Q &= \mathrm{d}t\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned}$$

Adapting the method of [19] the following result can be easily proved.

Theorem 2 *There exist $\lambda_0 > 0$, $\tau_0 > 0$ and a positive constant C such that for $\lambda \geq \lambda_0$, $\tau \geq \tau_0$ and for $s \geq -3$ the inequality*

$$\begin{aligned}
&\int_Q \left(\frac{1}{\lambda}\left|\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial a}\right|^2 + \frac{1}{\lambda}\left|\Delta\rho\right|^2 + \lambda\tau^2\varphi^2\left|\nabla\rho\right|^2 + \lambda^4\tau^4\varphi^4\left|\rho\right|^2\right)\varphi^{2s-1}e^{-2\alpha}\,\mathrm{d}Q\\
&\leq C\left(\tau\int_Q\left|\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial a} \pm \Delta\rho\right|^2\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q + \lambda^4\tau^4\int_{Q_{\omega_0}}\left|\rho\right|^2\varphi^{2s+3}e^{-2\alpha}\,\mathrm{d}Q\right)
\end{aligned} \tag{2.1}$$

holds for any function $\rho \in \mathcal{V}$ such that the term on the right-hand side of the inequality (2.1) *is finite.*

Following [13], for given λ and τ as in Theorem 2, we consider the functional

$$I(s,\rho)=\int_Q\left(\frac{1}{\lambda}\left|\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}\right|^2+\frac{1}{\lambda}\left|\Delta\rho\right|^2+\lambda\tau^2\varphi^2\left|\nabla\rho\right|^2+\lambda^4\tau^4\varphi^4\left|\rho\right|^2\right)\varphi^{2s-1}e^{-2\alpha}\,\mathrm{d}Q.$$

Lemma 2 *Let C be the constant given by Theorem 2. For any $\lambda\geq\lambda_0$, $\tau\geq\tau_1=\frac{AT^2}{4}\left(\frac{6C}{{\lambda_0}^4}\right)^{\frac{1}{3}}\|(a_\mu,\,b_\mu)\|_\infty^{\frac{2}{3}}$ and $s\geq-3$, we have for all $\rho=(\rho_1,\rho_2)\in\mathcal{W}$,*

$$\begin{aligned}\lambda^4&\int_Q\left(|\rho_1|^2+|\rho_2|^2\right)(\tau\varphi)^{2s+3}e^{-2\alpha}\,\mathrm{d}Q\\&\leq 2C\bigg(3\int_Q\left(|M(\rho)|^2+|N(\rho)|^2\right)(\tau\varphi)^{2s}e^{-2\alpha}\,\mathrm{d}Q\\&\quad+\lambda^4\int_{Q_{\omega_0}}\left(|\rho_1|^2+|\rho_2|^2\right)(\tau\varphi)^{2s+3}e^{-2\alpha}\,\mathrm{d}Q\bigg).\end{aligned}\tag{2.2}$$

Proof. Let $\rho=(\rho_1,\rho_2)\in\mathcal{W}$. Applying (2.1) to ρ_i, $i=1,2$, and then adding the results, we deduce the following relation

$$\begin{aligned}I(s,\rho_1)+I(s,\rho_2)\leq C\bigg(&\tau\int_Q\left(|L^*\rho_1|^2+|L^*\rho_2|^2\right)\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q\\&+\lambda^4\tau^4\int_{Q_{\omega_0}}\left(|\rho_1|^2+|\rho_2|^2\right)\varphi^{2s+3}e^{-2\alpha}\,\mathrm{d}Q\bigg).\end{aligned}\tag{2.3}$$

The first term on the right-hand side of (2.3) can be estimated as follows

$$\begin{aligned}&\int_Q\left(|L^*\rho_1|^2+|L^*\rho_2|^2\right)\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q\\&\quad\leq 2\int_Q\left(|M(\rho)|^2+|N(\rho)|^2\right)\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q+4\|(a_\mu,b_\mu)\|_\infty^2\int_Q\left(|\rho_1|^2+|\rho_2|^2\right)\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q.\end{aligned}$$

This, together with (2.3), yields

$$\begin{aligned}I(s,\rho_1)+I(s,\rho_2)\leq C\bigg(&2\tau\int_Q\left(|M(\rho)|^2+|N(\rho)|^2\right)\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q\\&+4\tau\|(a_\mu,b_\mu)\|_\infty^2\int_Q\left(|\rho_1|^2+|\rho_2|^2\right)\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q\\&+\lambda^4\tau^4\int_0^T\int_0^A\int_{\omega_0}\left(|\rho_1|^2+|\rho_2|^2\right)\varphi^{2s+3}e^{-2\alpha}\,\mathrm{d}Q\bigg).\end{aligned}\tag{2.4}$$

For $\rho=(\rho_1,\rho_2)$ let us define $I_1(s,\rho)$ and $I_2(s,\rho)$ by

$$\begin{aligned}I_1(s,\rho)&=\tau\int_Q\left|M(\rho)-a_\mu\rho_1-b_\mu\rho_2\right|\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q\\I_2(s,\rho)&=\tau\int_Q\left|M(\rho)-b_\mu\rho_1-a_\mu\rho_2\right|\varphi^{2s}e^{-2\alpha}\,\mathrm{d}Q.\end{aligned}$$

We have

$$I_1(s,\rho) \leq 3\tau \int_Q |M(\rho)|^2 \varphi^{2s} e^{-2\alpha}\,\mathrm{d}Q + 3\tau \int_Q \left(\|a_\mu\|_\infty^2 |\rho_1|^2 + \|b_\mu\|_\infty^2 |\rho_2|^2\right) \varphi^{2s} e^{-2\alpha}\,\mathrm{d}Q,$$

$$I_2(s,\rho) \leq 3\tau \int_Q |N(\rho)|^2 \varphi^{2s} e^{-2\alpha}\,\mathrm{d}Q + 3\tau \int_Q \left(\|b_\mu\|_\infty^2 |\rho_1|^2 + \|a_\mu\|_\infty^2 |\rho_2|^2\right) \varphi^{2s} e^{-2\alpha}\,\mathrm{d}Q.$$

Summing these two above inequalities (after some estimations), from Remark 1 and the relation (2.3), we deduce that

$$\left(\lambda^4\tau^4 - 3\tau C \left(\frac{AT^2}{4}\right)^3 \|(a_\mu, b_\mu)\|_\infty^2\right) \int_Q \left(|\rho_1|^2 + |\rho_2|^2\right) \varphi^{2s+3} e^{-2\alpha}\mathrm{d}Q$$
$$\leq C \left(3\tau \int_Q \left(|M(\rho)|^2 + |N(\rho)|^2\right) \varphi^{2s} e^{-2\alpha}\mathrm{d}Q + \lambda^4\tau^4 \int_G \left(|\rho_1|^2 + |\rho_2|^2\right) \varphi^{2s+3} e^{-2\alpha}\mathrm{d}Q\right).$$

Note that if $\tau \geq \tau_1$ and $\lambda \geq \lambda_0$, then $\left(\frac{AT^2}{4}\right)^3 3C\|(a_\mu, b_\mu)\|_\infty^2 \leq \frac{1}{2}\lambda_0^4\tau^3$ and, consequently, $3\tau C\left(\frac{AT^2}{4}\right)^3 \|(a_\mu, b_\mu)\|_\infty^2 \leq \frac{1}{2}\lambda^4\tau^4$. Therefore, for $\tau \geq \tau_1$ and $\lambda \geq \lambda_0$ we have

$$\lambda^4\tau^4 \int_Q \left(|\rho_1|^2 + |\rho_2|^2\right) \varphi^{2s+3} e^{-2\alpha}\mathrm{d}Q$$
$$\leq 2C\left(3\tau \int_Q \left(|M(\rho)|^2 + |N(\rho)|^2\right) \varphi^{2s} e^{-2\alpha}\mathrm{d}Q + \lambda^4\tau^4 \int_{Q_{\omega_0}} \left(|\rho_1|^2 + |\rho_2|^2\right) \varphi^{2s+3} e^{-2\alpha}\mathrm{d}Q\right).$$

To obtain (2.2) it suffices now to multiply both sides of the above inequality by τ^{2s-1}. $\square$

Let us consider the following adjoint problem associated to (1.8):

$$\begin{cases} \dfrac{\partial \phi_1}{\partial t} + \dfrac{\partial \phi_1}{\partial a} - \Delta\phi_1 + a_\mu\phi_1 + b_\mu\phi_2 = 0 & \text{in } Q, \\ \dfrac{\partial \phi_2}{\partial t} + \dfrac{\partial \phi_2}{\partial t} - \Delta\phi_2 + a_\mu\phi_2 + b_\mu\phi_1 = 0 & \text{in } Q, \\ \phi_1 = \phi_2 = 0 & \text{on } \Sigma, \\ \phi(0,a,x) = \phi(0,a,x) = 0 & \text{in } Q_A, \\ \phi_1(t,0,x) = \displaystyle\int_0^A (a_\beta\phi_1 + b_\beta\phi_2)\,\mathrm{d}a & \text{in } Q_T, \\ \phi_2(t,0,x) = \displaystyle\int_0^A (a_\beta\phi_2 + b_\beta\phi_1)\,\mathrm{d}a & \text{in } Q_T. \end{cases} \tag{2.5}$$

One obtains the following result.

Theorem 3 *Let all the assumptions of Lemma 2 be satisfied. Further, assume that $\tau_1 \geq 1$ and that there exists a constant $b_0 > 0$ and a set ω_b such that*

$$\overline{\omega_b} \subset \omega \text{ and } |b_\mu| \geq b_0 \text{ in } (0,T)\times(0,A)\times\omega_b. \tag{2.6}$$

Then, for all $r \in [0,2)$, there exists a constant $C = C(A, T, \|(a_\mu, b_\mu)\|_\infty, b_0, r)$ such that for all $\phi = (\phi_1, \phi_2) \in \mathcal{W}$ we have:

$$\begin{aligned} &\int_0^T \int_0^A \int_{\omega'} \left(|\phi_1|^2 + |\phi_2|^2\right) e^{-2\alpha} \mathrm{d}Q \\ &\quad \le C\left(\int_G |\phi_1|^2 e^{-r\alpha} \mathrm{d}Q + \int_Q \left(|M(\phi)|^2 + |N(\phi)|^2\right) \varphi^{2s} e^{-2\alpha} \mathrm{d}Q \right) \end{aligned} \tag{2.7}$$

for all ω' such that $\overline{\omega'} \subset \omega_b \subset \omega$.

Proof. The aim is to estimate $\int_{U\times\omega'} |\phi_2|^2 e^{-2\alpha} \mathrm{d}Q$ by $\int_G |\phi_1|^2 e^{-r\alpha} \mathrm{d}Q$ for all $r \in [0,2)$ and for all $\phi = (\phi_1, \phi_2) \in \mathcal{W}$.

Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a function such that

$$\begin{cases} \zeta(x) = 1, & \text{if } x \in \omega', \\ \zeta(x) \in (0,1], & \text{if } x \in \omega'', \\ \zeta(x) = 0, & \text{if } x \in \mathbb{R}^n \setminus \omega'', \end{cases} \tag{2.8}$$

with $\overline{\omega'} \subset \overline{\omega''} \subset \overline{\omega_b} \subset \omega$. Let us suppose that

$$-b_\mu \ \ge \ b_0 \ > 0 \ \text{ in } \ (0,T) \times (0,A) \times \omega_b. \tag{2.9}$$

Set

$$\eta = \zeta^6 \tag{2.10}$$

and for all $\lambda_0, \lambda_1, p, q > 0$ let

$$\Lambda(t,a) = \int_\Omega \left(e^{-p\alpha} \eta^{\frac{4}{3}} |\phi_2|^2 - \lambda_0 e^{-2\alpha} \eta \phi_1 \phi_2 + \lambda_1 e^{-q\alpha} \eta^{\frac{2}{3}} |\phi_1|^2 \right) \mathrm{d}x; \tag{2.11}$$

recall that $\alpha = \tau \frac{e^{\frac{4}{3}\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{at(T-t)}$ in Q. Let us remark that if instead of (2.9) we had $b_\mu \ge b_0 > 0$, then in the expression for Λ one should take $\lambda_0 e^{-2\alpha} \eta \phi_1 \phi_2$ instead of $-\lambda_0 e^{-2\alpha} \eta \phi_1 \phi_2$. From the definition of the function α one deduces that

$$\Lambda(0^+, 0^+) = \Lambda(0^+, a) = \Lambda(t, 0^+) = \Lambda(T^-, a) = 0 \text{ for } (t,a) \in (0,T) \times (0,A). \tag{2.12}$$

So,

$$\begin{aligned} \frac{\partial \Lambda}{\partial t}(t,a) = &\int_\Omega \eta^{\frac{2}{3}} \frac{\partial \alpha}{\partial t} \left(-p\eta^{\frac{2}{3}} |\phi_2|^2 e^{-p\alpha} - q\lambda_1 |\phi_1|^2 e^{-q\alpha} \right) \mathrm{d}x \\ &+ \int_\Omega \eta \frac{\partial \phi_2}{\partial t} \left(2\eta^{\frac{1}{3}} \rho_2 e^{-p\alpha} - \lambda_0 \phi_1 e^{-2\alpha} \right) \mathrm{d}x \\ &+ \int_\Omega \eta^{\frac{2}{3}} \frac{\partial \phi_1}{\partial t} \left(-\lambda_0 \eta^{\frac{1}{3}} \phi_2 e^{-2\alpha} + 2\lambda_1 \phi_1 e^{-q\alpha} \right) \mathrm{d}x \\ &+ 2\lambda_0 \int_\Omega \eta e^{-2\alpha} \frac{\partial \alpha}{\partial t} \phi_1 \phi_2 \, \mathrm{d}x. \end{aligned} \tag{2.13}$$

Let us integrate (2.13) with respect to the variable t over $(0, T)$, and use (2.12). Then, integrating the obtained result with respect to a over $(0, A)$, we get

$$\begin{aligned} 0 = &\int_Q \eta^{\frac{2}{3}} \frac{\partial \alpha}{\partial t} \left(-p\eta^{\frac{2}{3}} |\rho_2|^2 e^{-p\alpha} - q\lambda_1 |\phi_1|^2 e^{-q\alpha} \right) \mathrm{d}Q \\ &+ \int_Q \eta \frac{\partial \phi_2}{\partial t} \left(2\eta^{\frac{1}{3}} \phi_2 e^{-p\alpha} - \lambda_0 \phi_1 e^{-2\alpha} \right) \mathrm{d}Q \\ &+ \int_Q \eta^{\frac{2}{3}} \frac{\partial \phi_1}{\partial t} \left(-\lambda_0 \eta^{\frac{1}{3}} \phi_2 e^{-2\alpha} + 2\lambda_1 \phi_1 e^{-q\alpha} \right) \mathrm{d}Q \\ &+ 2\lambda_0 \int_Q \eta e^{-2\alpha} \frac{\partial \alpha}{\partial t} \phi_1 \phi_2 \mathrm{d}Q. \end{aligned} \tag{2.14}$$

Let us differentiate Λ with respect to a. Using the same approach as before, but this time integrating $\frac{\partial \Lambda}{\partial a}$ first with respect to a over $[0, A]$ and then with respect to t over $[0, T]$, and using (2.12), we obtain

$$\begin{aligned} \int_0^T \Lambda(t, A) \mathrm{d}t = &\int_Q \eta^{\frac{2}{3}} \frac{\partial \alpha}{\partial a} \Big(-p\eta^{\frac{2}{3}} |\phi_2|^2 e^{-p\alpha} - q\lambda_1 |\phi_1|^2 e^{-q\alpha} \Big) \mathrm{d}Q \\ &+ \int_Q \eta \frac{\partial \phi_2}{\partial a} \Big(2\eta^{\frac{1}{3}} \phi_2 e^{-p\alpha} - \lambda_0 \phi_1 e^{-2\alpha} \Big) \mathrm{d}Q \\ &+ \int_Q \eta^{\frac{2}{3}} \frac{\partial \phi_1}{\partial a} \Big(-\lambda_0 \eta^{\frac{1}{3}} \phi_2 e^{-2\alpha} + 2\lambda_1 \phi_1 e^{-q\alpha} \Big) \mathrm{d}Q \\ &+ \lambda_0 \int_Q \eta e^{-2\alpha} \frac{\partial \alpha}{\partial a} \phi_1 \rho_2 \mathrm{d}Q. \end{aligned} \tag{2.15}$$

From (2.5), for $i = 1, 2$, we have $\frac{\partial \phi_i}{\partial a} = \Delta \phi_i - a_\mu \phi_i - b_\mu \phi_j$, where $j \in \{1, 2\}$ and $j \neq i$. Now, let us replace $\frac{\partial \phi_i}{\partial a}$ by $\Delta \phi_i - a_\mu \phi_i - b_\mu \phi_j$ in (2.15) and sum the equalities (2.14) and (2.15) term by term. We obtain

$$-\lambda_0 \int_Q \eta b_\mu e^{-2\alpha} |\rho_2|^2 \mathrm{d}Q = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7, \tag{2.16}$$

where the terms J_i, $i = 1, \ldots, 7$, will be defined (and estimated) below.

Suppose now that for every $r \in [0, 2)$,

$$p > 2,\ q > 1 + \frac{r}{2},\ \lambda_0 \geq 1,\ \lambda_1 \geq 1. \tag{2.17}$$

We are going to estimate the terms J_i, $i = 1, \ldots, 7$. Let us begin with the sum $J_1 + J_2 + J_3$, where

$$J_1 = \int_Q \left(\lambda_1 \eta^{\frac{1}{3}} e^{-(q-r)\alpha} \left\{ -2a_\mu - q\left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\} + \lambda_0 b_\mu \eta^{\frac{2}{3}} e^{-(2-r)\alpha} \right) \eta^{\frac{1}{3}} e^{-r\alpha} |\phi_1|^2 \mathrm{d}Q,$$

$$J_2 = \int_Q \left(\eta^{\frac{1}{3}} e^{-(p-2)\alpha} \left\{ -2a_\mu - p\left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\} \right) \eta e^{-2\alpha} |\phi_2|^2 \mathrm{d}Q,$$

$$J_3 = 2 \int_Q \left(\lambda_0 \eta e^{-2\alpha} \left\{ a_\mu + \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\} - \lambda_1 \eta^{\frac{2}{3}} b_\mu e^{-q\alpha} - b_\mu \eta^{\frac{4}{3}} e^{-p\alpha} \right) \phi_1 \phi_2 \mathrm{d}Q.$$

We have

$$J_3 \leq \int_Q \Bigg(\lambda_0^2 \eta^{\frac{2}{3}} e^{-(2-r)\alpha} \left\{ a_\mu + \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\}^2 + \lambda_1^2 |b_\mu|^2 e^{-[2(q-1)-r]\alpha} + |b_\mu|^2 \eta^{\frac{4}{3}} e^{-[2(p-1)-r]\alpha} \Bigg) \eta^{\frac{1}{3}} e^{-r\alpha} |\phi_1|^2 \mathrm{d}Q + 3 \int_Q \eta e^{-2\alpha} |\phi_2|^2 \, \mathrm{d}Q.$$

Thus,

$$\begin{aligned} J_1 + J_2 + J_3 \leq \int_Q \Bigg(& \lambda_0^2 \eta^{\frac{2}{3}} e^{-(2-r)\alpha} \left\{ a_\mu + \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\}^2 + \lambda_1^2 \left|b_\mu\right|^2 e^{-[2(q-1)-r]\alpha} \\ & + \left|b_\mu\right|^2 \eta^{\frac{4}{3}} e^{-[2(p-1)-r]\alpha} + \lambda_1 \eta^{\frac{1}{3}} e^{-(q-r)\alpha} \left\{ -2a_\mu - q \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\} \\ & + \lambda_0 b_\mu \eta^{\frac{2}{3}} e^{-(2-r)\alpha} \Bigg) \eta^{\frac{1}{3}} e^{-r\alpha} \left|\phi_1\right|^2 \mathrm{d}Q \\ & + \int_Q \left(3 + \eta^{\frac{1}{3}} e^{-(p-2)\alpha} \left\{ -2a_\mu - p \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \right\} \right) \eta e^{-2\alpha} \left|\phi_2\right|^2 dQ. \end{aligned}$$

The function $\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a}$ is not bounded in U. From (2.12) and from the definitions of α and η, one deduces that $\eta^j \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) e^{-\ell \alpha} \in L^\infty(Q)$ for all $j > 0$ and $\ell > 0$, and thus, in particular, for $\ell \in \left\{ (2-r), 2(q-1)-r, 2(p-1)-r, q-r, p-2, 2 \right\}$. Applying Holder's formula to the previous inequality, we get

$$\begin{aligned} & J_1 + J_2 + J_3 \\ & \leq \Bigg[2\lambda_0^2 \left(\|a_\mu\|_\infty^2 \times \left\| \eta^{\frac{1}{3}} e^{-(1-\frac{r}{2})\alpha} \right\|_\infty^2 + \left\| \eta^{\frac{1}{3}} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) e^{-(1-\frac{r}{2})\alpha} \right\|_\infty^2 \right) \\ & + \lambda_1^2 \left(\|b_\mu\|_\infty^2 + \left\| \eta^{\frac{1}{6}} e^{-[(q-1)-r]\alpha} \right\|_\infty^2 \right) + \|b_\mu\|_\infty^2 + \left\| \eta^{\frac{2}{3}} e^{-[p-1-\frac{r}{2}]\alpha} \right\|_\infty^2 \\ & + \lambda_1 \left(\|a_\mu\|_\infty^2 + \frac{1}{2} \left\| \eta^{\frac{1}{3}} e^{-(q-r)\alpha} \right\|_\infty^2 + q \left\| \eta^{\frac{1}{3}} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) e^{-(q-r)\alpha} \right\|_\infty \right) \\ & + \frac{1}{2} \lambda_0 \left(\|b_\mu\|_\infty^2 + \left\| \eta^{\frac{2}{3}} e^{-(2-r)\alpha} \right\|_\infty^2 \right) \Bigg] \int_Q \eta^{\frac{1}{3}} e^{-r\alpha} |\rho_1|^2 \, \mathrm{d}Q \\ & + \left(3 + \|a_\mu\|_\infty^2 + \frac{1}{2} \left\| \eta^{\frac{1}{3}} e^{-(p-2)\alpha} \right\|_\infty^2 + p \left\| \eta^{\frac{1}{3}} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) \eta^{\frac{1}{3}} e^{-(p-2)\alpha} \right\|_\infty \right) \int_Q \eta e^{-2\alpha} |\rho_2|^2 \, \mathrm{d}Q. \end{aligned}$$

From the definition of η and from (2.17), one deduces that there exists a positive constant $C = C\left(p, q, r, ||\eta||_\infty\right)$ such that

$$\begin{aligned} & J_1 + J_2 + J_3 \\ & \leq C \Bigg\{ \Bigg[\lambda_0^2 \left(1 + \|a_\mu\|_\infty^2 + \|b_\mu\|_\infty^2 + \left\| \eta^{\frac{1}{3}} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) e^{-(1-\frac{r}{2})\alpha} \right\|_\infty^2 \right) \\ & \quad + \lambda_1^2 \left(1 + \left\|a_\mu\right\|_\infty^2 + 2\|b_\mu\|_\infty^2 + q \left\| \eta^{\frac{1}{3}} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) e^{-(q-r)\alpha} \right\|_\infty \right) \Bigg] \\ & \quad \times \int_Q \eta^{\frac{1}{3}} e^{-r\alpha} |\rho_1|^2 \, \mathrm{d}Q + \left(1 + \|a_\mu\|_\infty^2 + p \left\| \eta^{\frac{1}{3}} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a} \right) e^{-(p-2)\alpha} \right\|_\infty \right) \\ & \quad \times \int_Q \eta e^{-2\alpha} \left|\rho_2\right|^2 \mathrm{d}Q \Bigg\}. \end{aligned} \tag{2.18}$$

To conclude this estimation, we need to study the functions $\left(\frac{\partial\alpha}{\partial t}+\frac{\partial\alpha}{\partial a}\right)e^{-\ell\alpha}$ in U for $\ell>0$. For all $\ell\in\big\{(2-r),2(q-1)-r,2(p-1)-r,q-r,p-2,2\big\}$ we get the following inequalities

$$\left|\frac{\partial\alpha}{\partial a}e^{-\ell\alpha}\right|\leq\tau\frac{e^{\frac{4}{3}\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}}{t(T-t)}\times F_t,\quad\left|\frac{\partial\alpha}{\partial t}e^{-\ell\alpha}\right|\leq\frac{\tau(T-2t)}{a}\big(e^{\frac{4}{3}\|\psi\|_\infty}-e^{\lambda\psi(x)}\big)\times F_a,$$

where the functions F_a and F_t are defined respectively on (0,T) and $(0,A]$ by

$$F_a(t)=\frac{1}{t^2(T-t)^2}e^{-\ell\kappa_1\frac{1}{t(T-t)}}\quad\text{and}\quad F_t(a)=\frac{1}{a^2}e^{-\ell\kappa_2\frac{1}{a}}$$

with

$$\kappa_1(a,x)=\tau\frac{\big(e^{\frac{4}{3}\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}\big)}{a}\text{ in }Q_A,\quad\kappa_2(t,x)=\tau\frac{\big(e^{\frac{4}{3}\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}\big)}{t(T-t)}\text{ in }Q_T.$$

We must study the functions F_a and F_t on $(0,T)$ and $(0,A]$, respectively. Set $z_1=\frac{1}{t(T-t)}$ and $z_2=\frac{1}{a}$. Then, $z_2\geq\frac{1}{A}$. Note that $z_1=\frac{1}{t(T-t)}$ is equivalent to $z_1t^2-(z_1T)t+1=0$. The above equation admits a solution if $z_1\geq\frac{4}{T^2}$. Consequently, the study of functions F_a and F_t, respectively, on $(0,T)$ and $(0,A)$ can be reduced to the study of functions $F_1(z)=z^2e^{-\ell\kappa_1 z}$ and $F_2(z)=z^2e^{-\ell\kappa_2 z}$, respectively, on $[\frac{4}{T^2},+\infty)$ and $[\frac{1}{A},+\infty)$. The function $F(z)=z^2e^{-\ell\kappa z}$ $z\geq\max(\frac{4}{T^2},\frac{1}{A})$ is decreasing on $[\frac{2}{\ell\kappa},+\infty)$ and increasing on $\big[\max\big(\frac{4}{T^2},\frac{1}{A}\big),\frac{2}{\ell\kappa}\big]$. Thus, in particular, if $\frac{2}{\ell\kappa_1}\geq\frac{4}{T^2}$, then $F_1(z)\leq F_1\big(\frac{2}{\ell\kappa_1}\big)=\big(\frac{2e^{-1}}{\ell\kappa_1}\big)^2$. Consequently,

$$\left|\frac{\partial\alpha}{\partial t}e^{-\ell\alpha}\right|\leq\frac{12AT^2e^{-2}}{\tau\ell^2\big(e^{\frac{4}{3}\lambda\|\psi\|_\infty}-e^{\lambda\|\psi(x)\|_\infty}\big)}.$$

So, there exists a constant $C_1>0$, which depends on $\ell,\|\psi\|_\infty,A,\lambda$ and T, such that

$$\left|\frac{\partial\alpha}{\partial t}e^{-\ell\alpha}\right|\;\leq\;C_1\,.\tag{2.19}$$

If $\frac{2}{\ell\kappa_1}\leq\frac{4}{T^2}$, the function F_1 is decreasing on $[\frac{4}{T^2},+\infty)$, and so $F_1(z)\leq F_1(\frac{4}{T^2})$. Thus,

$$\left|\frac{\partial\alpha}{\partial t}e^{-\ell\alpha}\right|\leq\frac{68\tau\kappa_1}{T^2}e^{-\frac{4\ell\kappa_1}{T^2}}\leq\frac{17B}{a}e^{-\frac{B}{a}}\quad\text{with}\quad B=\frac{4\ell\tau}{T^2}\big(e^{\frac{4}{3}\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}\big).$$

The function $a\mapsto\frac{1}{a}e^{-\dfrac{B}{a}}$ admits a maximum at B and the maximal value is $\frac{1}{B}e^{-1}$. Therefore, there exists a positive constant C_2, which depends on $\ell,\|\psi\|_\infty,\tau,\lambda,A$ and T, such that

$$\left|\frac{\partial\alpha}{\partial t}e^{-\ell\alpha}\right|\leq C_2.\tag{2.20}$$

From (2.19) and (2.20) for $C_3=\min\left(C_1,C_2\right)$, one obtains

$$\left|\frac{\partial\alpha}{\partial t}e^{-\ell\alpha}\right|\leq C_3.\tag{2.21}$$

Now, we come back to the function F_2. For $\frac{2}{\ell\kappa_2}\geq\frac{1}{A}$, the function F_2 is decreasing on $[\frac{2}{\ell\kappa_2},+\infty)$ and increasing on $[\frac{1}{A},\frac{2}{\ell\kappa_2}]$. So, $F_2(z)\leq F_2(\frac{2}{\ell\kappa_2})$ and $\left|\frac{\partial\alpha}{\partial a}e^{-\ell\alpha}\right|\leq\tau\frac{e^{\frac{4}{3}\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}}{t(T-t)}\times F_2(\frac{2}{\ell\kappa_2})$.

Proceeding in the same way as in the case when $\frac{2}{\ell\kappa_1} \geq \frac{4}{T^2}$, one can prove that there exists a positive constant C_4, which depends on $\ell, \|\psi\|_\infty, \tau, \lambda, A$ and T, such that

$$\left|\frac{\partial\alpha}{\partial a}e^{-\ell\alpha}\right| \leq C_4\,. \tag{2.22}$$

For $\frac{2}{\ell\kappa_2} \leq \frac{1}{A} \leq z$, the function F_2 is decreasing $[\frac{1}{A}, +\infty)$. So, $F_2(z) \leq F_2(A)$ and one deduces that

$$\left|\frac{\partial\alpha}{\partial a}e^{-\ell\alpha}\right| \leq \frac{\kappa_2}{A^2}e^{-\frac{\ell\kappa_2}{A}} \leq \frac{D}{t(T-t)}e^{-\frac{\ell D}{\tau A}\frac{1}{t(T-t)}}.$$

Setting $z = \frac{1}{t(T-t)}$, we consider the function $F(z) = ze^{-\frac{\ell D}{\tau A}z}$ defined on $[\frac{4}{T^4}, +\infty)$. One deduces from the above estimates done for the function F_1, that there exists a constant $C_5 = C(\ell, \|\psi\|_\infty, \tau, \lambda, A, T) > 0$, and for $C_6 = \min(C_4, C_5)$ one has

$$\left|\frac{\partial\alpha}{\partial a}e^{-\ell\alpha}\right| \leq C_6. \tag{2.23}$$

From (2.21) and (2.23), one deduces that there exists C such that

$$\left|\left(\frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial a}\right)e^{-\ell\alpha}\right| \leq C. \tag{2.24}$$

Consequently, there exists a constant $\mathcal{C} = C(p, q, r, \| \eta \|_\infty, \lambda_m^2)$, where $\lambda_m^2 = \lambda_1^2 + \lambda_0^2$, such that the relation (2.18) becomes

$$\begin{aligned} J_1 + J_2 + J_3 \leq \mathcal{C}\Bigg\{ &\left(1 + \|(a_\mu, b_\mu)\|_\infty^2 + \frac{\tau^4}{T^4}\right)\int_Q \eta^{\frac{1}{3}}e^{-r\alpha}|\rho_1|^2\,\mathrm{d}Q \\ &+ \left(1 + \|a_\mu\|_\infty^2 + \frac{\tau^2}{T^2}\right)\int_Q \eta e^{-2\alpha}|\rho_2|^2\,\mathrm{d}Q\Bigg\}. \end{aligned} \tag{2.25}$$

Now, we will estimate

$$\begin{aligned} J_4 &= -\lambda_0\int_Q \eta e^{-2\alpha}\left(\phi_1\Delta\phi_2 + \phi_2\Delta\phi_1\right)\,\mathrm{d}Q, \\ J_5 &= 2\lambda_1\int_Q \eta^{\frac{2}{3}}e^{-q\alpha}\phi_1\Delta\phi_1\,\mathrm{d}Q, \\ J_6 &= 2\int_Q \eta^{\frac{4}{3}}e^{-p\alpha}\phi_2\Delta\phi_2\,\mathrm{d}Q. \end{aligned}$$

Similar methods to those used in [12, 15], give us the following inequalities

$$\left\|\eta^{-\frac{2}{3}}e^{(1+\frac{r}{2})\alpha}\Delta\left(\eta e^{-2\alpha}\right)\right\|_\infty \leq C \tag{2.26}$$

and

$$\begin{aligned} J_4 \leq \lambda_0^2 C&\left(1 + \frac{\tau^4}{T^4}\right)\int_Q \eta^{\frac{1}{3}}e^{-r\alpha}|\rho_1|^2\mathrm{d}Q \\ &+ \frac{1}{2}\int_Q \eta e^{-2\alpha}|\rho_2|\mathrm{d}Q - 2\lambda_0\int_Q \eta e^{-2\alpha}\nabla\rho_1.\nabla\rho_2\mathrm{d}Q, \end{aligned} \tag{2.27}$$

$$J_5 \leq \lambda_1 C\left(1+\frac{\tau^2}{T^2}\right)\int_Q \eta^{\frac{1}{3}} e^{-r\alpha}|\rho_1|^2\,\mathrm{d}Q - 2\lambda_1\int_Q \eta^{\frac{2}{3}} e^{-q\alpha}|\nabla\rho_1|^2\,\mathrm{d}Q, \tag{2.28}$$

$$J_6 \leq C\left(1+\frac{\tau^2}{T^2}\right)\int_Q \eta e^{-2\alpha}|\rho_2|^2\,\mathrm{d}Q - 2\int_Q \eta^{\frac{4}{3}} e^{-p\alpha}|\nabla\rho_2|^2\,\mathrm{d}Q. \tag{2.29}$$

Finally, let us estimate $J_7 = -\int_0^T \Lambda(t,A)\,\mathrm{d}t$. Moreover, suppose that

$$p+q\leq 4 \quad \text{and} \quad \lambda_0^2 \leq 2\lambda_1. \tag{2.30}$$

One deduces that

$$J_7 \leq -\frac{1}{2}\int_0^T\int_\Omega e^{-p\alpha}\eta^{\frac{4}{3}}|\rho_2(t,A,x)|^2\,\mathrm{d}Q_T \leq 0. \tag{2.31}$$

From relations (2.16), (2.25), (2.27), (2.28), (2.29), (2.31) and (2.9), one obtains

$$\lambda_0 b_0\int_{U\times\omega_b}\eta e^{-2\alpha}|\rho_2|^2\,\mathrm{d}Q \leq -\lambda_0\int_Q \eta b_\mu e^{-2\alpha}|\rho_2|^2\mathrm{d}Q \leq J_1+J_2+J_3+J_4+J_5+J_6.$$

Now, on the left-hand side of the inequality, we will regroup all integral terms that contain $\eta e^{-2\alpha}|\rho_2|^2$; one has

$$\begin{aligned}
&\left[b_0\lambda_0 - C(p,q,r,\|\eta\|_\infty,\lambda_m)\left(1+\|a_\mu\|_\infty^2+\frac{\tau^2}{T^2}\right) - C\left(1+\frac{\tau^2}{T^2}\right)-\frac{1}{2}\right]\int_Q \eta|\rho_2|^2 e^{-2\alpha}\mathrm{d}Q\\
&\leq \left[C(p,q,r,\|\eta\|_\infty,\lambda_m)\left(1+\|(a_\mu,b_\mu)\|_\infty^2+\frac{\tau^4}{T^4}\right)+\lambda_0^2 C\left(1+\frac{\tau^4}{T^4}\right)+\lambda_1 C\left(1+\frac{\tau^2}{T^2}\right)\right]\\
&\times\int_Q \eta^{\frac{1}{3}}e^{-r\alpha}|\rho_1|^2\mathrm{d}Q - 2\int_Q\left(\eta^{\frac{4}{3}}e^{-q\alpha}|\nabla\rho_2|^2 - \lambda_0\eta e^{-2\alpha}\nabla\rho_2.\nabla\rho_1+\lambda_1\eta^{\frac{2}{3}}e^{-q\alpha}|\nabla\rho_1|^2\right)\mathrm{d}Q.
\end{aligned}$$

The inequality (2.30) shows that the latest term on the left-hand side of the above inequality is negative. So, choosing b_0 so that

$$b_0 > \frac{1}{\lambda_0}\left[-C\left(p,q,r,\|\eta\|_\infty,\lambda_m\right)\left(1+\|a_\mu\|_\infty^2+\frac{\tau^2}{T^2}\right)-C\left(1+\frac{\tau^2}{T^2}\right)-\frac{1}{2}\right],$$

we see that there exists a positive constant $C = C(p,q,r,\|\eta\|_\infty,\lambda_m,T)$ such that

$$\int_Q \eta|\rho_2|^2 e^{-2\alpha}\,\mathrm{d}Q \leq C\int_Q \eta^{\frac{1}{3}}e^{-r\alpha}|\rho_1|^2\,\mathrm{d}Q.$$

From the properties of the function η, we deduce that

$$\int_{\omega'}|\rho_2|^2 e^{-2\alpha}\,\mathrm{d}Q \leq C\int_G e^{-r\alpha}|\rho_1|^2\,\mathrm{d}Q. \tag{2.32}$$

The relation (2.32) allows to deduce (2.7). □

Thanks to (2.2), we can write the following result.

Corollary 1 *Under the hypothesis of Theorem 3, for all $r\in[0,2)$ there exists a positive constant $C = C(A,T,\|(a_\mu,b_\mu)\|_\infty,c_0,r)$ such that for all $\rho=(\rho_1,\rho_2)$ belonging to the space $\mathcal{W}(U)$ we have*

$$\int_Q(|\rho_1|^2+|\rho_2|^2)e^{-2\alpha}\,\mathrm{d}Q \leq C\left(\int_G|\rho_1|^2\,e^{-r\alpha}\,\mathrm{d}Q+\int_Q(|M(\rho)|^2+|N(\rho)|^2)e^{-2\alpha}\,\mathrm{d}Q\right). \tag{2.33}$$

Setting $\theta = e^{\alpha}$ and $\delta = e^{(1-\frac{\tau}{2})\alpha}$, we obtain the following proposition.

Proposition 1 *Under the hypothesis of Theorem* 3*, for all* $\rho = (\rho_1, \rho_2) \in \mathcal{W}(U)$ *there exists a positive constant* C *such that*

$$\int_Q \frac{1}{\theta^2} \left(|\rho_1|^2 + |\rho_2|^2\right) \, \mathrm{d}Q \leq C \left(\int_Q \left(|M(\rho)|^2 + |N(\rho)|^2\right) \, \mathrm{d}Q + \int_G \delta^2 |\rho_1|^2 \, \mathrm{d}Q \right). \tag{2.34}$$

Lemma 3 *Under the hypothesis of Proposition* 1*, there exists a positive constant* C *such that for all* $\rho = (\rho_1, \rho_2) \in \mathcal{W}(U)$ *we have*

$$\begin{aligned}
&\int_0^T \int_\Omega \left(|\rho_1(t,0,x)|^2 + |\rho_2(t,0,x)|^2\right) \, \mathrm{d}Q_T \\
&\quad + \int_{Q_A} \left(|\rho_1(0,a,x)|^2 + |\rho_2(0,a,x)|^2\right) \mathrm{d}Q_A + \int_Q \frac{1}{\theta^2} \left(|\rho_1|^2 + |\rho_2|^2\right) dQ \\
&\quad \leq C \left(\int_Q \left(|M(\rho)|^2 + |N(\rho)|^2\right) \mathrm{d}Q + \int_G \delta^2 \, |\rho_1|^2 \, \mathrm{d}Q \right).
\end{aligned} \tag{2.35}$$

Proof. In order to proceed as in [15], let us consider the set $J_{TA} = [\frac{T}{4}, \frac{3T}{4}] \times [\frac{A}{4}, A]$. Then, α is bounded on $J_{TA} \times \Omega$ and there exists a constant C such that

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\frac{A}{4}}^{A} \int_\Omega \left(\frac{1}{\theta} |\rho_i| \right)^2 \, \mathrm{d}Q \geq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\frac{A}{4}}^{A} \int_\Omega |\rho_i|^2 \, \mathrm{d}Q, \quad i = 1, 2. \tag{2.36}$$

From (2.36) and Proposition 1, we deduce that

$$\int_{J_{TA} \times \Omega} \frac{1}{\theta^2} \left(|\rho_1| + |\rho_2|\right)^2 \, \mathrm{d}Q \leq C \left(\int_Q \left(|M(\rho)|^2 + |N(\rho)|^2\right) \mathrm{d}Q + \int_G \delta^2 |\rho_1|^2 \, \mathrm{d}Q \right). \tag{2.37}$$

Let us consider the function $\Upsilon \in C^\infty(U)$ such that

$$\begin{cases}
\Upsilon(t,a) = 1, & \text{if } (t,a) \in [0, \frac{T}{4}) \times [0, A) \\
\Upsilon(t,a) \in (0,1), & \text{if } (t,a) \in [\frac{T}{4}, \frac{3T}{4}] \times [\frac{A}{4}, A), \\
\Upsilon(t,a) = 0, & \text{if } (t,a) \in [\frac{T}{4}, T] \times (0, \frac{A}{4}] \cup (\frac{3T}{4}, T] \times [\frac{A}{4}, A].
\end{cases} \tag{2.38}$$

For $\rho_i \in \mathcal{V}$, $i = 1, 2$, and $p \in \mathbb{R}$ let us set $\xi(t,a,x) = \Upsilon(t,a) e^{-p(a+t)} \rho_i(t,a,x)$. Then, $\xi_i|_{\sum} = 0$, $\xi_i(T,a,x) = \xi_i(t,A,x) = 0$, $\xi_i(t,0,x) = e^{-pt}\rho_i(t,0,x)$ and $\xi_i(0,a,x) = e^{-pa}\rho_i(0,a,x)$. One obtains

$$L(\xi_1) + (a_\mu - 2p)\xi_1 + b_\mu \xi_2 = \left(-\frac{\partial \Upsilon}{\partial t} - \frac{\partial \Upsilon}{\partial a} \right) e^{-p(a+t)} \rho_1 + \Upsilon e^{-p(a+t)}. \tag{2.39}$$

Let us multiply (2.39) by ξ_1 and integrate by parts over Q. We obtain

$$\begin{aligned}
&\frac{1}{2} \int_0^T \int_\Omega |\xi_1(t,0,x)|^2 \, \mathrm{d}t \, \mathrm{d}x + \frac{1}{2} \int_0^A \int_\Omega |\xi_1(0,a,x)|^2 \, \mathrm{d}a \, \mathrm{d}x + \int_\Omega |\nabla \xi_1|^2 \, \mathrm{d}x \\
&\quad + \int_Q (a_\mu - 2p) \, |\xi_1|^2 \, \mathrm{d}Q + \int_Q b_\mu \xi_1 \xi_2 \, \mathrm{d}Q = - \int_Q \left(\frac{\partial \Upsilon}{\partial t} + \frac{\partial \Upsilon}{\partial a} \right) e^{-p(a+t)} \rho_1 \xi_1 \, \mathrm{d}Q.
\end{aligned} \tag{2.40}$$

In the same way, we get

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega |\xi_2(t,0,x)|^2\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{2}\int_0^A\int_\Omega |\xi_2(0,a,x)|^2\,\mathrm{d}a\,\mathrm{d}x + \int_\Omega |\nabla\xi_2|^2\,\mathrm{d}x\\ + \int_Q (b_\mu - 2p)|\xi_2|^2\,\mathrm{d}Q \int_Q a_\mu\xi_1\xi_2\,\mathrm{d}Q = -\int_Q \left(\frac{\partial\Upsilon}{\partial t} + \frac{\partial\Upsilon}{\partial a}\right) e^{-p(a+t)}\rho_2\xi_2\,\mathrm{d}Q.\end{aligned} \tag{2.41}$$

Summing (2.40) and (2.41), one deduces that

$$\begin{aligned}\frac{1}{2}\int_{Q_T}\left(|\xi_1(t,0,x)|^2 + |\xi_2(t,0,x)|^2\right)\mathrm{d}Q_T + \frac{1}{2}\int_{Q_A}\left(|\xi_1(0,a,x)|^2 + |\xi_2(0,a,x)|^2\right)\,\mathrm{d}Q_A\\ + \int_\Omega\left(|\nabla\xi_1|^2 + |\nabla\xi_2|^2\right)\,\mathrm{d}x + \int_Q (a_\mu - 2p)\,|\xi_1|^2\,\mathrm{d}Q + \int_Q (b_\mu - 2p)\,|\xi_2|^2\,\mathrm{d}Q\\ \le \int_Q\left|\frac{\partial\Upsilon}{\partial t} + \frac{\partial\Upsilon}{\partial a}\right|\left(\rho_1^2 + \rho_2^2\right)e^{-2p(a+t)}\,\mathrm{d}Q - \int_Q (a_\mu + b_\mu)\,\xi_1\xi_2\,\mathrm{d}Q.\end{aligned} \tag{2.42}$$

According to Young's inequality, there exist $k_1, k_2 \in \mathbb{R}_+^*$ such that

$$-\int_Q (a_\mu + b_\mu)\,\xi_1\xi_2\,\mathrm{d}Q \le \frac{1}{2k_1}\int_Q \left|(a_\mu + b_\mu)\,\xi_1\right|^2\mathrm{d}Q + \frac{k_2}{2}\int_Q \left|\xi_2\right|^2\mathrm{d}Q. \tag{2.43}$$

From the definition of the function Υ, we see that $\frac{\partial\Upsilon}{\partial t} \neq 0$, $\frac{\partial\Upsilon}{\partial a} \neq 0$ in $[\frac{T}{4}, \frac{3T}{4}] \times [\frac{A}{2}, A]$ and, moreover, that the function $\left|\frac{\partial\Upsilon}{\partial t} + \frac{\partial\Upsilon}{\partial a}\right|$ is bounded. There exists a positive constant C such that

$$\int_Q\left|\frac{\partial\Upsilon}{\partial t} + \frac{\partial\Upsilon}{\partial a}\right|\left(|\rho_1|^2 + |\rho_2|^2\right)e^{-2p(a+t)}\,\mathrm{d}Q \le \int_{(J_{TA})\times\Omega}\left(|\rho_1|^2 + |\rho_2|^2\right)\,\mathrm{d}Q. \tag{2.44}$$

Using the inequalities (2.43)–(2.44), from the inequality (2.42), one deduces that

$$\begin{aligned}\frac{1}{2}\int_{Q_T}\left(|\xi_1(t,0,x)|^2 + |\xi_2(t,0,x)|^2\right)\,\mathrm{d}Q_T + \frac{1}{2}\int_{Q_A}\left(|\xi_1(0,a,x)|^2 + \left|\xi_2(0,a,x)\right|^2\right)\,\mathrm{d}Q_A\\ + \int_Q\left(a_\mu - 2p - \frac{\|(a_\mu,b_\mu)\|_\infty^2}{2k_1}\right)\left|\xi_1\right|^2\mathrm{d}Q + \int_Q\left(b_\mu - 2p - \frac{k_2}{2}\right)|\xi_2|^2\mathrm{d}Q\\ \le C\int_{(J_{TA})\times\Omega}\left(|\rho_1|^2 + \left|\rho_2\right|^2\right)\mathrm{d}Q.\end{aligned}$$

For $p \in \mathbb{R}$ such that $p < \frac{1}{2}a_\mu - \frac{\|(a_\mu,b_\mu)\|_\infty^2}{4k_1}$ and $p < \frac{1}{2}b_\mu - \frac{k_2}{4}$, one obtains

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega\left(|\rho_1(t,0,x)|^2 + |\rho_2(t,0,x)|^2\right)e^{-2pt}\,\mathrm{d}Q_T\\ + \frac{1}{2}\int_{Q_A}\left(|\rho_1(0,a,x)|^2 + |\rho_2(0,a,x)|^2\right)e^{-2pt}\,\mathrm{d}Q_A \le C\int_{(J_{TA})\times\Omega}\left(|\rho_1|^2 + |\rho_2|^2\right)\,\mathrm{d}Q.\end{aligned}$$

The function $t \mapsto e^{-2pt}$ is bounded, respectively, on $[0,T]$ and $[0,A]$, and so there exists a constant C' such that

$$\begin{aligned}\int_0^T\int_\Omega\left(|\rho_1(t,0,x)|^2 + |\rho_2(t,0,x)|^2\right)\,\mathrm{d}Q_T\\ + \frac{1}{2}\int_{Q_A}\left(|\rho_1(0,a,x)|^2 + |\rho_2(0,a,x)|^2\right)\,\mathrm{d}Q_A \le C'\int_{(J_{TA})\times\Omega}\left(|\rho_1|^2 + |\rho_2|^2\right)\,\mathrm{d}Q.\end{aligned} \tag{2.45}$$

From (2.33), (2.37) and (2.45) we get (2.35). □

3 Existence of the control

In this section we solve the problem (1.8)–(1.10). Let us consider the auxiliary problem: for $\widehat{f} \in L^2(Q), a_\mu, b_\mu \in L^\infty(Q)$ find $((\widehat{p}_1, \widehat{p}_2), \widehat{k})$ in $\mathcal{W}(U) \times L^2(G)$ such that

$$\begin{cases} -\dfrac{\partial \widehat{p}_1}{\partial t} - \dfrac{\partial \widehat{p}_1}{\partial a} - \Delta \widehat{p}_1 + a_\mu \widehat{p}_1 + b_\mu \widehat{p}_2 = \widehat{f} + \widehat{k}\chi_\omega & \text{in } Q, \\ -\dfrac{\partial \widehat{p}_2}{\partial t} - \dfrac{\partial \widehat{p}_2}{\partial a} - \Delta \widehat{p}_2 + a_\mu \widehat{p}_2 + b_\mu \widehat{p}_1 = 0 & \text{in } Q, \\ \widehat{p}_1 = \widehat{p}_2 = 0 & \text{on } \Sigma, \\ \widehat{p}_1(T, a, x) = \widehat{p}_2(T, a, x) = 0 & \text{in } Q_A, \\ \widehat{p}_1(0, a, x) = \widehat{p}_2(0, a, x) = 0 & \text{in } Q_A, \\ \widehat{p}_1(t, A, x) = \widehat{p}_2(t, A, x) = 0 & \text{in } Q_T. \end{cases} \tag{3.1}$$

The adjoint problem associated to (3.1) reads as follows:

$$\begin{cases} \dfrac{\partial \phi_1}{\partial t} + \dfrac{\partial \phi_1}{\partial a} - \Delta \phi_1 + a_\mu \phi_1 + b_\mu \phi_2 = 0 & \text{in } Q, \\ \dfrac{\partial \phi_2}{\partial t} + \dfrac{\partial \phi_2}{\partial a} - \Delta \phi_2 + a_\mu \phi_2 + b_\mu \phi_1 = 0 & \text{in } Q, \\ \phi_1 = \phi_2 = 0 & \text{on } \Sigma. \end{cases} \tag{3.2}$$

The problems (2.5) and (3.2) give the same Carleman's inequalities.

For all $\rho = (\rho_1, \rho_2) \in \mathcal{W}(U)$ and $\sigma = (\sigma_1, \sigma_2) \in \mathcal{W}(U)$ let us consider the bilinear form

$$\mathcal{B}_1(\rho, \sigma) := \int_Q M(\rho) M(\sigma) \, \mathrm{d}Q + \int_Q N(\rho) N(\sigma) \, \mathrm{d}Q + \int_G \delta^2 \rho_1 \sigma_1 \, \mathrm{d}Q$$

and the linear form

$$\mathcal{L}(\rho) := \int_Q f \rho_1 \, \mathrm{d}Q.$$

One deduces from (2.35) that $\mathcal{B}_1$ is a scalar product on $\mathcal{W}$. Let $\mathcal{H}$ be the completion of the space $\mathcal{W}$ with respect to the norm $\|\rho\|_{\mathcal{H}} := \sqrt{\mathcal{B}_1(\rho, \rho)}$. Then, $\mathcal{H} = \overline{\mathcal{W}(U)}$ is a Hilbert space.

Lemma 4 *Let $\widehat{\theta}$ be a weight function such that $\widehat{\theta}\widehat{f} \in L^2(Q)$. Then, for any $\widehat{\rho} = (\widehat{\rho}_1, \widehat{\rho}_2) \in W(U)$ we have*

$$M(\widehat{\rho})(t, 0, x), N(\widehat{\rho})(t, 0, x) \in L^2(Q_T), \tag{3.3}$$

$$\|M(\widehat{\rho})\|_{L^2(Q)} \le C \|\widehat{f}\|_{L^2(Q)}, \tag{3.4}$$

$$\|N(\widehat{\rho})\|_{L^2(Q)} \le C \|\widehat{f}\|_{L^2(Q)}. \tag{3.5}$$

Proof. Let us consider the problem

$$\mathcal{B}_1(\rho, \sigma) = \mathcal{L}(\sigma). \tag{3.6}$$

According to (2.34), $\mathcal{L}$ is continuous on $\mathcal{W}$. Since the bilinear form B_1 is a scalar product on $\mathcal{W}$, we deduce from the Lax–Milgram theorem that there exists a unique function $\widehat{\rho} \in \mathcal{H}$ which solves the problem $\mathcal{B}_1(\widehat{\rho},\sigma)=\mathcal{L}(\sigma)$ for all $\sigma\in\mathcal{W}$. Set $\widehat{p}_1=M(\widehat{\rho})$, $\widehat{p}_2=N(\widehat{\rho})$, and $\widehat{k}=-\widehat{\delta}\widehat{\rho}_1\chi_\omega$. As $\widehat{\rho}\in\mathcal{H}$, we have $\widehat{p}_1,\widehat{p}_2\in\mathcal{W}(U)$ and $\widehat{k}\in L^2(Q)$. Let us replace in (3.6), $M(\widehat{\rho})$ by $\widehat{p}_1$, $N(\widehat{\rho})$ by $\widehat{p}_2$ and $-\delta\widehat{\rho}_1\chi_\omega$ by $\widehat{k}$. For all $\sigma\in\mathcal{H}$, after integrating by parts over Q, we have

$$\begin{aligned}
&\int_Q\big[\big(M^*(\widehat{p}_1)-\widehat{f}-\widehat{\delta}\chi_\omega\big)\big]\sigma_1\,\mathrm{d}Q+\int_Q\left[N^*(\widehat{p}_1)\right]\sigma_2\,\mathrm{d}Q\\
&\quad+\int_{Q_T}\left[\widehat{p}_1(t,A,x)\sigma_1(t,A,x)-\widehat{p}_1(t,0,x)\sigma_1(t,0,x)\right]\mathrm{d}Q_T\\
&\quad+\int_{Q_A}\left[\widehat{p}_1(T,a,x)\sigma_1(T,a,x)-\widehat{p}_1(0,a,x)\sigma_1(0,a,x)\right]\mathrm{d}Q_A\\
&\quad+\int_{Q_T}\left[\widehat{p}_2(t,A,x)\sigma_2(t,A,x)-\widehat{p}_2(t,0,x)\sigma_2(t,0,x)\right]\mathrm{d}Q_T\\
&\quad+\int_{Q_A}\left[\widehat{p}_2(T,a,x)\sigma_2(T,a,x)-\widehat{p}_2(0,a,x)\sigma_2(0,a,x)\right]\mathrm{d}Q_A=0.
\end{aligned}\tag{3.7}$$

In particular, for $\sigma=(\sigma_1,0)\in\mathcal{D}(Q)\times\mathcal{D}(Q)$ and $\sigma=(0,\sigma_2)\in\mathcal{D}(Q)\times\mathcal{D}(Q)$, respectively, from (3.7) one deduces that

$$-\frac{\partial\widehat{p}_1}{\partial t}-\frac{\partial\widehat{p}_1}{\partial a}-\Delta\widehat{p}_1+a_\mu\widehat{p}_1+b_\mu\widehat{p}_2=\widehat{f}+\widehat{\delta}\widehat{k}\chi_\omega\ \text{ in } Q \tag{3.8}$$

and

$$-\frac{\partial\widehat{p}_2}{\partial t}-\frac{\partial\widehat{p}_2}{\partial a}-\Delta\widehat{p}_2+a_\mu\widehat{p}_2+b_\mu\widehat{p}_1=0\ \text{ in } Q. \tag{3.9}$$

Since $\widehat{p}_1,\widehat{p}_2\in L^2(Q)$, we see that $\frac{\partial\widehat{p}_1}{\partial t},\frac{\partial\widehat{p}_2}{\partial a}\in H^{-1}(U;L^2(\Omega))$. Using (3.8)–(3.9), one has $-\Delta\widehat{p}_1,-\Delta\widehat{p}_2\in H^{-1}(U;L^2(Q))$. We also have the existence of restrictions $\widehat{p}_{1|\Sigma},\widehat{p}_{2|\Sigma}$ which belong to $H^{-1}(U;H^{-\frac{1}{2}}(\Gamma))$ and the existences of the restrictions $\frac{\partial\widehat{p}_1}{\partial\nu}|_\Sigma$, $\frac{\partial\widehat{p}_2}{\partial\nu}|_\Sigma$ which belong to $H^{-1}(U;H^{-\frac{3}{2}}(\Gamma))$. It follows that $\Delta\widehat{p}_1,\Delta\widehat{p}_2\in L^2(U;H^{-2}(\Omega))$. So, from (3.8)–(3.9), we get

$$\frac{\partial\widehat{p}_i}{\partial t},\frac{\partial\widehat{p}_i}{\partial a}\in L^2(U;H^{-2}(\Omega)),\ i=1,2.$$

Then, the functions $(t,a)\mapsto\widehat{p}_i(t,a,x)$, $i=1,2$, with values in $H^{-1}(\Omega)$ are continuous on U. So, $\widehat{p}_i(T,a,x),\widehat{p}_i(t,A,x),\widehat{p}_i(t,0,x),\widehat{p}_i(0,a,x)$, $i=1,2$, exist in $H^{-1}(\Omega)$. Let $\sigma_1,\sigma_2\in C^\infty(\overline{Q})$. Multiply, respectively, (3.8) and (3.9) by σ_1 and σ_2 and integrate by parts over Q. One deduces from (3.6) that for $\sigma_i|_\Sigma=0$, $i=1,2$, it is:

$$\begin{aligned}
&-\big\langle\widehat{p}_i(t,A,\cdot),\sigma_i(t,A,\cdot)\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\big\langle\widehat{p}_i(t,0,\cdot),\sigma_i(t,0,\cdot)\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\
&\quad-\big\langle\widehat{p}_i(T,a,\cdot),\sigma_i(T,a,\cdot)\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\big\langle p_i(0,a,\cdot),\sigma_i(0,a,\cdot)\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\
&\quad-\Big\langle\widehat{p}_i,\frac{\partial\sigma_i}{\partial\nu}\Big\rangle_{H^{-1}((U;H^{-1}(\Gamma^{-\frac{1}{2}})),H^1(U,H^1(\Gamma^{\frac{1}{2}}))}=0.
\end{aligned}\tag{3.10}$$

It follows that

$$\widehat{p}_i|_\Sigma=0, i=1,2, \tag{3.11}$$

$$\widehat{p}_i(T,a,x)=0, i=1,2,\ \text{ in } Q_A, \tag{3.12}$$

$$\widehat{p}_i(t,A,x)=0, i=1,2, \text{ in } Q_T, \tag{3.13}$$

$$\widehat{p}_i(0,a,x)=0, i=1,2, \text{ in } Q_A. \tag{3.14}$$

From the relations (3.8)–(3.9) and (3.11)–(3.14), one deduces that $(\widehat{\delta k},\widehat{p}_1,\widehat{p}_2)$ satisfies the auxiliary problem.

Since $\mathcal{B}_1(\widehat{\rho},\widehat{\rho})=\|\widehat{p}_1\|^2_{L^2(Q)}+\|\widehat{p}_2\|_{L^2(Q)}\|+\|\widehat{k}\|_{L^2(Q)}=\|\widehat{\rho}\|^2_{\mathcal{W}}$, from

$$\mathcal{B}_1(\widehat{\rho},\widehat{\rho})=\int_Q \widehat{f}\widehat{\rho}_1 \,\mathrm{d}Q\leq C\|\widehat{\theta}\widehat{f}\|_{L^2(Q)}\Big\|\frac{1}{\widehat{\theta}}\widehat{\rho}_1\Big\|_{\mathcal{W}}$$

and from Proposition 1, one deduces that

$$\|\widehat{\rho}\|_{\mathcal{W}}\leq C\|\widehat{\theta}\widehat{f}\|_{L^2(Q)}, \tag{3.15}$$

$$\|\widehat{p}_1\|_{\mathcal{W}}\leq C\|\widehat{\theta}\widehat{f}\|_{L^2(Q)}, \tag{3.16}$$

$$\|\widehat{p}_2\|_{\mathcal{W}}\leq C\|\widehat{\theta}\widehat{f}\|_{L^2(Q)}, \tag{3.17}$$

$$\|\widehat{k}\|\leq \|\widehat{\theta}\widehat{f}\|_{L^2(Q)}. \tag{3.18}$$

From (3.16) and (3.17), we obtain (3.4)–(3.5).

Multiply the first and second equation of (3.1) by $\widehat{p}_1$ and $\widehat{p}_2$, respectively. After integrating by parts over Q, we get

$$\frac{1}{2}\int_{Q_T}|\widehat{p}_1(t,0,x)|^2\,\mathrm{d}Q_T+\int_Q(|\nabla\widehat{p}_1|^2+a_\mu\,|\widehat{p}_1|^2)\,\mathrm{d}Q+\int_Q b_\mu\widehat{p}_1\widehat{p}_2\,\mathrm{d}Q=\int_Q(\widehat{f}+\widehat{\delta k})\widehat{p}_1\,\mathrm{d}Q, \tag{3.19}$$

$$\frac{1}{2}\int_{Q_T}|\widehat{p}_2(t,0,x)|^2\,\mathrm{d}Q_T+\int_Q(|\nabla\widehat{p}_2|^2+a_\mu\,|\widehat{p}_2|^2)\,\mathrm{d}Q+\int_Q b_\mu\widehat{p}_1\widehat{p}_2\,\mathrm{d}Q=0. \tag{3.20}$$

Applying Young's inequality to $\int_Q b_\mu\widehat{p}_1\widehat{p}_2\,\mathrm{d}Q$, we deduce from (3.15), (3.16) and (3.17), (3.18) the following inequalities

$$\frac{1}{2}\int_{Q_T}|\widehat{p}_1(t,0,x)|^2\,\mathrm{d}Q_T\leq C\|\widehat{\theta}\widehat{f}\|^2_{L^2(Q)}, \tag{3.21}$$

$$\frac{1}{2}\int_{Q_T}|\widehat{p}_2(t,0,x)|^2\,\mathrm{d}Q_T\leq C\|\widehat{\theta}\widehat{f}\|^2_{L^2(Q)}. \tag{3.22}$$

That gives (3.3). □

Lemma 5 *Assume that $a_\beta, b_\beta\in L^\infty(Q)$, $f\in L^2(Q)$ are such that $\theta a_\beta,\theta b_\beta\in L^\infty(Q)$, $\theta f\in L^2(Q)$. For all $\rho=(\rho_1,\rho_2),\sigma=(\sigma_1,\sigma_2)\in\mathcal{H}$ we define the bilinear form*

$$\begin{aligned}\mathcal{B}(\rho,\sigma)=\mathcal{B}_1(\rho,\sigma)&-\int_Q[a_\beta M(\rho)(t,0,x)+b_\beta N(\rho)(t,0,x)]\,\sigma_1(t,a,x)\,\mathrm{d}Q\\&-\int_Q[a_\beta N(\rho)(t,0,x)+b_\beta M(\rho)(t,0,x)]\,\sigma_2(t,a,x)\,\mathrm{d}Q.\end{aligned} \tag{3.23}$$

Then, there exists a unique $\rho_\theta=(\rho_{1_\theta},\rho_{2_\theta})\in\mathcal{H}$ such that for all $\sigma\in\mathcal{H}$ we have

$$\mathcal{B}(\rho_\theta,\sigma)=\mathcal{L}(\sigma). \tag{3.24}$$

Proof. As $\mathcal{B}$ is a continuous and coercive bilinear form, and $\mathcal{L}$ a continuous linear form, according to the Lax–Milgram theorem, there exists a unique $\rho_\theta = (\rho_{1_\theta}, \rho_{2_\theta}) \in \mathcal{H}$ such that $\mathcal{B}(\rho_\theta, \sigma) = \mathcal{L}(\sigma)$ for all $\sigma \in \mathcal{H}$. This ends the proof of Lemma 5. □

Proposition 2 *Assume that $a_\beta, b_\beta \in L^\infty(Q)$, $f \in L^2(Q)$ are such that $\theta a_\beta, \theta b_\beta \in L^2(Q)$, $\theta f \in L^2(Q)$. Let $\rho_\theta = (\rho_{1_\theta}, \rho_{2_\theta})$ be the unique solution of* (3.24). *Let us set*

$$k_\theta = -\delta \rho_{1_\theta} \chi_\omega, \tag{3.25}$$

$$p_{1_\theta} = M(\rho_\theta), \tag{3.26}$$

$$p_{2_\theta} = N(\rho_\theta). \tag{3.27}$$

Then, $(\delta k_\theta, (p_{1_\theta}, p_{2_\theta}))$ is a solution of the null controllability problem (1.8)–(1.10). *Moreover, there exists a constant $C = C(A, T, \|(a_\mu, b_\mu)\|_\infty, \|(\theta a_\beta, \theta b_\beta)\|_\infty, c_0, r) > 0$ such that*

$$\|\rho_\theta\|_{\mathcal{H}} \leq C\big(\|\theta f\|_{L^2(Q)} + \|p_{1_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)} + \|p_{2_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)}\big), \tag{3.28}$$

$$\|p_{1_\theta}\|_{L^2(Q)} \leq C\big(\|\theta f\|_{L^2(Q)} + \|p_{1_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)} + \|p_{2_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)}\big), \tag{3.29}$$

$$\|p_{2_\theta}\|_{L^2(Q)} \leq C\big(\|\theta f\|_{L^2(Q)} + \|p_{1_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)} + \|p_{2_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)}\big), \tag{3.30}$$

$$\|k_\theta\|_{L^2(G)} \leq C\big(\|\theta f\|_{L^2(Q)} + \|p_{1_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)} + \|p_{2_\theta}(\cdot,0,\cdot)\|_{L^2(Q_T)}\big). \tag{3.31}$$

Proof. Since $\rho_\theta \in \mathcal{H}$, we have $p_{1_\theta}, p_{2_\theta} \in L^2(Q)$, $p_{1_\theta}(\cdot,0,\cdot), p_{2_\theta}(\cdot,0,\cdot) \in L^2(Q_T)$ (see Lemma 4) and $k_\theta \in L^2(G)$. Let us replace $M(\rho_\theta)$ by $p_{1\theta}$, $N(\rho_\theta)$ by $p_{2\theta}$ and $-\delta\rho_{1\theta}\chi_\omega$ by k_θ in (3.24). Let $\sigma \in \mathcal{H}$. Proceeding in the same way as in the proof of Lemma 4, from (3.8)–(3.9) and (3.10)–(3.14), one deduces that $(\delta k_\theta, (p_{1_\theta}, p_{2_\theta}))$ satisfies (1.8)–(1.10). From (3.15)–(3.18) we obtain the inequalities (3.28)–(3.31). □

Proposition 3 *Under the hypothesis of Lemma 2, there exists a unique control $\widehat{k}$ such that*

$$\|\widehat{k}\| = \min_{\tilde{k} \in \vartheta} \|\tilde{k}\|, \tag{3.32}$$

where $\vartheta = \big\{\tilde{k} \in L^2(G) : (\tilde{k}, \tilde{\rho})$ satisfies (1.8)–(1.10)$\big\}$.

Proof. The set ϑ is non empty according to Proposition 2. Furthermore, ϑ is a closed and convex subset of $L^2(G)$. So, there exists a unique control in ϑ which is of minimal norm in $L^2(G)$. □

4 Proof of the main theorem

This section is entirely devoted to the demonstration of Theorem 1. The proof will be done in three steps.

Step 1. For a positive real number ε, let $\mathcal{A}$ be the set of all pairs (k, ρ), where $k \in L^2(G)$ and $\rho = (\rho_1, \rho_2) \in L^2(Q)^2$ which satisfy the following conditions

$$M(\rho) - a_\beta \rho_1(t,0,x) - b_\beta \rho_2(t,0,x) \in L^2(Q),$$
$$N(\rho) - a_\beta \rho_2(t,0,x) - b_\beta \rho_1(t,0,x) \in L^2(Q),$$
$$\rho_{i|\Sigma} = 0,\ i = 1,2,$$
$$\rho_i(0,a,x) = 0 \text{ in } Q_A,\ i = 1,2,$$
$$\rho_i(T,a,x) = 0 \text{ in } Q_A,\ i = 1,2,$$
$$\rho_i(t,A,x) = 0 \text{ in } Q_T,\ i = 1,2,$$

and for all $(k,\rho) \in \mathcal{A}$ we define the functional

$$\begin{aligned} J_\varepsilon(k,\rho) = \frac{1}{2}\|k\|^2_{L^2(G)} &+ \frac{1}{2\varepsilon}\|M(\rho) - a_\beta \rho_1(\cdot,0,\cdot) - b_\beta \rho_2(\cdot,0,\cdot) - f - k\chi_\omega\|^2_{L^2(Q)} \\ &+ \frac{1}{2\varepsilon}\|N(\rho) - a_\beta \rho_2(\cdot,0,\cdot) - b_\beta \rho_1(\cdot,0,\cdot)\|^2_{L^2(Q)}. \end{aligned} \tag{4.1}$$

Let us consider the following optimal control problem

$$\inf\{J_\varepsilon(k,\rho) : (k,\rho) \in \mathcal{A}\}. \tag{4.2}$$

Note that the set $\mathcal{A}$ is non-empty because $(\delta k_\theta, \rho_\theta) \in \mathcal{A}$ according to Proposition 3. As $\mathcal{A}$ is a non-empty closed and convex subset of $L^2(G) \times L^2(Q)^2$ and J_ε is strictly convex, lower semi-continuous and coercive on $\mathcal{A}$, there exists a unique $(k_\varepsilon, \rho_\varepsilon)$ that solves (4.2).

Step 2. The goal is to give the system of optimality that characterizes the solution of the problem of minimization. According to the Euler–Lagrange optimality conditions, we have

$$\frac{dJ_\varepsilon}{d\lambda}(k_\varepsilon + \lambda k, \rho_\varepsilon) = 0 \quad \text{for all } k \in L^2(G)$$

and

$$\frac{dJ_\varepsilon}{d\lambda}(k_\varepsilon, \rho_\varepsilon + \lambda) = 0 \quad \text{for all } \rho \in L^2(Q)^2,$$

with $(k,\rho) \in \mathcal{A}$ and $(k_\varepsilon, \rho_\varepsilon)$ being a solution of (4.2). Namely, for all k, ρ such that $(k,\rho) \in \mathcal{A}$, we have

$$\int_Q [k_\varepsilon - \tfrac{1}{\varepsilon}\left(M(\rho) - a_\beta \rho_1(\cdot,0,\cdot) - b_\beta \rho_2(\cdot,0,\cdot) - f - k\chi_\omega\right)] k\chi_\omega \,\mathrm{d}Q = 0, \tag{4.3}$$

$$\begin{aligned} &\int_Q \eta_{1_\varepsilon}\big(M(\rho) - a_\beta \rho_1(\cdot,0,\cdot) - b_\beta \rho_2(\cdot,0,\cdot)\big)\,\mathrm{d}Q \\ &\quad + \int_Q \eta_{2_\varepsilon}\big(N(\rho) - a_\beta \rho_2(\cdot,0,\cdot) - b_\beta \rho_1(\cdot,0,\cdot)\big)\,\mathrm{d}Q = 0, \end{aligned} \tag{4.4}$$

with

$$\eta_{1_\varepsilon} = \frac{1}{\varepsilon}\big(M(\rho_\varepsilon) - a_\beta \rho_{1\varepsilon}(\cdot,0,\cdot) - b_\beta \rho_{2\varepsilon}(\cdot,0,\cdot) - f - k_\varepsilon \chi_\omega\big), \tag{4.5}$$

$$\eta_{2_\varepsilon} = \frac{1}{\varepsilon}\big(N(\rho_\varepsilon) - a_\beta \rho_{2\varepsilon}(\cdot,0,\cdot) - b_\beta \rho_{1\varepsilon}(\cdot,0,\cdot)\big), \tag{4.6}$$

$\eta_{1_\varepsilon}\eta_{2_\varepsilon} \in L^2(Q)$. Thus, similar reasoning to the one used in [12, 15] leads to the following conclusion: $(k_\varepsilon, \rho_\varepsilon)$ is a solution of the problem of optimisation if and only if there exists a function $\eta_\varepsilon = (\eta_{1_\varepsilon}, \eta_{1_\varepsilon})$ such that $(k_\varepsilon, \rho_\varepsilon, \eta_\varepsilon)$ satisfies

$$k_\varepsilon = \eta_{1_\varepsilon}\chi_\omega \tag{4.7}$$

and

$$\begin{cases} M(\rho_\varepsilon) = a_\beta \rho_{1\varepsilon}(\cdot,0,\cdot) + b_\beta \rho_{2\varepsilon}(\cdot,0,\cdot) + f + k_\varepsilon \chi_\omega + \varepsilon \eta_{1_\varepsilon} & \text{in } Q, \\ N(\rho_\varepsilon) = a_\beta \rho_{2\varepsilon}(\cdot,0,\cdot) + b_\beta \rho_{1\varepsilon}(\cdot,0,\cdot) + \varepsilon \eta_{2_\varepsilon} & \text{in } Q, \\ \rho_{1\varepsilon} = \rho_{2\varepsilon} = 0 & \text{on } \Sigma, \\ \rho_{1\varepsilon}(0,a,x) = \rho_{2\varepsilon}(0,a,x) = 0 & \text{in } Q_A, \\ \rho_{1\varepsilon}(T,a,x) = \rho_{2\varepsilon}(T,a,x) = 0 & \text{in } Q_A, \\ \rho_{1\varepsilon}(t,A,x) = \rho_{2\varepsilon}(t,A,x) = 0 & \text{on } Q_T, \end{cases} \tag{4.8}$$

as well as

$$\begin{cases} M(\eta_\varepsilon) = 0 & \text{in } Q, \\ N(\eta_\varepsilon) = 0 & \text{in } Q, \\ \eta_{1_\varepsilon} = \eta_{2_\varepsilon} = 0 & \text{in } \Sigma, \\ \eta_{1_\varepsilon}(t,0,x) = \displaystyle\int_0^A (a_\beta \eta_{1_\varepsilon} + b_\beta \eta_{2_\varepsilon})\, \mathrm{d}a & \text{in } Q_T, \\ \eta_{2_\varepsilon}(t,0,x) = \displaystyle\int_0^A (a_\beta \eta_{2_\varepsilon} + b_\beta \eta_{1_\varepsilon})\, \mathrm{d}a & \text{in } Q_T. \end{cases} \tag{4.9}$$

Step 3. We will demonstrate the uniqueness of the solution for the problem of optimisation. That will complete the proof of the main theorem. From (3.32) and Proposition 2, one deduces that

$$\|k_\varepsilon\|_{L^2(G)} \leq C\|\delta k_\theta\|_{L^2(G)}, \tag{4.10}$$

$$\|M(\rho_\varepsilon) - a_\beta \rho_{1_\varepsilon}(\cdot,0,\cdot) - b_\beta \rho_{2_\varepsilon}(\cdot,0,\cdot) - f - k_\varepsilon \chi_\omega\|_{L^2(Q)} \leq C\sqrt{\varepsilon}\|\delta k_\theta\|_{L^2(G)}, \tag{4.11}$$

$$\|N(\rho_\varepsilon) - a_\beta \rho_{2_\varepsilon}(\cdot,0,\cdot) - b_\beta \rho_{1_\varepsilon}(\cdot,0,\cdot)\|_{L^2(Q)} \leq C\sqrt{\varepsilon}\|\delta k_\theta\|_{L^2(G)}. \tag{4.12}$$

So,

$$\|\rho_\varepsilon\|_{L^2(U,H_0^1(\Omega))} \leq C, \tag{4.13}$$

$$\|\eta_{1_\varepsilon}\chi_\omega\|_{L^2(Q)} \leq C\|\delta k_\theta\|_{L^2(G)}, \tag{4.14}$$

and η_ε is a solution of (4.9), then

$$\|\eta_\varepsilon\|_{\mathcal{H}} \leq C\|\delta k_\theta\|_{L^2(G)}, \tag{4.15}$$

according to Proposition 1 and we obtain that

$$\left\|\frac{1}{\theta}\eta_{1_\varepsilon}\right\|_{L^2(Q)} \leq C\|\delta \eta_{1_\varepsilon}\|_{L^2(G)}. \tag{4.16}$$

From (4.14), one obtains

$$\|\eta_{1_\varepsilon}\|_{L^2(G)} \leq C\|\delta \eta_{1_\theta}\|_{L^2(Q)}\,. \tag{4.17}$$

We extract from the sequences $(k_\varepsilon)_\varepsilon, (\rho_\varepsilon)_\varepsilon, (\eta_\varepsilon)_\varepsilon$ subsequences, denoted again by $(k_\varepsilon)_\varepsilon, (\rho_\varepsilon)_\varepsilon, (\eta_\varepsilon)_\varepsilon$, such that

$$k_\varepsilon \rightharpoonup \tilde{k} \text{ weakly in } L^2(G), \tag{4.18}$$

$$\rho_{i_\varepsilon} \rightharpoonup \tilde{\rho}_i \text{ weakly in } L^2(U, H_0^1(\Omega)),\ i = 1, 2, \tag{4.19}$$

$$\eta_{i_\varepsilon} \rightharpoonup \tilde{\eta}_i \text{ weakly in } \mathcal{V},\ i = 1,2, \tag{4.20}$$

$$\eta_{1_\varepsilon} \rightharpoonup \tilde{\eta}_1 \text{ weakly in } L^2(G). \tag{4.21}$$

One deduces from (4.21) that $\eta_{1_\varepsilon}\chi_\omega \rightharpoonup \tilde{\eta}_1$ weakly in $L^2(G)$. From the compactness of the injection of $L^2(U; H_0^2(\Omega))$ into $L^2(Q)$, one deduces that $(\tilde{k}, (\tilde{\rho}_1, \tilde{\rho}_2))$ satisfies the simultaneous null controllability problem (1.8)–(1.10). Consequently, $\tilde{k} \in \vartheta$. According to Proposition 3, for the only one $\widehat{k} \in \vartheta$ satysfying (3.32), we have

$$\|\widehat{k}\|_{L^2(G)} \leq \|\tilde{k}\|_{L^2(G)}. \tag{4.22}$$

Let $\widehat{\rho} = (\widehat{\rho}_1, \widehat{\rho}_2)$ be the solution of (1.8) associated to $\widehat{k}$. Since $(k_\varepsilon, \rho_\varepsilon)$ satisfies (4.1), then

$$\frac{1}{2}\|k_\varepsilon\|^2_{L^2(G)} \leq J_\varepsilon\left(k_\varepsilon, \rho_\varepsilon\right) \leq J_\varepsilon(\widehat{k}, \widehat{\rho}) = \frac{1}{2}\|\widehat{k}\|^2_{L^2(G)}.$$

On the other hand,

$$\frac{1}{2}\|\tilde{k}\|^2_{L^2(G)} \leq \liminf_{\varepsilon\to 0} \frac{1}{2}\|k_\varepsilon\|^2_{L^2(G)} = \frac{1}{2}\|\widehat{k}\|_{L^2(G)}.$$

Consequently,

$$\widehat{k} = \tilde{k}.$$

Finally, $\tilde{k}$ satisfies (4.7) with $\tilde{\eta}_1$ instead of η_{1_ε}, and thanks to (4.9), (4.18)–(4.21), it follows that $\tilde{\eta}$ satisfies (1.12). This ends the proof of the main theorem.

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