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EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL SYSTEMS

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Abstract. In this paper, we establish some existence and uniqueness results for a class of fractional dynamical systems with Caputo fractional derivative. The arguments are based upon the Banach contraction principle and Schaefer's fixed point theorem. An illustrative example is included to show the applicability of our results.

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1 Introduction

Recently, fractional differential equations have been studied intensively [1, 2, 3, 4, 5, 9, 11, 16, 17, 18, 20]. The mathematical modelling of many real world phenomena based on fractional-order operators is regarded as better and improved than the one depending on integer-order operators. In particular, fractional calculus has played a significant role in the recent development of special functions and integral transforms, signal processing, control theory, bioengineering and biomedicine, viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, etc. The motivation for this work arises from both the development of the theory of fractional calculus itself and its wide applications to various fields of science, such as physics, chemistry, biology, electromagnetism of complex media, robotics, economics, etc.

Much attention has been paid to the existence and uniqueness of solutions of fractional dynamical systems [6, 7, 8, 13, 15, 21, 23] due to the fact that existence is the fundamental problem and a necessary condition for considering some other properties for fractional dynamical systems, such as controllability, stability, etc. Chai [10] provided sufficient conditions for the existence of solutions to a class of anti-periodic boundary value problems for fractional differential equations, while Sheng and Jiang [19] considered a class of initial value problems for fractional differential systems.

Motivated by the works of Chai [10] and Sheng and Jiang [19], we will establish in this paper existence and uniqueness results of the solutions of the fractional dynamical system with Caputo fractional derivative

$$ {}^cD^{\alpha}_{0^+}x(t) - A\,{}^cD^{\beta}_{0^+}x(t) = f\big(t, x(t), {}^cD^{\beta}_{0^+}x(t), {}^cD^{\alpha}_{0^+}x(t)\big), \;\; t \in J := [0,T], \tag{1.1} $$

$$ x(0) = x_0, \;\; x'(0) = x_0^*, \tag{1.2} $$

where $T > 0$, ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f\colon J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, $x_0, x_0^* \in \mathbb{R}^n$, A is an $\mathbb{R}^{n\times n}$ matrix and $0 < \beta \leq 1 < \alpha \leq 2$ with $1 + \beta < \alpha$. The present work is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about fractional calculus and auxiliary results. The proofs of the main results, based on the Banach fixed point theorem and Schaefer's fixed point theorem, are presented in Section 3. An illustrative example is presented in Section 4.

The present results extend those obtained in [19] in the case of a right-hand side independent of the fractional derivatives.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $|\cdot|$ be a suitable norm in $\mathbb{R}^n$ and $\|\cdot\|$ be the matrix norm. By $C(J,\mathbb{R}^n)$ we denote the Banach space of continuous functions from J into $\mathbb{R}^n$ with the norm $\|x\|_\infty = \sup\big\{|x(t)| : t \in J\big\}$. Denote by $L^1(J,\mathbb{R}^n)$ the space of Lebesgue-integrable functions $x\colon J \to \mathbb{R}^n$ with the norm

$$ \|x\|_{L^1} = \int_0^T |x(t)|\,\mathrm{d}t. $$

Let $C^1(J,\mathbb{R}^n) = \{x\colon J \to \mathbb{R}^n : x' \text{ exists and } x' \in C(J,\mathbb{R}^n)\}$ be the Banach space with the norm $\|x\|_1 = \max\big\{\|x\|_\infty, \|x'\|_\infty\big\}$, and set

$$ AC^n(J) = \{x\colon J \to \mathbb{R}^n : x, x', \dots, x^{(n-1)} \in C(J,\mathbb{R}^n) \text{ and } x^{(n-1)} \text{ is absolutely continuous}\}. $$

Definition 1 ([16]) *The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $h \in L^1((a,b],\mathbb{R})$, $0 \le a < b$, is given by*

$$I^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,\mathrm{d}s, \quad t \in (a,b],$$

where Γ is the Euler gamma function defined by $\Gamma(\zeta) = \int_0^{+\infty} t^{\zeta-1}e^{-t}\,\mathrm{d}t$, $\zeta > 0$.

Definition 2 ([16]) *The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $h \in L^1((a,b],\mathbb{R})$ is given by*

$$D^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \frac{h(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, \quad t \in (a,b],$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 3 ([16]) *The Caputo fractional derivative of order $\alpha > 0$ of a function $h \in AC^n([a,b])$ on $(a,b]$ is defined via the above Riemann–Liouville derivatives by*

$${}^{c}D^{\alpha}_{a^+}h(t) = \left(D^{\alpha}_{a^+}\left[h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right]\right)(t), \quad t \in (a,b].$$

Definition 4 ([16]) *The Caputo fractional derivative of order α of a function $h \in AC^n(J)$ is defined by*

$${}^{c}D^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s.$$

Remark 1 ([16]) *From the definitions of fractional integrals and Caputo derivatives, we have*

$$I^{\alpha}_{0^+}\left({}^{c}D^{\alpha}_{0^+}h(t)\right) = h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(0)}{k!}t^k, \quad t > 0, \; n-1 < \alpha < n.$$

Especially, if $1 < \alpha < 2$, then we have $I^{\alpha}_{0^+}\left({}^{c}D^{\alpha}_{0^+}h(t)\right) = h(t) - h(0) - th'(0)$.

Lemma 1 ([16]) *Let $\alpha > 0$ and $h \in C(J,\mathbb{R})$. Then, the equality ${}^{c}D^{\alpha}_{0^+}\left(I^{\alpha}_{0^+}h(t)\right) = h(t)$ holds on $[a,b]$.*

Lemma 2 ([12]) *Let $0 < \beta < 1 < \alpha < 2$. Then, we have*

$$I^{\alpha}_{0^+}\left({}^{c}D^{\beta}_{0^+}x(t)\right) = I^{\alpha-\beta}_{0^+}x(t) - \frac{x(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.$$

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 3 ([22]) *Let $\upsilon\colon\ [0,T] \longrightarrow [0,+\infty)$ be a real function and let $\omega(\cdot)$ be a non-negative, locally integrable function on $[0,T]$. Assume that there are constants a > 0 and $0 < \alpha \leq 1$ such that*

$$\upsilon\left(t\right) \leq \omega\left(t\right) + a\int_0^t \left(t-s\right)^{-\alpha} \upsilon\left(s\right)\, \mathrm{d}s.$$

Then, there exists a constant $K = K\left(\alpha\right)$ such that

$$\upsilon\left(t\right) \leq \omega\left(t\right) + Ka\int_0^t \left(t-s\right)^{-\alpha} \omega\left(s\right)\, \mathrm{d}s \text{ for every } t \in \left[0,T\right].$$

Theorem 1 (Banach's fixed point theorem, see [14]) *Let C be a non-empty closed subset of a Banach space X. Then, any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2 (Schaefer's fixed point theorem, see [14]) *Let X be a Banach space, and let $N\colon X \longrightarrow X$ be a completely continuous operator. If the set $\mathcal{E} = \{y \in X : y = \lambda N y \text{ for some } \lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3 Existence of solutions

Lemma 4 *For any $x \in C^1(J,\mathbb{R}^n)$ and $0 < \beta \leq 1$, we have*

$$\|{}^cD^{\beta}_{0^+}x\|_\infty \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x'\|_\infty, \text{ and so } \|{}^cD^{\beta}_{0^+}x\|_\infty \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x\|_1.$$

Proof. Obviously, when $\beta = 1$, the conclusions are true. So, we only consider the case $0 < \beta < 1$. We observe that for all $0 \leq t \leq T$,

$$\int_0^t (t-s)^{-\beta}\,\mathrm{d}s = \frac{1}{1-\beta}t^{1-\beta} \leq \frac{1}{1-\beta}T^{1-\beta} = \int_0^T (T-s)^{-\beta}\,\mathrm{d}s.$$

Then, by Definition 4, for any $x \in C^1(J,\mathbb{R}^n)$ and $t \in J$, we have

$$\begin{aligned} |{}^cD^{\beta}_{0^+}x(t)| &= \frac{1}{\Gamma(1-\beta)}\left|\int_0^t (t-s)^{-\beta}x'(s)\,\mathrm{d}s\right| \\ &\leq \|x'\|_\infty\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\,\mathrm{d}s \\ &\leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x'\|_\infty. \end{aligned}$$

The proof is complete. □

Lemma 5 *Let $0 < \beta \leq 1 < \alpha \leq 2$ and $h \in C(J, \mathbb{R}^n)$. The function x is a solution of the following fractional dynamical system*

$$^cD^{\alpha}_{0+}x(t) - A^cD^{\beta}_{0+}x(t) = h(t),\ t \in J, \tag{3.1}$$

$$x(0) = x_0,\ \ x'(0) = x_0^*, \tag{3.2}$$

if and only if $x \in C^1(J, \mathbb{R}^n)$ is a solution of the integral equation

$$\begin{aligned} x(t) &= x_0 + tx_0^* - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0 + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Proof. Let $x \in C^1(J, \mathbb{R}^n)$ be a solution of the dynamical system (3.1)–(3.2). Then, by applying I^{α}_{0+} to both sides of equation (3.1) and using Remark 1, we get

$$\begin{aligned} x(t) &= x(0) + tx'(0) - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,\mathrm{d}s. \end{aligned}$$

Applying the conditions (3.2), we obtain $x(0) = x_0$ and $x'(0) = x_0^*$. Thus, x is a solution of the integral equation (3.3).

Conversely, assume that x satisfies the fractional integral equation (3.3). Using the fact that $^cD^{\alpha}$ is the left inverse of I^{α}, and the fact that $^cD^{\alpha}C = 0$, where C is a constant, we get

$$^cD^{\alpha}_{0+}x(t) - A^cD^{\beta}_{0+}x(t) = h(t) \ \text{ for each } t \in J.$$

Also, we can easily show that $x(0) = x_0$ and $x'(0) = x_0^*$. □

We are now in a position to state and prove our existence result for the dynamical system (1.1)–(1.2) using Banach's fixed point theorem. Let us introduce the following hypotheses.

(H1) The function $f\colon J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(H2) There exist constants $L_1, L_2 > 0$ and $0 < L_3 < 1$ such that

$$|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})| \leq L_1|u-\bar{u}| + L_2|v-\bar{v}| + L_3|w-\bar{w}|$$

for any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^n$ and $t \in J$.

Set

$$R_1 = \frac{T^{\alpha}L_1\Gamma(2-\beta) + T^{1-\beta+\alpha}\left(L_2 + L_3\|A\|\right)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)} + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$$

and

$$R_2 = \frac{T^{\alpha-1}L_1\Gamma(2-\beta) + T^{\alpha-\beta}\left(L_2 + L_3\|A\|\right)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)} + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}.$$

Theorem 3 *Assume that* **(H1)** *and* **(H2)** *hold. If*

$$\max\{R_1, R_2\} < 1, \tag{3.4}$$

then the dynamical system (1.1)–(1.2) *has a unique solution on* J.

Proof. Transform the dynamical system (1.1)–(1.2) into a fixed point problem. Consider the operator $N\colon C^1(J,\mathbb{R}^n) \to C^1(J,\mathbb{R}^n)$ defined by

$$\begin{aligned}(Nx)(t) &= x_0 + tx_0^* - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0 + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)\,\mathrm{d}s,\end{aligned} \tag{3.5}$$

where $g \in C(J,\mathbb{R}^n)$ is such that

$$g(t) = f\big(t, x(t), {}^cD_{0+}^{\beta}x(t), g(t) + A{}^cD_{0+}^{\beta}x(t)\big).$$

For every $x \in C^1(J,\mathbb{R}^n)$ and $t \in J$, we have

$$\begin{aligned}(Nx)'(t) &= x_0^* - \frac{(\alpha-\beta)At^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}x_0 + \frac{A}{\Gamma(\alpha-\beta-1)}\int_0^t (t-s)^{\alpha-\beta-2}x(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}g(s)\,\mathrm{d}s,\end{aligned} \tag{3.6}$$

and it is clear that $(Nx)' \in C(J,\mathbb{R}^n)$ due to **(H1)**. Consequently, N is well-defined.

Clearly, the fixed points of the operator N are solutions of the dynamical system (1.1)–(1.2).

Similar to the observation we made in the first part of the proof of Lemma 4, we observe that for each $t \in J$,

$$\frac{1}{\Gamma(\alpha-\beta-1)}\int_0^t (t-s)^{\alpha-\beta-2}\,\mathrm{d}s \le \frac{1}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\,\mathrm{d}s$$

and

$$\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-\beta-1}\,\mathrm{d}s \le \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\,\mathrm{d}s.$$

Let $x, y \in C^1(J,\mathbb{R}^n)$. Then, for each $t \in J$, we have

$$\begin{aligned}|(Nx)(t) - (Ny)(t)| &\le \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}|x(s)-y(s)|\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|g(s)-h(s)|\,\mathrm{d}s\end{aligned}$$

for $h \in C(J,\mathbb{R}^n)$ such that

$$h(t) = f\big(t, y(t), {}^cD_{0+}^{\beta}y(t), h(t) + A{}^cD_{0+}^{\beta}y(t)\big).$$

From **(H2)** for each $t \in J$ we have

$$\begin{aligned}|g(t)-h(t)| &\le L_1|x(t)-y(t)| + L_2\big|{}^cD^{\beta}_{0^+}x(t) -{}^c D^{\beta}_{0^+}y(t)\big|\\ &\quad + L_3\big|g(t)+A{}^cD^{\beta}_{0^+}x(t)-h(t)-A{}^cD^{\beta}_{0^+}y(t)\big|\\ &\le L_1|x(t)-y(t)| + L_2|g(t)-h(t)| + L_2\big|{}^cD^{\beta}_{0^+}x(t) -{}^c D^{\beta}_{0^+}y(t)\big|\\ &\quad + L_3\|A\|\big|{}^cD^{\beta}_{0^+}x(t) -{}^c D^{\beta}_{0^+}y(t)\big|\\ &\le L_1|x(t)-y(t)| + L_3|g(t)-h(t)| + (L_3\|A\|+L_2)\left|{}^cD^{\beta}_{0^+}\left(x(t)-y(t)\right)\right|.\end{aligned}$$

Thus,

$$\begin{aligned}|g(t)-h(t)| &\le \frac{L_1}{1-L_3}|x(t)-y(t)| + \frac{L_3\|A\|+L_2}{1-L_3}\left|{}^cD^{\beta}_{0^+}\left(x(t)-y(t)\right)\right|\\ &\le \frac{L_1}{1-L_3}\|x-y\|_{\infty} + \frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}\left(x-y\right)\right\|_{\infty}.\end{aligned}$$

Then, by using Lemma 4, we have

$$\begin{aligned}|g(t)-h(t)| &\le \frac{L_1}{1-L_3}\|x-y\|_1 + \frac{T^{1-\beta}(L_3\|A\|+L_2)}{\Gamma(2-\beta)(1-L_3)}\|x-y\|_1\\ &= \frac{L_1\Gamma(2-\beta)+T^{1-\beta}\left(L_3\|A\|+L_2\right)}{\Gamma(2-\beta)(1-L_3)}\|x-y\|_1.\end{aligned} \tag{3.7}$$

Therefore, for each $t \in J$,

$$\begin{aligned}&|(Nx)(t)-(Ny)(t)|\\ &\quad\le \left[\frac{T^{\alpha}L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}\left(L_3\|A\|+L_2\right)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)} + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]\|x-y\|_1 = R_1\|x-y\|_1.\end{aligned}$$

On the other hand, for each $t \in J$,

$$\begin{aligned}|(Nx)'(t)-(Ny)'(t)| &\le \frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}|x(s)-y(s)|\,\mathrm{d}s\\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}|g(s)-h(s)|\,\mathrm{d}s.\end{aligned}$$

Using (3.7), we get

$$\begin{aligned}&|(Nx)'(t)-(Ny)'(t)|\\ &\quad\le \left[\frac{T^{\alpha-1}L_1\Gamma(2-\beta)+T^{\alpha-\beta}\left(L_3\|A\|+L_2\right)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)} + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right]\|x-y\|_1 = R_2\|x-y\|_1.\end{aligned}$$

Thus,

$$\|N(x)-N(y)\|_1 \le \max\{R_1,R_2\}\|x-y\|_1.$$

By (3.4), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is the unique solution of the dynamical systems (1.1)–(1.2). □

Our second result is based on Schaefer's fixed point theorem. Set

$$R = \frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-1}(T+\alpha)}{\Gamma(\alpha+1)}\left(\frac{L_1}{1-L_3} + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right).$$

Theorem 4 *Assume that* **(H1)** *and* **(H2)** *hold. If* $R < 1$, *then the dynamical system* (1.1)–(1.2) *has at least one solution.*

Proof. Let N be the operator defined in (3.5). We shall use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.* Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C^1(J, \mathbb{R}^n)$. Then, for each $t \in J$,

$$\begin{aligned}|(Nx)(t) - (Nx_n)(t)| \leq &\frac{\|A\|}{\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} |x(s) - x_n(s)| \, \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |g(s) - g_n(s)| \, \mathrm{d}s,\end{aligned}$$

where $g_n \in C(J, \mathbb{R}^n)$ is such that

$$g_n(t) = f\big(t, x_n(t), {}^c D_{0^+}^{\beta} x_n(t), g_n(t) + A^c D_{0^+}^{\beta} x_n(t)\big).$$

From **(H2)** for each $t \in J$ we have

$$\begin{aligned}|g(t) - g_n(t)| &\leq L_1 |x(t) - x_n(t)| + L_3 \big|g(t) + A^c D_{0^+}^{\beta} x(t) - g_n(t) - A^c D_{0^+}^{\beta} x_n(t)\big| \\ &\quad + L_2 \big|{}^c D_{0^+}^{\beta} x(t) - {}^c D_{0^+}^{\beta} x_n(t)\big| \\ &\leq L_1 |x(t) - x_n(t)| + L_2 |g(t) - g_n(t)| \\ &\quad + L_3 \|A\| \big|{}^c D_{0^+}^{\beta} x(t) - {}^c D_{0^+}^{\beta} x_n(t)\big| + L_2 \big|{}^c D_{0^+}^{\beta} x(t) - {}^c D_{0^+}^{\beta} x_n(t)\big| \\ &\leq L_1 |x(t) - x_n(t)| + L_3 |g(t) - g_n(t)| \\ &\quad + (L_3 \|A\| + L_2) \big|{}^c D_{0^+}^{\beta} (x(t) - x_n(t))\big|.\end{aligned}$$

Thus,

$$\begin{aligned}|g(t) - g_n(t)| &\leq \frac{L_1}{1 - L_3} |x(t) - x_n(t)| + \frac{L_3 \|A\| + L_2}{1 - L_3} \big|{}^c D_{0^+}^{\beta} (x(t) - x_n(t))\big| \\ &\leq \frac{L_1}{1 - L_3} \|x - x_n\|_{\infty} + \frac{L_3 \|A\| + L_2}{1 - L_3} \big\|{}^c D_{0^+}^{\beta} (x - x_n)\big\|_{\infty}.\end{aligned}$$

Then, by using Lemma 4, we have

$$\begin{aligned}|g(t) - g_n(t)| &\leq \frac{L_1}{1 - L_3} \|x - x_n\|_1 + \frac{T^{1-\beta}(L_3 \|A\| + L_2)}{\Gamma(2-\beta)(1 - L_3)} \|x - x_n\|_1 \\ &= \frac{L_1 \Gamma(2-\beta) + T^{1-\beta}(L_3 \|A\| + L_2)}{\Gamma(2-\beta)(1 - L_3)} \|x - x_n\|_1.\end{aligned} \tag{3.8}$$

Therefore, for each $t \in J$ we get

$$\begin{aligned}&|(Nx)(t) - (Nx_n)(t)| \\ &\quad \leq \left[\frac{T^{\alpha} L_1 \Gamma(2-\beta) + T^{1-\beta+\alpha}(L_3 \|A\| + L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1 - L_3)} + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] \|x - x_n\|_1 = R_1 \|x - x_n\|_1.\end{aligned}$$

On the other hand, for each $t \in J$ we have

$$|(Nx)'(t) - (Nx_n)'(t)| \leq \frac{\|A\|}{\Gamma(\alpha-\beta-1)} \int_0^T (T-s)^{\alpha-\beta-2} |x(s) - x_n(s)| \, \mathrm{d}s + \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} |g(s) - g_n(s)| \, \mathrm{d}s.$$

Using (3.8), we get

$$|(Nx)'(t) - (Nx_n)'(t)| \leq \left[\frac{T^{\alpha-1} L_1 \Gamma(2-\beta) + T^{\alpha-\beta} \left(L_3\|A\| + L_2\right)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)} + \frac{\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right] \|x - x_n\|_1 = R_2 \|x - x_n\|_1.$$

Thus, $\|Nx - Nx_n\|_1 \to 0$ as $n \to \infty$, which implies that the operator N is continuous.

Step 2: *N maps bounded sets into bounded sets in $C^1(J, \mathbb{R}^n)$.* Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $x \in B_{\eta^*} = \{x \in C^1(J, \mathbb{R}^n) : ||x||_1 \leq \eta^*\}$ we have $\|N(x)\|_1 \leq \ell$. We have for each $t \in J$,

$$\begin{aligned} |g(t)| &= \left| f\left(t, x(t), {}^c D_{0^+}^{\beta} x(t), g(t) + A {}^c D_{0^+}^{\beta} x(t)\right) - f(t,0,0,0) + f(t,0,0,0) \right| \\ &\leq L_1 |x(t)| + L_3 \left| g(t) + A {}^c D_{0^+}^{\beta} x(t) \right| + L_2 \left| {}^c D_{0^+}^{\beta} x(t) \right| + |f(t,0,0,0)| \\ &\leq L_1 \|x\|_\infty + L_3 |g(t)| + (L_3\|A\| + L_2) \left\| D_{0^+}^{\beta} x \right\|_\infty + f^*, \end{aligned}$$

where $f^* = \sup_{t \in J} |f(t,0,0,0)|$. Thus,

$$|g(t)| \leq \frac{L_1}{1-L_3} \|x\|_\infty + \frac{L_3\|A\| + L_2}{1-L_3} \|D_{0^+}^{\beta} x\|_\infty + \frac{f^*}{1-L_3}.$$

Then, by using Lemma 4, we have

$$\begin{aligned} |g(t)| &\leq \frac{L_1}{1-L_3} \|x\|_\infty + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \|x'\|_\infty + \frac{f^*}{1-L_3} \\ &\leq \frac{L_1}{1-L_3} \|x\|_1 + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \|x\|_1 + \frac{f^*}{1-L_3} \\ &\leq \frac{L_1}{1-L_3} \eta^* + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \eta^* + \frac{f^*}{1-L_3} := M_1. \end{aligned} \tag{3.9}$$

Thus, (3.5) implies

$$|(Nx)(t)| \leq |x_0| + T|x_0^*| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \eta^* + \frac{T^{\alpha}}{\Gamma(\alpha+1)} M_1 := \ell_1.$$

On the other hand, for each $t \in J$ we have

$$|(Nx)'(t)| \leq |x_0^*| + \frac{(\alpha-\beta)\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \eta^* + \frac{T^{\alpha-1}}{\Gamma(\alpha)} M_1 := \ell_2.$$

Thus, $\|Nx\|_1 \leq \max\{\ell_1, \ell_2\} := \ell$.

Step 3: *N maps bounded sets into equicontinuous sets of* $C^1(J,\mathbb{R}^n)$. Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of $C^1(J,\mathbb{R}^n)$ as in Step 2, and let $x \in B_{\eta^*}$. Then,

$$\begin{aligned}
|(Nx)(t_2) - (Nx)(t_1)| &\leq |x_0^*|(t_2 - t_1) + \frac{\|A\||x_0|}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}) \\
&\quad + \frac{\|A\|\eta^*}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-\beta-1}\,\mathrm{d}s \\
&\quad + \frac{\|A\|\eta^*}{\Gamma(\alpha-\beta)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-\beta-1} - (t_1 - s)^{\alpha-\beta-1}\right]\mathrm{d}s \\
&\quad + \frac{M_1}{\Gamma(\alpha)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\,\mathrm{d}s \right. \\
&\quad \left. + \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right]\mathrm{d}s\right] \\
&\leq |x_0^*|(t_2 - t_1) + \frac{\|A\||x_0|}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}) \\
&\quad + \frac{\|A\|\eta^*}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}) + \frac{M_1}{\Gamma(\alpha+1)}(t_2^{\alpha} - t_1^{\alpha}).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Now, from (3.6) we have

$$\begin{aligned}
|(Nx)'(t_2) - (Nx)'(t_1)| &\leq \frac{(\alpha-\beta)\|A\||x_0|}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) \\
&\quad + \frac{\|A\|\eta^*}{\Gamma(\alpha-\beta-1)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-\beta-2}\,\mathrm{d}s \\
&\quad + \frac{\|A\|\eta^*}{\Gamma(\alpha-\beta-1)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-\beta-2} - (t_1 - s)^{\alpha-\beta-2}\right]\mathrm{d}s \\
&\quad + \frac{M_1}{\Gamma(\alpha-1)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\alpha-2}\,\mathrm{d}s \right. \\
&\quad \left. + \int_0^{t_1} \left[(t_2 - s)^{\alpha-2} - (t_1 - s)^{\alpha-2}\right]\,\mathrm{d}s\right] \\
&\leq \frac{(\alpha-\beta)\|A\||x_0|}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) \\
&\quad + \frac{\|A\|\eta^*}{\Gamma(\alpha-\beta)}(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) + \frac{M_1}{\Gamma(\alpha)}(t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli–Arzelà theorem, we can conclude that $N\colon C^1(J,\mathbb{R}^n) \to C^1(J,\mathbb{R}^n)$ is completely continuous.

Step 4: *A priori bounds.* Now, it remains to show that the set

$$E = \{x \in C^1(J,\mathbb{R}^n) : x = \lambda N(x) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $x \in E$. Then, $x = \lambda N(x)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$x(t) = x_0 + \lambda t x_0^* - \frac{\lambda A t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} x_0 + \frac{\lambda A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s)\,\mathrm{d}s$$
$$+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,\mathrm{d}s.$$

From **(H2)** for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= \left| f\big(t, x(t), {}^c D_{0^+}^{\beta} x(t), g(t) + A {}^c D_{0^+}^{\beta} x(t)\big) - f(t,0,0,0) + f(t,0,0,0) \right| \\ &\le L_1 |x(t)| + L_3 \left| g(t) + A {}^c D_{0^+}^{\beta} x(t) \right| + L_2 \left| {}^c D_{0^+}^{\beta} x(t) \right| + |f(t,0,0,0)| \\ &\le L_1 \|x\|_\infty + L_3 |g(t)| + (L_3 \|A\| + L_2) \left\| D_{0^+}^{\beta} x \right\|_\infty + f^*.\end{aligned}$$

Thus,

$$|g(t)| \le \frac{L_1}{1-L_3} \|x\|_\infty + \frac{L_3\|A\| + L_2}{1-L_3} \|D_{0^+}^{\beta} x\|_\infty + \frac{f^*}{1-L_3}.$$

Then, by using Lemma 4, we have

$$\begin{aligned}|g(t)| &\le \frac{L_1}{1-L_3} \|x\|_\infty + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_2)\Gamma(2-\beta)} \|x'\|_\infty + \frac{f^*}{1-L_3} \\ &\le \left[\frac{L_1}{1-L_3} + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \right] \|x\|_1 + \frac{f^*}{1-L_3}.\end{aligned}$$

Thus, for each $t \in J$ we have

$$\begin{aligned}|x(t)| &\le |x_0| + T|x_0^*| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|x\|_1 \\ &\quad + \frac{T^\alpha}{\Gamma(\alpha+1)} \left[\frac{L_1}{1-L_3} + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \right] \|x\|_1 + \frac{f^* T^\alpha}{(1-L_3)\Gamma(\alpha+1)} \\ &\le \left[\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \left(\frac{L_1}{1-L_3} + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \right) \right] \|x\|_1 \\ &\quad + |x_0| + T|x_0^*| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{f^* T^\alpha}{(1-L_3)\Gamma(\alpha+1)}.\end{aligned}$$

This implies that

$$\begin{aligned}\|x\|_\infty \le \Bigg[&\frac{\|A\| T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)} \\ &+ \frac{T^{\alpha-1}(T+\alpha)}{\Gamma(\alpha+1)} \left(\frac{L_1}{1-L_3} + \frac{(L_3\|A\| + L_2) T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} \right) \Bigg] \|x\|_1 \\ &+ |x_0| + (T+1)|x_0^*| \\ &+ \frac{\|A\| T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{f^* T^{\alpha-1}(T+\alpha)}{(1-L_3)\Gamma(\alpha+1)}.\end{aligned} \tag{3.10}$$

On the other hand, for each $t \in J$ we have

$$
\begin{aligned}
|x'(t)| &\leq |x_0^*| + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\|x\|_1 \\
&\quad + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{L_1}{1-L_3} + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right]\|x\|_1 + \frac{f^*T^{\alpha-1}}{(1-L_3)\Gamma(\alpha)} \\
&\leq \left[\frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{L_1}{1-L_3} + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right)\right]\|x\|_1 \\
&\quad + |x_0^*| + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{f^*T^{\alpha-1}}{(1-L_3)\Gamma(\alpha)},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x'\|_\infty \leq \Bigg[&\frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)} \\
&+ \frac{T^{\alpha-1}(T+\alpha)}{\Gamma(\alpha+1)}\left(\frac{L_1}{1-L_3} + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right)\Bigg]\|x\|_1 \\
&+ |x_0| + (T+1)|x_0^*| \\
&+ \frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{f^*T^{\alpha-1}(T+\alpha)}{(1-L_3)\Gamma(\alpha+1)}.
\end{aligned} \tag{3.11}
$$

Now, from (3.10) and (3.11) we get

$$
\begin{aligned}
\|x\|_1 &\leq \Bigg[\frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)} \\
&\quad + \frac{T^{\alpha-1}(T+\alpha)}{\Gamma(\alpha+1)}\left(\frac{L_1}{1-L_2} + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right)\Bigg]\|x\|_1 \\
&\quad + |x_0| + (T+1)|x_0^*| + \frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{f^*T^{\alpha-1}(T+\alpha)}{(1-L_3)\Gamma(\alpha+1)} \\
&= R\|x\|_1 + |x_0| + (T+1)|x_0^*| \\
&\quad + \frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{f^*T^{\alpha-1}(T+\alpha)}{(1-L_3)\Gamma(\alpha+1)}.
\end{aligned}
$$

Since $R < 1$, we obtain

$$
\begin{aligned}
\|x\|_1 &\leq \frac{|x_0| + (T+1)|x_0^*|}{1-R} + \frac{\|A\|T^{\alpha-\beta-1}(T+\alpha-\beta)}{(1-R)\Gamma(\alpha-\beta+1)}|x_0| \\
&\quad + \frac{f^*T^{\alpha-1}(T+\alpha)}{(1-R)(1-L_3)\Gamma(\alpha+1)} \\
&=: \psi.
\end{aligned}
$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the dynamical system (1.1)–(1.2). $\square$

4 An example

Consider the following fractional dynamical system

$$ {}^cD^{\alpha}_{0^+}x(t) - A\,{}^cD^{\beta}_{0^+}x(t) = f\big(t, x(t), {}^cD^{\beta}_{0^+}x(t), {}^cD^{\alpha}_{0^+}x(t)\big), \;\; t \in [0,1], \tag{4.1} $$

$$ x(0) = (1,0), \;\; x'(0) = (0,1), \tag{4.2} $$

where $f\colon [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is such that $f = (f_1, f_2)$ with

$$ f_i(t,u,v,w) = \frac{c_i t^2}{1 + \|u\| + \|v\| + \|w\|}, \; i = 1,2, $$

$0 < c_i < 1$, $u, v, w \in \mathbb{R}^2$, $\|(x_1,x_2)\| = \max\{|x_1|, |x_2|\}$, $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\beta = \frac{1}{4}$ and $\alpha = \frac{3}{2}$.

It is clear that the hypotheses **(H1)** and **(H2)** are satisfied. Indeed, the function f is jointly continuous, and for any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^2$ and $t \in [0,1]$, we get

$$ \|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})\| \le L_1\|u - \bar{u}\| + L_2\|v - \bar{v}\| + L_3\|w - \bar{w}\| $$

with $L_1 = L_2 = L_3 = \max\{c_1, c_2\}$. Simple computations show that the condition $\max\{R_1, R_2\} < 1$ is satisfied for suitable choices of the constants c_1 and c_2. Then, all conditions of Theorem 3 are satisfied. It follows that the problem (4.1)–(4.2) has a unique solution defined on $[0,1]$.

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