Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2019, Number 7, pp. 115–133 (August 2020) http://www.jneea.com

NONLINEAR PARABOLIC EQUATIONS WITH SOFT MEASURE DATA

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Received on July 20, 2018, revised version on May 17, 2019

Accepted on June 30, 2019

Communicated by Alexander Pankov

Abstract. In this paper we prove the existence and uniqueness of renormalized solutions for nonlinear parabolic problems with Dirichlet boundary values whose model is

$$\begin{cases} b(u)_t - \Delta_p u = \mu & \text{in } (0,T)\times\Omega, \\ b(u(0,x)) = b(u_0) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplace operator, b is an increasing C^1-function and μ is a finite measure which does not charge sets of zero parabolic p-capacity. Furthermore, we discuss main properties of such solutions.

Keywords: A priori estimates, equi-diffuse measure, parabolic p-capacity, porous media equation, renormalized solutions.

2010 Mathematics Subject Classification: 65J15, 28A12, 35B45, 35A35, 35Q35.

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1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$ ($N \geq 2$), let T be a positive real number, $p > 1$, and let us consider the model problem

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \Delta_p u = \mu & \text{in } (0,T)\times\Omega, \\ b(u) = b(u_0) & \text{on } \{0\}\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.1}$$

where u_0 is a measurable function such that $b(u_0) \in L^1(\Omega)$ and μ is a bounded Radon measure on $Q = (0,T)\times\Omega$. It is well-known that if $b(u) = u$, $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, J.-L. Lions [18] proved the existence and uniqueness of a weak solution to (1.1). Under the general assumptions that μ and u_0 are bounded measures, the existence of a distributional solution was proved in [4], by approximating (1.1) with problems having regular data and using compactness arguments. Due to the lack of regularity of the solutions, the distributional formulation is not strong enough to provide uniqueness, as it can be proved by adapting the counterexample of J. Serrin to the parabolic case. However, for nonlinear operators with L^1-data, a new concept of solutions was studied in [5] and in [24] (see also [12]), where the notions of a renormalized solution and an entropy solution, respectively, were introduced. If μ is a measure that does not charge sets of zero parabolic p-capacity (that is, the so-called diffuse measures), the notion of a renormalized solution was introduced in [17]. In [16] a similar notion of an entropy solution was also defined, and proved to be equivalent to the renormalized one. The case in which b is a strictly increasing C^1-function and Δ_p is a $p-$Laplace operator (*i.e.*, (1.1)) was studied in [10] with μ being a diffuse measure (see also [20] for the case when μ is general). All these latest results are strongly based on a decomposition theorem given in [17], the key point in the existence result being the proof of the strong compactness of suitable truncates of the approximating solutions in the energy space. Recently, in [22] (see also [23]) the authors proposed a new approach to the same problem with diffuse measures as data. This approach avoids using the particular structure of the decomposition of the measure and it seems more flexible to handle a fairly general class of problems. In order to do that, the authors introduced a definition of a renormalized solution which is closer to the one used for conservation laws in [3] and to one of the existing formulations in the elliptic case (see [14, 15]). Our goal is to extend the approach from [23] to the framework of the so-called standard porous medium equation of the type $v_t - \Delta_p\psi(v)$ with $\psi(v) = u$ and $\psi^{-1} = b$; here ψ is a strictly increasing function.

The paper is organized as follows. In Section 2 we give some preliminaries on the notion of parabolic p-capacity and on the functional spaces, and some basic notations and properties. Section 3 is devoted to setting the main assumptions and the new renormalized formulation of problem (1.1). In Section 4, we prove that the definition of a renormalized solution does not depend on the classical decomposition of μ. In Section 5 we give the proof of the main result (Theorem 5.1). We will briefly sketch in Section 6 the proof of the uniqueness of the solution.

2 Preliminaries on parabolic capacity

Given a bounded open set $\Omega \subset \mathbb{R}^N$ and $T > 0$, let $Q = (0,T)\times\Omega$. We recall that for every $p > 1$ and every open subset $U \subset Q$, the parabolic p-capacity of U (see [17, 19] and [22]) is given by

$$\text{cap}_p(U) = \inf\big\{\|u\|_W : u \in W,\ u \geq \chi_U \text{ a.e. in } Q\big\}, \tag{2.1}$$

where

$$W = \{u \in L^p(0,T;V) : u_t \in L^{p'}(0,T;V')\}. \tag{2.2}$$

Let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ is endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and that V' is its dual space. As usual W is endowed with the norm

$$\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}.$$

Moreover, we set $\inf \emptyset = +\infty$. The parabolic capacity cap_p can be then extended to an arbitrary Borel subset B of Q putting

$$\mathrm{cap}_p(B) = \inf\{\mathrm{cap}_p(U) : B \subset U \text{ and } U \subset Q \text{ is open}\}.$$

We denote by $\mathcal{M}_b(Q)$ the set of all Radon measures with bounded variation on Q equipped with the norm $\|\mu\|_{\mathcal{M}_b(Q)} = |\mu|(Q)$. We call a measure μ diffuse if $\mu(E) = 0$ for every Borel set $E \subset Q$ such that $\mathrm{cap}_p(E) = 0$. By $\mathcal{M}_0(Q)$ we will denote the subspace of all diffuse measures in Q. Diffuse measures play an important role in the study of boundary value problems with measures as source terms. Indeed, for such measures one expects to obtain counterparts (in some generalized framework) of the existence and uniqueness results known in the variational setting. Properties of diffuse measures in connection with the resolution of nonlinear parabolic problems have been investigated in [17]. In that paper, the authors proved that for every $\mu \in \mathcal{M}_0(Q)$ there exists $f \in L^1(Q)$, $g \in L^p(0,T;V)$ and $\chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ such that

$$\mu = f + g_t + \chi \text{ in } \mathcal{D}'(Q). \tag{2.3}$$

Note that the decomposition in (2.3) is not uniquely determined and the presence of the term g_t is essentially due to the presence of the diffuse measure which charges sections of the parabolic cylinder Q and gives some extra difficulties in the study of this type of problems; in particular, the parabolic case with absorption term $h(u)$. The main reason is that a solution of

$$u_t - \Delta_p u + h(u) = \mu = f + \chi + g_t \text{ in } Q$$

is meant in the sense that $v = u - g$ satisfies

$$v_t - \Delta_p(v+g) + h(v+g) = f + \chi \text{ in } Q.$$

However, since no growth restriction is made on h, the proof is a hard technical issue if g is not bounded. For further considerations on this fact we refer to [8] (see also [6, 22]) and the references therein.

In [22], the authors also proved the following approximation theorem for an arbitrary diffuse measure that is essentially independent on the decomposition of the measure data.

Theorem 2.1 *Let $\mu \in \mathcal{M}_0(Q)$. Then, for every $\epsilon > 0$ there exists $\nu \in \mathcal{M}_0(Q)$ such that*

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \epsilon \text{ and } \nu = w_t - \Delta_p w \text{ in } \mathcal{D}'(Q), \tag{2.4}$$

where $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Note that the function w is constructed as the truncation of a nonlinear potential of μ.

To prove the existence of a solution we will use a density argument, so we need the following preliminary result whose proof can be found, for instance, in [22] (see also Appendix).

Proposition 2.2 *Given $\mu \in \mathcal{M}(Q) \cap L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$, let $u \in W$ be the (unique) weak solution of*

$$\begin{cases} u_t - \Delta_p u = \mu & \textit{in } Q, \\ u = u_0 & \textit{on } \{0\} \times \Omega, \\ u = 0 & \textit{on } (0,T) \times \partial\Omega. \end{cases} \tag{2.5}$$

Then,

$$\operatorname{cap}_p(\{|u| > k\}) \leq C \max\left\{ \frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}} \right\} \textit{ for every } k \geq 1, \tag{2.6}$$

where $C > 0$ is a constant depending on $\|\mu\|_{\mathcal{M}(Q)}$, $\|u_0\|_{L^1(\Omega)}$ and p.

Note that the proof of the corresponding proposition in our case is postponed to Section 7.

Definition 2.3 *A sequence of measures (μ_n) in Q is equi-diffuse, if for every $\eta > 0$ there exists $\delta > 0$ such that*

$$\operatorname{cap}_p(E) < \delta \Longrightarrow |\mu_n|(E) < \eta \quad \textit{for every } n \geq 1.$$

The following result is proved in [23].

Lemma 2.4 *Let (ρ_n) be a sequence of mollifiers on Q. If $\mu \in \mathcal{M}_0(Q)$, then the sequence $(\rho_n * \mu_n)$ is equi-diffuse.*

Here is some notation we will use throughout the paper. We consider a sequence of mollifiers (ρ_n) such that for any $n \geq 1$,

$$\rho_n \in C_c^\infty(\mathbb{R}^{N+1}),\ \operatorname{Supp} \rho_n \subset B_{\frac{1}{n}}(0),\ \rho_n \geq 0 \text{ and } \int_{\mathbb{R}^{N+1}} \rho_n = 1. \tag{2.7}$$

Given $\mu \in \mathcal{M}(Q)$, we define μ_n as a convolution $\rho_n * \mu$ for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ by

$$\mu_n(t,x) = (\rho_n * \mu)(t,x) = \int_Q \rho_n(t-s, x-y)\, \mathrm{d}\mu(s,y). \tag{2.8}$$

For any non-negative real number, we denote by $T_k(r) = \min\{k, \max\{r, -k\}\}$ the truncation function at level k. For every $r \in \mathbb{R}$, let $\overline{T}_k(z) = \int_0^z T_k(s)\, \mathrm{d}s$. Finally by $\langle \cdot, \cdot \rangle$ we mean the duality between suitable spaces in which functions are involved. In particular, we will consider both the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and the duality between $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $W^{-1,p'}(\Omega) + L^1(Q)$. And we denote by $\omega(h, n, \delta, \cdots)$ any quantity that vanishes as the parameters go to their limit point.

3 Main assumptions and renormalized formulation

Let us state our basic assumptions. Let Ω be a bounded, open subset of $\mathbb{R}^N$, T a positive number and $Q = (0,T) \times \Omega$. We will actually consider a larger class of problems involving Leray–Lions type operators of the form $-\operatorname{div}(a(t,x,\nabla u))$ (the same argument as above still holds for more general nonlinear operators – see [7]), and the nonlinear parabolic problem

$$\begin{cases} b(u)_t - \operatorname{div}(a(t,x,\nabla u)) = \mu & \text{in } (0,T) \times \Omega, \\ b(u) = b(u_0) & \text{on } \{0\} \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.1}$$

where $a\colon (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (*i.e.*, $a(\cdot,\cdot,\zeta)$ is measurable on Q for every ζ in $\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t,x) in Q) such that the following assumptions hold:

$$a(t,x,\zeta) \cdot \zeta \geq \alpha |\zeta|^p, \quad p > 1, \tag{3.2}$$

$$|a(t,x,\zeta)| \leq \beta [L(x,t) + |\zeta|^{p-1}], \tag{3.3}$$

$$[a(t,x,\zeta) - a(t,x,\eta)] \cdot (\zeta - \eta) > 0, \tag{3.4}$$

for almost every (t,x) in Q, for every ζ, η in $\mathbb{R}^N$ with $\zeta \neq \eta$; here α and β are two positive constants, and L is a non-negative function in $L^{p'}(Q)$.

In all the following, we assume that $b\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function which satisfies

$$b(0) = 0 \text{ and } 0 < b_0 \leq b'(s) \leq b_1 \text{ for every } s \in \mathbb{R}, \tag{3.5}$$

$$u_0 \text{ is a measurable function in } \Omega \text{ such that } b(u_0) \in L^1(\Omega), \tag{3.6}$$

and that μ is a diffuse measure, *i.e.*,

$$\mu \in \mathcal{M}_0(Q). \tag{3.7}$$

Let us give the notion of a renormalized solution for parabolic problem (3.1) using a different formulation. We recall that the following definition is the natural extension of the one given in [10] for diffuse measures.

Definition 3.1 *Let $\mu \in \mathcal{M}_0(Q)$. A measurable function u defined on Q is a renormalized solution of problem* (3.1) *if $T_k(b(u)) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$, and if there exists a sequence (λ_k) in $\mathcal{M}(Q)$ such that*

$$\lim_{k\to\infty} \|\lambda_k\|_{\mathcal{M}(Q)} = 0, \tag{3.8}$$

and

$$\begin{aligned} &-\int_Q T_k(b(u))\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \int_Q \varphi \,\mathrm{d}\mu + \int_Q \varphi\,\mathrm{d}\lambda_k + \int_\Omega T_k(b(u_0))\varphi(0,x)\,\mathrm{d}x \end{aligned} \tag{3.9}$$

for every $k > 0$ and $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Remark 3.2 *Several remarks are in order.*

(i) *Equation* (3.9) *implies that* $(T_k(b(u)))_t - \operatorname{div}(a(t,x,\nabla u))$ *is a bounded measure, and since* $T_k(b(u)) \in L^p(0,T;W_0^{1,p}(\Omega))$ *and* $\mu_0 \in \mathcal{M}_0(Q)$ *this means that*

$$(T_k(b(u)))_t - \operatorname{div}(a(t,x,\tfrac{1}{b'(u)}\nabla T_k(b(u)))) = \mu + \lambda^k \text{ in } \mathcal{M}(Q). \tag{3.10}$$

(ii) *Thanks to a result of* [23]*, the renormalized solution of problem* (3.1) *turns out to coincide with the renormalized solution of the same problem in the sense of* [10] (*see the proof of Theorem* 4.3 *bellow*).

(iii) *For every* $\varphi \in W^{1,\infty}(Q)$ *such that* $\varphi = 0$ *on* $(\{T\}\times\Omega)\cup((0,T)\times\partial\Omega)$*, we can use* φ *as a test function in* (3.9) *or in the approximate problem.*

(iv) *A remark on the assumption* (3.5) *is also necessary. As one could check later, essentially due to the presence of the term* g (*dependent on* t) *in the formulation of the renormalized solution* (i.e., *the term with* μ) *in Definition* 3.1*, we are forced to assume that* $b'(s) \geq b_0 > 0$*. We conjecture that this assumption is only technical to prove the equivalence and could be removed in order to deal with more general elliptic-parabolic problems* (*see* [1, 2] *and* [13]).

4 The formulation does not depend on the decomposition of μ

As we said before, for every measure $\mu \in \mathcal{M}_0(Q)$ there exist a decomposition (f,g,χ) not uniquely determined such that $f \in L^1(Q)$, $g \in L^p(0,T;V)$ and $\chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ with

$$\mu = f + g_t + \chi \text{ in } \mathcal{D}'(Q).$$

It is not known whether every measure which can be decomposed in this form is diffuse. However, in [23] the following result was proved.

Lemma 4.1 *Assume that* $\mu \in \mathcal{M}(Q)$ *satisfies* (2.3)*, where* $f \in L^1(Q)$*,* $g \in L^p(0,T;V)$ *and* $\chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))$*. If* $g \in L^\infty(Q)$*, then* μ *is diffuse.*

Proof. See [23, Proposition 3.1]. □

Recall the notion of a renormalized solution in the sense of [10].

Definition 4.2 *Let* $\mu \in \mathcal{M}_0(Q)$*. A measurable function defined on* Q *is a renormalized solution of problem* (3.1) *if*

$$b(u) - g \in L^\infty(0,T;L^1(\Omega)), \quad T_k(b(u)-g) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k>0, \tag{4.1}$$

$$\lim_{h\to\infty} \int_{\{h\leq|b(u)-g|\leq h+1\}} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.2}$$

and for every $S \in W^{2,\infty}(\mathbb{R})$ *such that* S' *has compact support,*

$$\begin{aligned}
&-\int_Q S(b(u)-g)\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_Q a(t,x,\nabla u)\cdot\nabla(S'(b(u)-g)\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q f S'(b(u)-g)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q G\cdot\nabla(S'(b(u)-g)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega S(b(u_0))\varphi(0,x)\,\mathrm{d}x
\end{aligned} \tag{4.3}$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Finally, we conclude by proving that Definition 3.1 implies that u is a renormalized solution in the sense of Definition 4.2; this proves that the formulations are actually equivalent.

Theorem 4.3 *Let μ be splitted as in* (2.3); *namely*

$$\mu = f - \operatorname{div}(G) + g_t, \quad f \in L^1(Q),\ G \in L^{p'}(Q) \text{ and } g \in L^p(0,T;V).$$

If u satisfies Definition 3.1*, then u satisfies Definition* 4.2*.*

Proof. We split the proof in two steps.

Step 1. Let $v = T_k(b(u) - g)$. We have $v \in L^p(0,T;V)$. Moreover, using the decomposition of μ in (2.3), and integrating by parts the term with g, we obtain

$$\begin{aligned} &-\int_Q v\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{1}{b'(u)} a(t,x,\nabla T_k(b(u))) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_Q f\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q G \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi \,\mathrm{d}\lambda_k + \int_\Omega T_k(b(u_0))\varphi(0,x)\,\mathrm{d}x \end{aligned}$$

for every $\varphi \in C_c^\infty([0,T] \times \Omega)$. Observe that for every $\varphi \in W^{1,\infty}(Q)$ the above equality remains true. We can choose $\varphi(t,x)$ such that

$$\varphi(t,x) = \left\{ \zeta(t,x) \frac{1}{h} \int_t^{t+h} \psi(v(s,x))\,\mathrm{d}s \right\},$$

where $\zeta \in C_c^\infty([0,T] \times \overline{\Omega})$, $\zeta \geq 0$, $\zeta\psi(0) = 0$ on $(0,T) \times \partial\Omega$, and ψ is a Lipschitz non-decreasing function. This, together with [9, Lemma 2.1], clearly implies that

$$\begin{aligned} &\liminf_{h \to 0} \left\{ -\int_Q (v - T_k(b(u_0))) \left(\zeta \frac{1}{h} \int_t^{t+h} \psi(v)\,\mathrm{d}s \right)_t \mathrm{d}x\,\mathrm{d}t \right\} \\ &\qquad \geq -\int_Q \left(\int_0^t \psi(r)\ \mathrm{d}r \right) \zeta_t \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \left(\int_0^{T_k(b(u_0))} \psi(r)\ \mathrm{d}r \right) \zeta(0,x)\,\mathrm{d}x. \end{aligned}$$

Indeed, since ψ is bounded, we have

$$\left| \int_Q \psi \,\mathrm{d}\lambda_k \right| \leq \|\zeta\|_\infty \|\psi\|_\infty \|\lambda_k\|_{\mathcal{M}(Q)},$$

and since ψ is Lipschitz, we have $\psi(v) \in L^p(0,T;W_0^{1,p}(\Omega))$. Notice that $(\psi(v))_h$ converges to $\psi(v)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ and weakly* in $L^\infty(Q)$. So that, as $h \to 0$,

$$\begin{aligned} &-\int_Q \left(\int_0^r \psi(r)\,\mathrm{d}r \right) \zeta_r \,\mathrm{d}x\,\mathrm{d}t + \int_Q a(t,x,\nabla T_k(u)) \cdot \nabla(\psi(r)\zeta)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \int_Q f\psi(v)\zeta \,\mathrm{d}x\,\mathrm{d}t + \int_Q G \cdot \nabla(\psi(v)\zeta)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_\Omega \left(\int_0^{T_k(b(u_0))} \psi(r)\ \mathrm{d}r \right) \zeta(0,x)\,\mathrm{d}x + \|\zeta\| \|\psi\|_\infty \|\lambda_k\|_{\mathcal{M}(Q)} \end{aligned} \tag{4.4}$$

for every ψ Lipschitz and non-decreasing. In order to obtain the reverse inequality, we only need to take

$$\varphi(x,t) = \left\{\zeta(t,x)\frac{1}{h}\int_{t-h}^{t}\psi(\tilde{v}(s,x))\,\mathrm{d}s\right\},$$

where $\tilde{v}(x,t) = v(x,t)$ when $t \geq 0$ and $\tilde{v} = U_j$ when $t < 0$; here $U_j \in C_c^\infty(\Omega)$ is such that $U_j \to T_k(b(u_0))$ strongly in $L^1(\Omega)$. Thus, using [9, Lemma 2.3], we obtain

$$\begin{aligned}
&\liminf_{h\to 0}\left\{-\int_Q (v - T_k(b(u_0)))\left(\zeta\frac{1}{h}\int_{t-h}^{t}\psi(v)\,\mathrm{d}s\right)_t \mathrm{d}x\,\mathrm{d}t\right\}\\
&\quad\leq -\int_Q\left(\int_0^r \psi(r)\,\mathrm{d}r\right)\zeta_t\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega\left(\int_0^{U_j}\psi(r)\ \mathrm{d}r\right)\zeta(0,x)\,\mathrm{d}x\\
&\quad\quad-\int_\Omega (T_k(b(u_0)) - U_j)\zeta(0,x)\,\mathrm{d}x.
\end{aligned}$$

Recalling that $\tilde{v} \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, when $h \to 0$, we can pass to the limit in the other terms as before, and we observe that

$$\begin{aligned}
&-\int_Q\left(\int_0^v \psi(r)\,\mathrm{d}r\right)\zeta_t\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(t,x,\nabla u)\cdot\nabla(\psi(v)\zeta)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geq \int_Q f\psi(v)\zeta\,\mathrm{d}x\,\mathrm{d}t + \int_Q G\cdot\nabla(\psi(v)\zeta)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega\left(\int_0^{U_0}\psi(r)\ \mathrm{d}r\right)\zeta(0,x)\,\mathrm{d}x\\
&\quad\quad+\int_\Omega (T_k(b(u_0) - U_j)\psi(U_j)\zeta(0,x)\,\mathrm{d}x - \|\zeta\|_\infty\|\psi\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)}.
\end{aligned}$$

Hence, from the fact that $U_j \to T_k(b(u_0))$, we obtain

$$\begin{aligned}
&-\int_Q\left(\int_0^v \psi(r)\,\mathrm{d}r\right)\zeta_t\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{b'(u)}\int_Q a(t,x,\nabla T_k(b(u)))\cdot\nabla(\psi(r)\zeta)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geq \int_Q f\psi(v)\zeta\,\mathrm{d}x\,\mathrm{d}t + \int_Q G\cdot\nabla(\psi(v))\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega\left(\int_0^{T_k(b(u_0))}\psi(r)\ \mathrm{d}r\right)\zeta(0,x)\,\mathrm{d}x\\
&\quad\quad-\|\zeta\|_\infty\|\psi\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)}.
\end{aligned}\tag{4.5}$$

Using equality (4.4) with $S \in W^{2,\infty}(\mathbb{R})$ and $\psi = \int_0^s (S''(t))^+\,\mathrm{d}t$ and equality (4.5) with $\psi = \int_0^s (S''(t))^-\,\mathrm{d}t$, we easily deduce by subtracting the two inequalities (observe that $S'(s) = \int_0^s (S''(t)^+ - S''(t)^-)\,\mathrm{d}t$) that

$$\begin{aligned}
&-\int_Q S(v)\zeta_t\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(t,x,\nabla u)\cdot\nabla(S'(v)\zeta)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq \int_Q fS'(v)\zeta\,\mathrm{d}x\,\mathrm{d}t + \int_Q G\nabla(S'(v)\zeta)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad+\int_\Omega S(T_k(b(u_0)))\zeta(0,x)\,\mathrm{d}x + 2\|\zeta\|_\infty\|S'\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)}
\end{aligned}\tag{4.6}$$

for every $S \in W^{2,\infty}(\mathbb{R})$ and for every non-negative ζ.

Step 2. Let us use $S'(\Theta_h(s))$ in (4.6) such that $\Theta_h = T_1(s - T_h(s))$ and $\zeta = \zeta(t)$. Then, we easily obtain by setting $R_h(s) = \int_0^s \Theta_h(\zeta)\,\mathrm{d}\zeta$,

$$
\begin{aligned}
&-\int_Q R_h(T_k(b(u)) - g)\zeta_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\{h<|b(u)-g|<h+k\}} a(t,x,\nabla u)\cdot\nabla(T_k(b(u)) - g)\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_Q f\Theta_h(T_k(b(u)) - g)\zeta \,\mathrm{d}x\,\mathrm{d}t + \int_{\{h<|b(u)-g|<h+k\}} G\cdot\nabla(T_k(b(u) - g))\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_\Omega R_h(T_k(b(u_0)))\zeta(0,x)\,\mathrm{d}x + 2\|\zeta\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)}.
\end{aligned}
$$

Moreover, we can use Young's inequality and assumptions (3.2)–(3.3) to get

$$
\begin{aligned}
&-\int_Q R_h(T_k(b(u) - g))\zeta_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\{h<|b(u)-g|<h+1\}} b'(u)|\nabla T_k(b(u))|^p\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_Q f\Theta_h(T_k(b(u)) - g)\zeta \,\mathrm{d}x\,\mathrm{d}t + C\int_{\{h<|b(u)-g|<h+1\}} \big(|G|^{p'} + |g|^p + |L|^{p'}\big)\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_\Omega R_h(T_k(b(u_0)))\zeta(0,x)\,\mathrm{d}x + 2\|\zeta\|_\infty\|\lambda_k\|_{\mathcal{M}(Q)}.
\end{aligned}
$$

Now, letting $k \to \infty$, thanks to (3.8) and Fatou's Lemma, we deduce that

$$
\begin{aligned}
&-\int_Q R_h(b(u) - g)\zeta_t \,\mathrm{d}x\,\mathrm{d}t + \alpha\int_{\{h<|b(u)-g|<h+1\}} b'(u)|\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_Q f\Theta_h(u - g)\zeta \,\mathrm{d}x\,\mathrm{d}t + C\int_{\{h<|b(u)-g|\le h+1\}} \big(|G|^{p'} + |g|^p + |L|^{p'}\big)\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_\Omega R_h(b(u_0))\zeta(0,x)\,\mathrm{d}x.
\end{aligned}
$$

Consider $\zeta = 1 - \frac{1}{\epsilon}T_\epsilon(t-\tau)^+$ for $\tau \in (0,T)$. Letting $\epsilon \to 0$, we claim that the estimate of $b(u) - g$ in $L^\infty(0,T;L^1(\Omega))$ is valid. By repeating the argument for the non-increasing $\zeta_\epsilon \in C_c^\infty([0,T])$, we are allowed to pass to the limit $\zeta_\epsilon \to 1$ to prove that

$$
\begin{aligned}
&\alpha b_0 \int_{\{h<|b(u)-g|<h+1\}} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_{\{|b(u)-g|>h\}} |f|\,\mathrm{d}x\,\mathrm{d}t + C\int_{\{h<|b(u)-g|<h+1\}} \big(|G|^{p'} + |g|^p + |L|^{p'}\big)\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\{|b(u_0)|>h\}} b(u_0)\,\mathrm{d}x,
\end{aligned}
$$

which implies (4.2). Finally, by using the regularity (4.1) and the fact that $S \in W^{2,\infty}(\mathbb{R})$ is such that S' has compact support and $\zeta \in C_c^\infty([0,T]\times\Omega)$, we can easily deduce (4.3) by passing to the limit in (4.6) and using (3.8). □

5 Existence of solutions

Now we are ready to prove the main results. Some of the reasonings are based on the ideas developed in [10] (see also [17, 23] and [21]). First we have to prove the existence of a renormalized solution for problem (3.1).

Theorem 5.1 *Under the assumptions* (3.1)–(3.7)*, there exists at least one renormalized solution* u *of problem* (3.1)*.*

Proof. We first introduce the approximate problems. For $n \geq 1$ fixed, we define

$$b_n(s) = b(T_{\frac{1}{n}}(s)) + ns \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R}, \tag{5.1}$$

$$u_0^n \in C_0^\infty(\Omega): \quad b_n(u_0^n) \to b(u_0) \text{ in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty. \tag{5.2}$$

We consider a sequence of mollifiers (ρ_n), and we define the convolution $\rho_n * \mu$ for every $(t,x) \in Q$ by

$$\mu^n(t,x) = (\rho_n * \mu)(t,x) = \int_Q \rho_n(t-s, x-y)\, \mathrm{d}\mu(s,y). \tag{5.3}$$

Then, we consider the approximate problem of (3.1) defined by

$$\begin{cases} (b_n(u_n))_t - \operatorname{div}(a(t,x,\nabla u_n)) = \mu_n & \text{in } (0,T)\times\Omega, \\ b_n(u_n) = b_n(u_0^n) & \text{on } \{0\}\times\Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.4}$$

By classical results (see [18]), we can find a non-negative weak solution $u_n \in L^p(0,T; W_0^{1,p}(\Omega))$ for problem (5.4). Our aim is to prove that a subsequence of these approximate solutions (u_n) converges increasingly to a measurable function u, which is a renormalized solution of problem (3.1). We will divide the proof into several steps. We present a self-contained proof for the sake of clarity and readability.

Step 1: *Basic estimates.* Choosing $T_k(b_n(u_n) - g_n)$ as a test function in (5.4), we have

$$\begin{aligned} &\int_\Omega \overline{T}_k(b_n(u_n) - g_n)\,\mathrm{d}x + \int_0^t \int_\Omega a(x,s,\nabla u_n)\cdot \nabla T_k(b_n(u_n)-g_n)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t\int_\Omega f_n T_k(b_n(u_n)-g_n)\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\int_\Omega G_n\cdot\nabla T_k(b_n(u_n)-g_n)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_\Omega \overline{T}_k(b_n(u_0^n))\,\mathrm{d}x \end{aligned} \tag{5.5}$$

for almost every t in $(0,T)$; here $\overline{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s$. It follows from the definition of $\overline{T}_k$, assumptions (3.2)–(3.3) and (3.6) that

$$\begin{aligned} &\int_\Omega \overline{T}_k(b_n(u_n)-g_n)\,\mathrm{d}x + \alpha\int_{\{|b_n(u_n)-g_n|\leq k\}} b_n'(u_n)|\nabla u_n|^p\,\mathrm{d}x\,\mathrm{d}s \\ &\leq k\|\mu_n\|_{L^1(Q)} + \beta\int_{\{|b_n(u_n)-g_n|\leq k\}} L(x,s)|\nabla g_n|\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \beta\int_{\{|b_n(u_n)-g_n|\leq k\}} |\nabla u_n|^{p-1}|\cdot|\nabla g_n|\,\mathrm{d}x\,\mathrm{d}s + k\|b_n(u_0^n)\|_{L^1(\Omega)}. \end{aligned} \tag{5.6}$$

Then, from (3.5) and Young's inequality

$$\begin{aligned}&\int_\Omega \overline{T}_k(b_n(u_n)-g_n)\,\mathrm{d}x + \frac{\alpha}{2}\int_{\{|b_n(u_n)-g_n|\le k\}} b_n'(u_n)|\nabla u_n|^p\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le k\|\mu_n\|_{L^1(Q)} + \beta\|L\|_{L^{p'}(Q)}\|\nabla g_n\|_{L^p(Q)} + C\|\nabla g_n\|^{p'}_{L^{p'}(Q)} + k\|b_n(u_0^n)\|_{L^1(\Omega)},\end{aligned} \tag{5.7}$$

where C is a positive constant. We will use the properties of $\overline{T}_k$ ($\overline{T}_k \ge 0$ and $\overline{T}_k(s) \ge |s|-1$ for every $s\in\mathbb{R}$), b_n, f_n, G_n, g_n, the boundedness of μ_n in $L^1(Q)$ and $b_n(u_0^n)$ in $L^1(\Omega)$ to deduce that

$$b_n(u_n)-g_n \text{ is bounded in } L^\infty(0,T;L^1(\Omega)). \tag{5.8}$$

Using Hölder's inequality and (3.5), we deduce that (5.7) implies that

$$T_k(b_n(u_n)-g_n) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{5.9}$$

independently of n for any $k\ge 0$.

Let us observe that for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support ($\mathrm{Supp}(S')\subset[-k,k]$),

$$S(b_n(u_n)-g_n) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \tag{5.10}$$

and

$$(S(b_n(u_n)-g_n))_t \text{ is bounded in } L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{5.11}$$

independently of n (cf. [5, 7]). In fact, thanks to (5.9) and Stampacchia's theorem, we easily deduce (5.10). To show that (5.11) holds true, we multiply (5.4) by $S'(b_n(u_n)-g_n)$ to obtain

$$\begin{aligned}(S(b_n(u_n)-g_n))_t &= \mathrm{div}(S(b_n(u_n)-g_n)a(t,x,\nabla u_n)) \\ &\quad - a(t,x,\nabla u_n)\cdot\nabla S'(b_n(u_n)-g_n) + f_nS'(b_n(u_n)-g_n) \\ &\quad - \mathrm{div}(G_nS'(b_n(u_n)-g_n)) + G_n\cdot\nabla S(b_n(u_n)-g_n) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{5.12}$$

Since each term on the right-hand side of (5.12) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ or in $L^1(Q)$, we obtain (5.11).

Moreover, arguing again as in [10] (see also [5, 7] and [11]), there exists a measurable function $u\in L^\infty(0,T;L^1(\Omega))$ such that $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$. And up to a subsequence, for any $k>0$ we have

$$\begin{cases} u_n \to u \text{ a.e. in } Q, \\ T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ b_n(u_n)-g_n \to b(u)-g \text{ a.e. in } Q, \\ T_k(b_n(u_n)-g_n) \rightharpoonup T_k(b(u)-g) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \end{cases} \tag{5.13}$$

as n tends to $+\infty$.

Step 2: *Estimates in $L^1(Q)$ on the energy term.* Let (ρ_n) be a sequence of mollifiers as in (2.7) and μ a non-negative measure such that $\mu_n(t,x)=(\rho_n*\mu)(t,x)$. In view of Lemma 2.4 (μ_n) is an equi-diffuse sequence of measures. Moreover, there exists a sequence (μ_n) with $\mu_n\in C^\infty(Q)$ such that

$$\|\mu\|_{L^1(Q)} \le \|\mu\|_{\mathcal{M}(Q)}$$

and

$$\mu_n \to \mu \text{ tightly in } \mathcal{M}(Q).$$

Let us fix $\eta > 0$ and define $S_{k,\eta}(s)\colon \mathbb{R} \to \mathbb{R}$ and $h_{k,\eta}(s)\colon \mathbb{R} \to \mathbb{R}$ by

$$S_{k,\eta}(s) = \begin{cases} 1 & \text{if } |s| \le k \\ 0 & \text{if } |s| > k+\eta \\ \text{affine} & \text{otherwise} \end{cases} \quad \text{and} \quad h_{k,\eta}(s) = 1 - S_{k,\eta}(u_n). \tag{5.14}$$

Let us denote by $T_{k,\eta}\colon \mathbb{R} \to \mathbb{R}$ the primitive function of $S_{k,\eta}$, that is,

$$T_{k,\eta}(s) = \int_0^s S_{k,\eta}(\sigma)\,\mathrm{d}\sigma.$$

Notice that $T_{k,\eta}(s)$ converges pointwise to $T_k(s)$ as η goes to zero and that using the admissible test function $h_{k,\eta}(b(u_n))$ in (5.4) leads to

$$\begin{aligned} &\int_\Omega \overline{h}_{k,\eta}(b(u_n)(T))\,\mathrm{d}x + \frac{1}{\eta}\int_{\{k<u_n<k+\eta\}} a(t,x,\nabla u_n)\cdot\nabla h_{k,\eta}(b(u_n)) \\ &\quad = \int_Q h_{k,\eta}(b(u_n))\mu_n\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \overline{h}_{k,\eta} b(u_0^n)\,\mathrm{d}x, \end{aligned} \tag{5.15}$$

where $\overline{h}_{k,\eta}(r) = \int_0^r h_{k,\eta}(s)\,\mathrm{d}s \ge 0$. Hence, using (5.2), (5.3) and dropping a non-negative term,

$$\begin{aligned} &\frac{1}{\eta}\int_{\{k<b(u_n)<k+\eta\}} b'(u_n)a(t,x,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \le \int_{\{|b(u_n)|>k\}} |\mu_n|\,\mathrm{d}x\,\mathrm{d}t + \int_{\{b(u_0^n)>k\}} |b(u_0^n)|\,\mathrm{d}x \le C. \end{aligned} \tag{5.16}$$

Thus, there exists a bounded Radon measures λ_k^n such that, as η tends to zero,

$$\lambda_k^{n,\eta} = \frac{1}{\eta} a(t,x,\nabla u_n)\cdot\nabla u_n \chi_{\{k\le b(u_n)\le k+\eta\}} \rightharpoonup \lambda_k^n \text{ *weakly in } \mathcal{M}(Q). \tag{5.17}$$

Step 3: *Equation for the truncations.* We are able to prove that (3.9) holds true. To see that, we multiply (5.4) by $S_{k,\eta}(b(u_n))\xi$ where $\xi \in C_c^\infty([0,T]\times\Omega)$ to obtain

$$\begin{aligned} &T_{k,\eta}(b(u_n))_t - \operatorname{div}(S_{k,\eta}(b(u_n))a(t,x,\tfrac{1}{b'(u_n)}\nabla T_{k,\eta}(b(u_n)))) \\ &\quad = \mu_n + (S_{k,\eta}(b(u_n)) - 1)\mu_n + \frac{1}{n} a(t,x,\nabla u_n)\cdot\nabla u_n \chi_{\{k<|b(u_n)|<k+\eta\}} \text{ in } \mathcal{D}'(Q). \end{aligned} \tag{5.18}$$

Passing to the limit in (5.18) as η tends to zero, and using the fact that $|S_{k,\eta}| \le 1$ and (5.17), we deduce that

$$T_k(b(u_n))_t - \operatorname{div}(a(t,x,\tfrac{1}{b'(u)}\nabla T_k(b(u_n)))) = \mu_n - \mu_n\chi_{\{|b(u_n)|\le k\}} + \lambda_k^n \text{ in } \mathcal{D}'(Q). \tag{5.19}$$

Now, using the properties of the convolution $\rho_n * \mu$, in view of (5.16)–(5.17), we deduce that $\Lambda_k^n = -\mu_n\chi_{\{|b(u_n)|<k\}} + \lambda_k^n$ is bounded in $L^1(Q)$. Then, there exists a bounded measures Λ_k such that $(-\mu_n\chi_{\{|b(u_n)|<k\}} + \lambda_k^n)_n$ converges to Λ_k *weakly in $\mathcal{M}(Q)$. Therefore, using (5.13) and (5.19) we deduce that u satisfies

$$T_k(b(u))_t - \operatorname{div}(a(t,x,\nabla u)) = \mu + \Lambda_k \text{ in } \mathcal{D}'(Q). \tag{5.20}$$

Step 4: *u is a renormalized solution.* In this step, Λ_k is shown to satisfy (3.8). From (5.16) and (5.17) we deduce that

$$\begin{aligned}\|\Lambda_k^n\|_{L^1(Q)} &= \| -\mu_n\chi_{\{|b(u_n)|>k\}} + \lambda_k^n\|_{L^1(Q)} \\ &\leq 2\int_{\{|b(u_n)|>k\}} |\mu_n|\,\mathrm{d}x\,\mathrm{d}t + \int_{\{|b(u_0^n)|>k\}} |b(u_0^n)|\,\mathrm{d}x.\end{aligned} \tag{5.21}$$

Since

$$\|\lambda_k\|_{\mathcal{M}(Q)} \leq \liminf_{n\to+\infty} \|\mu_n\chi_{\{|b(u_n)|>k\}} + \lambda_k^n\|_{\mathcal{M}(Q)},$$

the sequence (μ_n) is equi-diffuse, and the function $b(u_0^n)$ converges to $b(u_0)$ strongly in $L^1(\Omega)$, we deduce from Proposition 2.2 and (5.21) that $\|\Lambda_k\|_{\mathcal{M}(Q)}$ tends to zero as k tends to infinity. Then, we obtain (3.8), and hence u is a renormalized solution. □

6 Uniqueness of the renormalized solution

This section is devoted to establishing the uniqueness of the renormalized solution. As we already said, due to the presence of both the general monotone operator associated to a and the non-linearity of the term b, a standard approach (see for instance [17]) does not apply here. To overcome this difficulty, we are going to exploit the idea of [23] for which the uniqueness result comes from the following comparison principle.

Theorem 6.1 *Let u_1, u_2 be two renormalized solutions of problem* (3.1) *with data $(b(u_0^1), \mu_1)$ and $(b(u_0^2), \mu_2)$, respectively. Then, we have*

$$\int_\Omega (b(u_1) - b(u_2))^+(t)\,\mathrm{d}x \leq \|b(u_0^1) - b(u_0^2)\|_{L^1(\Omega)} + \|(\mu_1-\mu_2)^+\|_{\mathcal{M}(Q)} \tag{6.1}$$

for almost every $t \in [0,T]$. In particular, if $b(u_0^1) \leq b(u_0^2)$ and $\mu_1 \leq \mu_2$ (in the case of measures), we have $u_1 \leq u_2$ a.e. in Q. As a consequence, there exists at most one renormalized solution of problem (3.1).

Proof. Let $\lambda_{k_1}, \lambda_{k_2}$ be the measures given by Definition 3.1 corresponding to $b(u_1), b(u_2)$. We can extend the class of test functions (see [23, Proposition 4.2]) and obtain

$$\begin{aligned}&-\int_Q (T_k(b(u_1)) - T_k(b(u_2))v_t\,\mathrm{d}x\,\mathrm{d}t + \int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2))\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q v\,\mathrm{d}(\mu_1-\mu_2) + \int_Q v\,\mathrm{d}\lambda_{k,1} - \int_Q v\,\mathrm{d}\lambda_{k,2} + \int_\Omega (T_k(b(u_0^1)) - T_k(b(u_0^2)))v(0,x)\,\mathrm{d}x\end{aligned}$$

for every $v \in W \cap L^\infty(Q)$ such that $v(T) = 0$. Consider the function

$$\omega_h(t,x) = \frac{1}{h}\int_t^{t+h} \frac{1}{\epsilon} T_\epsilon(T_k(b(u_1)) - T_k(b(u_2)))^+(s,x)\,\mathrm{d}s.$$

Given $\zeta \in C_c^\infty([0,T))$, $\zeta \geq 0$, take $v = \omega_h\zeta$ as a test function. Observe that both ω_h and $(\omega_h)_t$ belong to $L^p(0,T;V)\cap L^\infty(Q)$ for $h > 0$ sufficiently small, and hence $\omega_h \in W\cap L^\infty(Q)$. Moreover, we have

$$\omega_h \to \frac{1}{\epsilon}T_\epsilon(T_k(b(u_1)) - T_k(b(u_2)))^+ \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)).$$

Since $0 \leq \omega_h \leq 1$ almost everywhere, and hence $0 \leq \omega_h \leq 1$ cap-quasi-everywhere (see [17]), we have

$$\begin{aligned}
&-\int_Q \left[(T_k(b(u_1)) - T_k(b(u_2)) - (T_k(b(u_0^1)) - T_k(b(u_0^2))\right] (\omega_h\zeta)_t \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2)) \cdot \nabla\omega_h\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \|\zeta\|_\infty \big(\|(\mu_1-\mu_2)^+\|_{\mathcal{M}(Q)} + \|\lambda_{k,1}\|_{\mathcal{M}(Q)} + \|\lambda_{k,2}\|_{\mathcal{M}(Q)}\big).
\end{aligned} \tag{6.2}$$

Using the monotonicity of $T_\epsilon(s)$, we obtain (see [9, Lemma 2.1])

$$\begin{aligned}
&\liminf_{h\to 0} \left\{-\int_Q [(T_k(b(u_1)) - T_k(b(u_2)) - (T_k(b(u_0^1)) - T_k(b(u_0^2)))](\omega_h\zeta_t)\,\mathrm{d}x\,\mathrm{d}t\right\} \\
&\quad \geq -\int_Q \widetilde{\Theta}_\epsilon(T_k(b(u_1))\zeta_t \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \widetilde{\Theta}_\epsilon(T_k(b(u_0^1)) - T_k(b(u_0^2))\zeta(0)\,\mathrm{d}x,
\end{aligned}$$

where $\widetilde{\Theta}_\epsilon(s) = \int_0^s \frac{1}{\epsilon}T_\epsilon(r)^+\,\mathrm{d}r$. Therefore, letting $h \to 0$ in (6.2), we obtain

$$\begin{aligned}
&-\int_Q \widetilde{\Theta}_\epsilon(T_k(b(u_1)) - T_k(b(u_2))\zeta_t \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\epsilon}\int_Q (a(t,x,\nabla u_1) - a(t,x,\nabla u_2)) \cdot \nabla T_\epsilon(T_k(b(u_1)) - T_k(b(u_2))\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_\Omega \widetilde{\Theta}_\epsilon(T_k(b(u_0^1)) - T_k(b(u_0^2))\zeta(0)\,\mathrm{d}x \\
&\qquad + \|\zeta\|_\infty \big(\|(\mu_1-\mu_2)^+\|_{\mathcal{M}(Q)} + \|\lambda_{k,1}\|_{\mathcal{M}(Q)} + \|\lambda_{k,2}\|_{\mathcal{M}(Q)}\big).
\end{aligned}$$

Using (3.4) and letting $\epsilon \to 0$, we deduce that

$$\begin{aligned}
&-\int_Q (T_k(b(u_1)) - T_k(b(u_2))^+\zeta_t \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_\Omega (T_k(b(u_0^1)) - T_k(b(u_0^2))^+\zeta(0)\,\mathrm{d}x \\
&\qquad + \|\zeta\|_\infty \big(\|(\mu_1-\mu_2)^+\|_{\mathcal{M}(Q)} + \|\lambda_{k,1}\|_{\mathcal{M}(Q)} + \|\lambda_{k,2}\|_{\mathcal{M}(Q)}\big).
\end{aligned}$$

And letting $k \to \infty$, we obtain, thanks to (3.8),

$$-\int_Q (b(u_1) - b(u_2))^+\zeta_t \,\mathrm{d}x\,\mathrm{d}t \leq \|\zeta\|_\infty \big(\|(b(u_0^1) - b(u_0^2)^+\|_{L^1(\Omega)} + \|(\mu_1-\mu_2)^+\|_{\mathcal{M}(Q)}\big)$$

for every non-negative $\zeta \in C_c^\infty([0,T)$. Of course, the same inequality holds for any $\zeta \in W^{1,\infty}(0,T)$ with compact support in $[0,T)$. Take then $\zeta(t) = 1 - \frac{1}{\epsilon}T_\epsilon(t-\tau)^+$, where $\tau \in (0,T)$; since $b(u_1), b(u_2) \in L^\infty(0,T;L^1(\Omega))$, by letting $\epsilon \to 0$, we have

$$-\int_Q (b(u_1) - b(u_2))^+\zeta_t \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\epsilon}\int_\tau^{\tau+\epsilon}\int_\Omega (b(u_1) - b(u_2))^+\,\mathrm{d}x\,\mathrm{d}t \to \int_\Omega (b(u_1) - b(u_2))^+(\tau)\,\mathrm{d}x$$

for almost every $\tau \in (0,T)$. Finally, using the fact that $\|\zeta\|_\infty \leq 1$, we get (6.1). $\square$

7 Appendix

Here we prove the extension of Proposition 2.2.

Proof. We still use the notations introduced in Section 2. In particular, we consider the condition $p > \frac{2N+1}{N+1}$. For simplicity, we assume in addition that $\mu \geq 0$ and $b(u_0) \geq 0$. Hence, we have $u \geq 0$ (the case $\mu \leq 0$ can be obtained similarly). Actually, the proof will be split into three parts with the first one devoted to obtaining some basic estimates.

Step 1: Estimates of $T_k(b(u))$ in the space $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. For every $\tau \in \mathbb{R}$ let

$$\overline{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s.$$

We recall that if $u \in W$, then u is a weak solution of (1.1) if

$$\int_0^t \langle b(u)_t, v\rangle\,\mathrm{d}t + \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x\,\mathrm{d}t = \int_0^t \langle \mu, v\rangle\,\mathrm{d}t \quad \text{for every } v \in W, \tag{7.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between V and V'. Note that, if $\mu \in \mathcal{M}(Q) \cap L^{p'}(0,T;W^{-1,p'}(\Omega))$, then (7.1) holds for every $v \in L^p(0,T;V)$, and we have

$$\int_s^t \langle b(u)_t, \psi'(u)\rangle\,\mathrm{d}t = \int_\Omega \psi(b(u)(t))\,\mathrm{d}x - \int_\Omega \psi(b(u)(s))\,\mathrm{d}x \tag{7.2}$$

for every $s,t \in [0,T]$ and every function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that ψ' is Lipschitz continuous and $\psi'(0) = 0$. Now, we choose $T_k(b(u))$ as a test function in (7.1). And using (7.2) with $\psi = \overline{T_k}$, $s = 0$ and $t = r$, we have

$$\int_\Omega \overline{T}_k(b(u))(r)\,\mathrm{d}x + \int_0^r\int_\Omega a(t,x,\nabla u)\cdot\nabla T_k(b(u))\,\mathrm{d}x\,\mathrm{d}t \leq k\|\mu\|_{\mathcal{M}(Q)} + \int_\Omega \overline{T}_k(b(u_0))\,\mathrm{d}x.$$

Let $E_k = \{(t,x) : |b(u)| \leq k\}$. Observing that $\frac{T_k(s)^2}{2} \leq \overline{T}_k(s) \leq k|s|$ for every $s \in \mathbb{R}$, we have

$$\begin{aligned}\int_\Omega \frac{|T_k(b(u))(r)|^2}{2}\,\mathrm{d}x + \int_0^r\int_\Omega \chi_{E_k} b'(u) a(t,x,\nabla u)\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t&\\ \leq k\big(\|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}\big)&\end{aligned} \tag{7.3}$$

for any $r \in [0,T]$. In particular, we deduce that

$$\|T_k(b(u))\|^2_{L^\infty(0,T;L^2(\Omega))} \leq 2kM, \tag{7.4}$$

where $M = \|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}$. From the assumption (3.2), we have

$$\alpha\int_{E_k} b'(u)|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^r\int_\Omega \chi_{E_k} b'(u)a(t,x,\nabla u)\cdot\nabla u \leq kM.$$

Note that

$$\begin{aligned}\int_{E_k} b'(u)|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t &= \int_{E_k} b'(u)|b'^{-1}\nabla b(u)|^p\,\mathrm{d}x\\ &= \int_{E_k}\frac{1}{(b')^{p-1}}|\nabla b(u)|^p\,\mathrm{d}x\\ &\geq \int_0^r\int_\Omega \frac{1}{(b_1)^{p-1}}|\nabla T_k b(u)|^p\,\mathrm{d}x.\end{aligned}$$

Then,

$$\|T_k(b(u))\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq CkM, \tag{7.5}$$

where

$$C = \frac{b_1^{p-1}}{\alpha} \quad \text{and} \quad M = \|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}. \tag{7.6}$$

Step 2: *Estimates in W.* Note that in view of [19] (see also [18]), any function $z \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ is a solution of the backward problem

$$\begin{cases} -z_t - \Delta_p z = -2\Delta_p T_k(b(u)) & \text{in } Q, \\ z = T_k(b(u)) & \text{on } \{T\} \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{7.7}$$

We can choose z as a test function in (7.7) and integrate t between τ and T. Using Young's inequality we have

$$\int_\Omega \frac{[z(\tau)]^2}{2}\,\mathrm{d}x + \frac{1}{2}\int_\tau^T\int_\Omega b'(u)|\nabla z|^p\,\mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \frac{[T_k(b(u))(T)]^2}{2}\,\mathrm{d}x + C\int_\tau^T\int_\Omega b'(u)|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t,$$

we deduce, using also (7.2) with $r = T$, that

$$\int_\Omega \frac{[z(\tau)]^2}{2}\,\mathrm{d}x + \frac{1}{2}\int_\tau^T\int_\Omega b'(u)|\nabla z|^p\,\mathrm{d}x\,\mathrm{d}t \leq Ck\big(\|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}\big) = CkM.$$

This implies the estimate for z

$$\|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq CkM. \tag{7.8}$$

By the definition of V (*i.e.*, $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$), we have

$$\|z\|^p_{L^p(0,T;V)} \leq C\Big(\|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|z\|^p_{L^p(0,T;L^2(\Omega))}\Big).$$

Then, from (7.8), we have that

$$\|z\|_{L^p(0,T;V)} \leq C\left[(kM)^{\frac{1}{p}} + (kM)^{\frac{1}{2}}\right]. \tag{7.9}$$

Using the equation (7.7), we obtain

$$\|z_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C\Big(\|z\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))} + \|T_k(b(u))\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))}\Big).$$

Hence, from (7.5) and (7.8), we get

$$\|z\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C(kM)^{\frac{1}{p'}}. \tag{7.10}$$

Putting together (7.9) and (7.10), we have the estimate

$$\|z\|_W \leq C\max\left\{(kM)^{\frac{1}{p}}, (kM)^{\frac{1}{p'}}\right\}, \tag{7.11}$$

where M is the constant defined in (7.6).

Step 3: *Proof completed.* Obtaining the energy inequality (7.11) was the main step in order to prove the estimate of the capacity (2.6). It should be noticed that we assume that $\mu \geq 0$ to obtain $b(u)_t - \Delta_p u \geq 0$, $u \geq 0$ in Q and that the following inequality holds

$$(T_k(b(u)))_t - \Delta_p T_k(b(u)) \geq 0. \tag{7.12}$$

Indeed, let $\varphi \in C_c^\infty(Q)$ be a non-negative function and choose $T'_{k,\eta}(b(u))\varphi$ a test function in (7.1). Using the facts that $\mu \geq 0$ and that $T_{k,\eta}(s)$ is concave for $s \geq 0$, we obtain

$$-\int_0^T \varphi_t T_{k,\eta}(b(u))\,\mathrm{d}t + \int_Q b'(u)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi S_{k,\eta}(u)\,\mathrm{d}x\,\mathrm{d}t \geq 0,$$

which yields (7.12) as η goes to 0. Therefore, the combination of (7.7) and (7.12) gives

$$-z_t - \Delta_p z \geq -(T_k(b(u)))_t - \Delta_p T_k(b(u)). \tag{7.13}$$

We are left to prove that $z \geq T_k(b(u))$ a.e. in Q (in particular, $z \geq k$ a.e. on $\{b(u) > k\}$). This is done by means of a standard comparison principle by multiplying both sides of (7.13) by $(z - T_k(b(u)))^-$ and using the fact that z and $T_k(u)$ belong to $L^p(0,T;W_0^{1,p}(\Omega))$. Indeed, we have u has a unique cap_p-quasi continuous representative (recall that u belongs to W); hence, the set $\{b(u) > k\}$ is cap-quasi open, and its capacity can be estimated with (2.1). So that

$$\mathrm{cap}_p(\{|b(u)| > k\}) \leq \left\|\frac{z}{k}\right\|_W.$$

Using (7.11) and the fact that the result is also true for $\mu \leq 0$, we conclude the extension of (2.6). □

Acknowledgements. The authors thank editors and referees for their care in reading this manuscript and their valuable comments.

References

[1] H. W. Alt, S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Mathematische Zeitschrift **183** (1983), no. 3, 311–341.

[2] K. Ammar, P. Wittbold, *Existence of renormalized solutions of degenerate elliptic-parabolic problems*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **133** (2003), no. 3, 477–496.

[3] P. Bénilan, J. Carrillo, P. Wittbold, *Renormalized entropy solutions of scalar conservation laws*, Annali Della Scuola Normale Superiore Di Pisa, Classe Di Scienze (4) **29** (2000), no. 2, 313–327.

[4] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, *Nonlinear parabolic equations with measure data*, Journal of Functional Analysis **147** (1997), no. 1, 237–258.

[5] D. Blanchard, F. Murat, *Renormalized solutions of nonlinear parabolic problems with L^1 data, existence and uniqueness*, Proceedings of the Royal Society of Edinburgh Section A **127** (1997), no. 6, 1137–1152.

[6] L. Boccardo, F. Murat, J.-P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Annali di Matematica Pura ed Applicata **152** (1988), no. 1, 183–196.

[7] D. Blanchard, F. Murat, H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems*, Journal of Differential Equations **177** (2001), no. 2, 331–374.

[8] D. Blanchard, A. Porretta, *Nonlinear parabolic equations with natural growth terms and measure initial data*, Annali Della Scuola Normale Superiore Di Pisa, Classe Di Scienze (4) **30** (2001), no. 3–4, 583–622.

[9] D. Blanchard, A. Porretta, *Stefan problems with nonlinear diffusion and convection*, Journal of Differential Equations **210** (2005), no. 2, 383–428.

[10] D. Blanchard, F. Petitta, H. Redwane, *Renormalized solutions of nonlinear parabolic equations with diffuse measure data*, Manuscripta Mathematica **141** (2013), no. 3–4, 601–635.

[11] D. Blanchard, H. Redwane, *Renormalized solutions of nonlinear parabolic evolution problems*, Journal de Mathématiques Pures et Appliquées **77** (1998), no. 2, 117–151.

[12] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Annali della Scuola Normale Superiore di Pisa **22** (1995), no. 2, 240–273.

[13] J. Carrillo, P. Wittbold, *Uniqueness of renormalized solutions of degenerate elliptic parabolic problems*, Journal of Differential Equations **156** (1999), no. 1, 93–121.

[14] G. Dal Maso, A. Malusa, *Some properties of reachable solutions of nonlinear elliptic equations with measure data*, Dedicated to Ennio De Giorgi. Annali Della Scuola Normale Superiore Di Pisa, Classe Di Scienze (4) **25** (1997), no. 1–2, 375–396.

[15] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Annali Della Scuola Normale Superiore Di Pisa, Classe Di Scienze **28** (1999), no. 1, 741–808.

[16] J. Droniou, A. Prignet, *Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data*, Nonlinear Differential Equations and Applications NoDEA **14** (2007), no. 1–2, 181–205.

[17] J. Droniou, A. Porretta, A. Prignet, *Parabolic capacity and soft measures for nonlinear equations*, Potential Analysis **19** (2003), no. 2, 99–161.

[18] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaire*, Dunod et Gauthier-Villars, 1969.

[19] M. Pierre, *Parabolic capacity and Sobolev spaces*, SIAM Journal on Mathematical Analysis **14** (1983), no. 3, 522–533.

[20] F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data*, Annali di Matematica Pura ed Applicata **187** (2008), no. 4, 187–563.

[21] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations*, Annali di Matematica Pura ed Applicata (IV) **177** (1999), no. 1, 143–172.

[22] F. Petitta, A. C. Ponce, A. Porretta, *Approximation of diffuse measures for parabolic capacities*, Comptes Rendus Mathématique Académie des Sciences, Paris **346** (2008), no. 3–4, 161–166.

[23] F. Petitta, A. C. Ponce, A. Porretta, *Diffuse measures and nonlinear parabolic equations*, Journal of Evolution Equations **11** (2011), no. 4, 861–905.

[24] A. Prignet, *Existence and uniqueness of entropy solutions of parabolic problems with L^1 data*, Nonlinear Analysis: Theory, Methods & Applications **28** (1997), no. 12, 1943–1954.