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PROBLEMS WITHOUT INITIAL CONDITIONS FOR NONLINEAR EVOLUTION INCLUSIONS WITH VARIABLE TIME-DELAY

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Abstract. The well-posedness of problems without initial conditions for nonlinear parabolic variational inequalities with variable time-delay is proved. The estimates of weak solutions are obtained.

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1 Introduction

Evolution equations and inclusions with time delays have often arisen in modeling in physical physiology, biology, population dynamics, etc. Delay differential equations allow taking into account past actions into mathematical models, thus making the model closer to the real-world phenomenon. Delay terms can be of different types: constant, time dependent, state dependent etc.

In this paper we consider problem without initial conditions for evolution variational inequalities (inclusions) with a time-depended delay. Let us introduce an example of the problem being studied here.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \in \mathbb{N})$, $\partial\Omega$ be a piecewise boundary of Ω. We put $Q := \Omega \times (-\infty\,, 0]$, $\Sigma := \partial\Omega \times (-\infty\,, 0]$, $\Omega_t := \Omega \times \{t\}$, $\forall\, t \in \mathbb{R}$. For an arbitrary measurable set $F \subset \mathbb{R}^k$ $(k = n$ or $k = n+1)$, let $L^2(F)$ be the standard Lebesgue space. Let $L^2_{\rm loc}(\overline{Q})$ be the space of functions defined on Q such that their restrictions on any bounded measurable set

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$Q' \subset Q$ belong to $L^2(Q')$. Denote by $H^1(\Omega)$ the standard Sobolev space, *i.e.*, $H^1(\Omega) = \{v \in L^2(\Omega) \mid v_{x_i} \in L^2(\Omega)\ (i = \overline{1,n})\}$ with scalar product $(v\,,\,w)_{H^1(\Omega)} = \int_\Omega [\nabla v \nabla w + v\,w]\,\mathrm{d}x$, where $\nabla u := (u_{x_1}, \ldots, u_{x_n})$.

Let K be a convex closed set in $H^1(\Omega)$ which contains 0. Let us consider the problem of finding a function $u \in L^2_{\mathrm{loc}}(\overline{Q})$ such that $u_{x_i} \in L^2_{\mathrm{loc}}(\overline{Q})$ $(i = \overline{1,n})$, $u_t \in L^2_{\mathrm{loc}}(\overline{Q})$, and, for a.e. $t \in (-\infty\,,\,0]$, $u(\cdot,\,t) \in K$ and

$$\int_{\Omega_t} \{u_t(v-u) + \nabla u \nabla(v-u) + u(v-u) + (v-u)\int_{t-\tau(t)}^{t} u(x\,,\,s)\ \mathrm{d}s\}\,\mathrm{d}x$$
$$\geq \int_{\Omega_t} f(v-u)\,\mathrm{d}x\,, \quad \forall\, v \in K\,, \tag{1.1}$$

$$\lim_{t\to-\infty} ||u(\cdot,\,t)||_{L^2(\Omega)} = 0\,, \tag{1.2}$$

where $f \in L^2_{\mathrm{loc}}(\overline{Q}),\ \tau \in C((-\infty\,,\,0]),\ \tau(t) \geq 0,\ \forall t \in (-\infty\,,\,0],\ \sup\limits_{t\in(-\infty\,,\,0]} \tau(t) < 1.$

As it will be shown in the sequel the solution of this problem, which we call problem (1.1)–(1.2), is unique and it exists if $f \in L^2(Q)$.

We remark that problem (1.1)–(1.2) can be written in more abstract way. Indeed, after appropriate identification of functions and functionals, we have continuous and dense imbedding

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))',$$

where $(H^1(\Omega))'$ is dual to space $H^1(\Omega)$. Clearly, for any $h \in L^2(\Omega)$ and $v \in H^1(\Omega)$ we have $\langle h\,,\,v\rangle = (h\,,\,v)$, where $\langle\cdot,\cdot\rangle$ is the notation for scalar product on dual pair $(H^1(\Omega))'$, $H^1(\Omega)$, and $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$. Thus, we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$.

Now, we denote $S := (-\infty\,,\,0]$, $V := H^1(\Omega)$, $H := L^2(\Omega)$ and define an operator $A : V \to V'$ as follows

$$(Av\,,\,w) = \int_\Omega \left[\nabla v \nabla w + v\,w\right]\mathrm{d}x\,, \quad v\,,\,w \in V\,.$$

Then problem (1.1)–(1.2) can be rewritten as following: find a function $u \in L^2_{\mathrm{loc}}(S;V)$ such that $u' \in L^2_{\mathrm{loc}}(S;H)$ and for a.e. $t \in S$, $u(t) \in K$ and

$$(u'(t) + Au(t) + \int_{t-\tau(t)}^{t} u(s)\,\mathrm{d}s\,,\,v - u(t)) \geq (f(t)\,,\,v - u(t))\,, \quad \forall\, v \in K\,. \tag{1.3}$$

Here $f \in L^2_{\mathrm{loc}}(S;H)$ is a given function, function τ is as above.

We remark that variational inequality (1.3) can be written as a subdifferential inclusion. For this purpose we put $I_K(v) := 0$ if $v \in K$, and $I_K(v) := +\infty$ if $v \in V \setminus K$, and also

$$\Phi(v) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + |v|^2\right)\mathrm{d}x + I_K(v)\,, \quad v \in V\,.$$

It is easy to verify that the functional $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ is convex and semi-lower-continuous. By the known results (see, *e.g.*, [21, pp. 83]) it follows that the problem of finding a solution of variational inequality (1.3) can be written as such subdifferential inclusion: to find a function $u \in L^2_{\mathrm{loc}}(S;V)$ such that $u' \in L^2_{\mathrm{loc}}(S;H)$ and for a.e. $t \in S$

$$u'(t) + \partial\Phi(u(t)) + \int_{t-\tau(t)}^{t} u(s)\,\mathrm{d}s \ni f(t) \quad \text{in} \quad H\,. \tag{1.4}$$

The aim of this paper is to investigate problems for inclusions of type (1.4).

We have mentioned that initial-valued problems for evolution inclusions with constant delay previously were studied in [19, 25, 26] and others. Many results on such problems were obtained by using the semi-group theory. We refer the reader to [25] for more comments and citations. In [19,26] fixed point theorems were used.

Problems without initial conditions for evolution equations and inclusions arise in modeling different nonstationary processes in nature, that started a long time ago and initial conditions do not affect on them in the actual time moment. Thus, we can assume that the initial time is $-\infty$, while 0 is the final time, and initial conditions can be replaced with the behaviour of the solution as time variable tends to $-\infty$. Such problems appear in modeling in many fields of science such as ecology, economics, physics, cybernetics, etc. The research of the problem without initial conditions for the evolution equations and variational inequalities (without delay) were conducted in the monographs [15, 17, 21] and the papers [3, 4, 6, 7, 10, 14, 16, 18, 22–24] and others. In particular, R.E. Showalter in the paper [22] proved the existence of a unique solution $u \in \mathrm{e}^{2\omega\cdot}H^1(S;H)$, where H is a Hilbert space, of the problem without initial condition

$$u'(t) + \mu\, u(t) + A\big(u(t)\big) \ni f(t)\,, \quad t \in S\,,$$

for every $\omega + \mu > 0$ and $f \in \mathrm{e}^{2\omega\cdot}H^1(S;H)$, in case when $A : H \to 2^H$ is maximal monotone operator such that $0 \in A(0)$. Moreover, if $A = \partial\varphi$, where $\varphi : H \to (-\infty\,, +\infty]$ is proper, convex, and lower-semi-continuous functional such that $\varphi(0) = 0 = \inf\{\varphi(v) : v \in H\}$, then this problem is uniquely solvable for each $\mu > 0$, $f \in L^2(S;H)$ and $\omega = 0$.

Note that the uniqueness of the solutions of problem without initial conditions for linear parabolic equations and variational inequalities is possible only under some restrictions on the behavior of solutions as time variable turns to $-\infty$. For the first time it was strictly justified by A. N. Tikhonov [24] in the case of heat equation. However, as it was shown by M. M. Bokalo [3], problem without initial conditions for some nonlinear parabolic equations has a unique solution in the class of functions without behavior restriction as time variable tends to $-\infty$. Similar results were also obtained for evolutionary variational inequalities in the paper [4].

Previously, problems without initial conditions of evolution equations with constant delay were studied in [5, 11] (see also references therein), and with variable delay, as far as we know, only in [13]. Let us note that problems without initial conditions for variational inequalities or inclusions with delay have not been considered in the literature, which serves as one of the motivations for the study of such problems.

The outline of this paper is as follows. In Section 2, we give notations, definitions of function spaces and auxiliary results. In Section 3, we formulate the problem and main result. In Section 4, we prove the main result.

2 Preliminaries

Set $S := (-\infty\,,\,0]$. Let V and H be separable Hilbert spaces with the scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)$ and norms $\|\cdot\|$, $|\cdot|$, respectively. Suppose that $V \subset H$ with dense, continuous and compact injection, *i.e.*, the closure of V in H coincides with H, and there exists a constant $\lambda > 0$ such that

$$\lambda|v|^2 \le \|v\|^2 \quad \forall v \in V\,, \tag{2.1}$$

and for every sequence $\{v_k\}_{k=1}^{\infty}$ bounded in V there exist an element $v \in V$ and a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ such that $v_{k_j} \underset{j\to\infty}{\longrightarrow} v$ strongly in H.

Let V' and H' be the dual spaces to V and H, respectively. We suppose (after appropriate identification of functionals), that the space H' is a subspace of V'. By the Riesz–Fréchet representation theorem, identifying the spaces H and H', we obtain dense and continuous embeddings

$$V \subset H \subset V'\,. \tag{2.2}$$

Note that in this case $\langle g\,,\,v\rangle_V = (g\,,\,v)$ for every $v \in V$, $g \in H$, where $\langle\cdot,\cdot\rangle_V$ is the scalar product for the duality V', V. Therefore, further we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle_V$.

We introduce some spaces of functions and distributions. Let X be an arbitrary Hilbert space with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$. By $C(S;X)$ we mean the linear space of continuous functions defined on S with values in X. We say that $w_m \underset{m\to\infty}{\longrightarrow} w$ in $C(S;X)$ if for each $t_1,\, t_2 \in S$ $(t_1 < t_2)$ we have $\|w - w_m\|_{C([t_1\,,\,t_2];X)} \underset{m\to\infty}{\longrightarrow} 0$.

Denote by $L^2_{\text{loc}}(S;X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[t_1\,,\,t_2] \subset S$ belong to the space $L^2(t_1\,,\,t_2;X)$. We say that a sequence $\{w_m\}$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to w) in $L^2_{\text{loc}}(S;X)$, if for each $t_1, t_2 \in S$ $(t_1 < t_2)$ the sequence of restrictions of $\{w_m\}$ to the segment $[t_1, t_2]$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to the restrictions of w to this segment) in $L^2(t_1, t_2; X)$.

Let $\nu \in \mathbb{R}$. Put by definition

$$L^2_\nu(S;X) := \left\{ f \in L^2_{\text{loc}}(S;X) \;\Big|\; \int_S e^{-2\nu t}\|f(t)\|_X^2\,\mathrm{d}t < \infty \right\}.$$

This space is a Hilbert space with the scalar product

$$(f\,,\,g)_{L^2_\nu(S;X)} = \int_S e^{-2\nu t}(f(t)\,,\,g(t))_X\,\mathrm{d}t$$

and the corresponding norm

$$\|f\|_{L^2_\nu(S;X)} := \Big(\int_S e^{-2\nu t}\|f(t)\|_X^2\,\mathrm{d}t \Big)^{1/2}.$$

Also we introduce the space

$$L^\infty_\nu(S;X) := \{ f \in L^\infty(S;X) \mid \operatorname*{ess\,sup}_{t\in S} \big[e^{-\nu t}\|f(t)\|_X \big] < \infty \}\,.$$

By $D'(-\infty\,,\,0;V'_w)$ we mean the space of continuous linear functionals on $D(-\infty\,,\,0)$ with values in V'_w (hereafter $D(-\infty\,,\,0)$ is space of test functions, that is, the space of infinitely differentiable on $(-\infty\,,\,0)$ functions with compact support, equipped with the corresponding topology, and V'_w is the linear space V' equipped with weak topology). It is easy to see (using (2.2)), that spaces $L^2_{\mathrm{loc}}(S;V)$, $L^2_{\mathrm{loc}}(S;H)$, $L^2_{\mathrm{loc}}(S;V')$ can be identified with the corresponding subspaces of $D'(-\infty\,,\,0;V'_w)$. In particular, this allows us to talk about derivatives w' of functions w from $L^2_{\mathrm{loc}}(S;V)$ or $L^2_{\mathrm{loc}}(S;H)$ in the sense of distributions $D'(-\infty\,,\,0;V'_w)$ and belonging of such derivatives to $L^2_{\mathrm{loc}}(S;H)$ or $L^2_{\mathrm{loc}}(S;V')$.

Let us define the spaces

$$H^1_{\mathrm{loc}}(S;H) := \{w \in L^2_{\mathrm{loc}}(S;H) \,\big|\, w' \in L^2_{\mathrm{loc}}(S;H)\}\ ,$$

$$W_{2\,,\,\mathrm{loc}}(S;V) := \{w \in L^2_{\mathrm{loc}}(S;V) \,\big|\, w' \in L^2_{\mathrm{loc}}(S;V')\}\ .$$

From known results (see., for example, [12, pp. 177–179]) it follows that $H^1_{\mathrm{loc}}(S;H) \subset C(S;H)$ and $W_{2\,,\,\mathrm{loc}}(S;V) \subset C(S;H)$. Moreover, for every w in $H^1_{\mathrm{loc}}(S;H)$ or $W_{2\,,\,\mathrm{loc}}(S;V)$ the function $t \to |w(t)|^2$ is absolutely continuous on any segment of the interval S and the following equality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 = 2(w'(t)\,,\,w(t)) \quad \text{for a.e.} \quad t \in S\ . \tag{2.3}$$

Denote

$$H^1_\nu(S;H) := \{w \in L^2_\nu(S;H) \,\big|\, w' \in L^2_\nu(S;H)\}\ , \quad \nu \in \mathbb{R}\ . \tag{2.4}$$

In this paper we use the following well-known facts.

Lemma 2.1 (Cauchy-Schwarz inequality [12, pp. 158]) *Suppose that* $t_1,\,t_2 \in \mathbb{R}$ $(t_1 < t_2)$, *and* X *is a Hilbert space with the scalar product* $(\cdot,\cdot)_X$. *Then, if* $v\,,\,w \in L^2(t_1\,,\,t_2;X)$, *we have* $(w(\cdot)\,,\,v(\cdot))_X \in L^1\big(t_1\,,\,t_2\big)$ *and*

$$\int_{t_1}^{t_2}(w(t)\,,\,v(t))_X\,\mathrm{d}t \le \|w\|_{L^2(t_1\,,\,t_2;X)}\|v\|_{L^2(t_1\,,\,t_2;X)}\ .$$

Lemma 2.2 ([27, pp. 173,179]) *Let* Y *be a Banach space with the norm* $\|\cdot\|_Y$, *and* $\{v_k\}_{k=1}^\infty$ *be a sequence of elements of* Y, *which is weakly or* $*$*-weakly convergent to* v *in* Y. *Then* $\varliminf\limits_{k\to\infty}\|v_k\|_Y \ge \|v\|_Y$.

Lemma 2.3 ([1, Aubin theorem], [2, pp. 393]) *Let* $q > 1,\ r > 1,\ t_1,\,t_2 \in \mathbb{R}$ $(t_1 < t_2)$, *and* $\mathcal{W},\,\mathcal{L},\,\mathcal{B}$ *are Banach spaces such that* $\mathcal{W} \overset{c}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$ *(here* $\overset{c}{\subset}$ *means compact embedding and* $\circlearrowleft$ *means continuous embedding). Then*

$$\{w \in L^q(t_1\,,\,t_2;\mathcal{W}) \mid w' \in L^r(t_1\,,\,t_2;\mathcal{B})\} \overset{c}{\subset} \Big(L^q(t_1\,,\,t_2;\mathcal{L}) \cap C([t_1\,,\,t_2];\mathcal{B})\Big)\ . \tag{2.5}$$

Note that, we understand embedding (2.5) as follows: if a sequence $\{w_m\}$ is bounded in the space $L^q(t_1\,,\,t_2;\mathcal{W})$ and the sequence $\{w'_m\}_{m\in\mathbb{N}}$ is bounded in the space $L^r(t_1\,,\,t_2;\mathcal{B})$, then there exist a function $w \in C([t_1\,,\,t_2];\mathcal{B}) \cap L^q(t_1\,,\,t_2;\mathcal{L})$ and a subsequence $\{w_{m_j}\}$ of the sequence $\{w_m\}$ such that $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ in $C([t_1\,,\,t_2];\mathcal{B})$ and strongly in $L^q(t_1\,,\,t_2;\mathcal{L})$.

Lemma 2.4 *If a sequence $\{w_m\}$ is bounded in the space $L^2_{\rm loc}(S;V)$ and the sequence $\{w'_m\}$ is bounded in the space $L^2_{\rm loc}(S;H)$, then there exist a function $w \in L^2_{\rm loc}(S;V)$, $w' \in L^2_{\rm loc}(S;H)$, and a subsequence $\{w_{m_j}\}$ of the sequence $\{w_m\}$ such that $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ in $C(S;H)$ and weakly in $L^2_{\rm loc}(S;V)$, and, $w'_{m_j} \underset{j\to\infty}{\longrightarrow} w'$ weakly in $L^2_{\rm loc}(S;H)$.*

Proof. Lemma 2.3 for $q = 2$, $r = 2$, $\mathcal{W} = V$, $\mathcal{L} = \mathcal{B} = H$ and reflexiveness of Hilbert spaces yield, for every $t_1,\, t_2 \in S$ $(t_1 < t_2)$ from the sequence of restrictions of the elements $\{w_m\}$ to the segment $[t_1\,,\, t_2]$ one can choose a subsequence which is convergent in $C([t_1\,,\, t_2];H)$ and weakly in $L^2(t_1\,,\, t_2;V)$, and the sequence of derivatives of the elements of this subsequence is weakly convergent in $L^2(t_1\,,\, t_2;H)$. For each $k \in \mathbb{N}$ we choose a subsequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$ of the given sequence which is convergent in $C([-k\,,\,0];H)$ and weakly in $L^2(-k\,,\,0;V)$ to some function $\widehat{w}_k \in C([-k\,,\,0];H) \cap L^2(-k\,,\,0;V)$, and the sequence $\{w'_{m_{k,j}}\}_{j=1}^{\infty}$ is weakly convergent to the derivative $\widehat{w}'_k$ in $L^2(-k\,,\,0;H)$. Making this choice we ensure that the sequence $\{w_{m_{k+1,j}}\}_{j=1}^{\infty}$ was a subsequence of the sequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$. Now, according to the diagonal process we select the desired subsequence as $\{w_{m_{j,j}}\}_{j=1}^{\infty}$, and we define the function w as follows: for each $k \in \mathbb{N}$ we take $w(t) := \widehat{w}_k(t)$ for $t \in (-k\,,\,-k+1]$. □

3 Statement of the problem and main result

Let $\Phi : V \to \mathbb{R}_\infty := (-\infty\,,\,+\infty]$ be a proper functional, *i.e.*, $\mathrm{dom}(\Phi) := \{v \in V :\ \Phi(v) < +\infty\} \neq \emptyset$, which satisfies the conditions:

$(\mathcal{A}_1)$ $\quad \Phi\big(\alpha\, v + (1-\alpha)w\big) \le \alpha\,\Phi(v) + (1-\alpha)\Phi(w) \quad \forall\, v,\, w \in V,\ \forall\, \alpha \in [0\,,\,1]\,,$
i.e., the functional Φ is *convex*,

$(\mathcal{A}_2)$ $\quad v_k \underset{k\to\infty}{\longrightarrow} v$ in $V \quad \Longrightarrow \quad \underset{k\to\infty}{\underline{\lim}}\ \Phi(v_k) \ge \Phi(v)\,,$
i.e., the functional Φ is *lower semicontinuous*.

Recall that the *subdifferential* of functional Φ is a mapping $\partial\Phi : V \to 2^{V'}$, defined as follows

$$\partial\Phi(v) := \{v^* \in V' \mid \Phi(w) \ge \Phi(v) + (v^*,\, w - v) \quad \forall\, w \in V\}\,, \quad v \in V\,,$$

and the *domain* of the subdifferential $\partial\Phi$ is the set $D(\partial\Phi) := \{v \in V \,|\, \partial\Phi(v) \neq \emptyset\}$. We identify the subdifferential $\partial\Phi$ with its graph, assuming that $[v\,,\, v^*] \in \partial\Phi$ if and only if $v^* \in \partial\Phi(v)$, *i.e.*, $\partial\Phi = \{[v\,,\, v^*] \mid v \in D(\partial\Phi),\ v^* \in \partial\Phi(v)\}$. R. Rockafellar in paper [20, Theorem A] proves that the subdifferential $\partial\Phi$ is a *maximal monotone operator*, that is,

$$(v_1^* - v_2^*\,,\, v_1 - v_2) \ge 0 \quad \forall\, [v_1\,,\, v_1^*],\ [v_2\,,\, v_2^*] \in \partial\Phi$$

and for every element $[v_1\,,\, v_1^*] \in V \times V'$ we have the implication

$$(v_1^* - v_2^*\,,\, v_1 - v_2) \ge 0 \quad \forall\, [v_2\,,\, v_2^*] \in \partial\Phi \quad \Longrightarrow \quad [v_1\,,\, v_1^*] \in \partial\Phi\,.$$

Let $c : S \times S \times H \to H$ be a function which satisfies the condition:

$(\mathcal{C})$ for any $v \in H$ the mapping $c(\cdot, \cdot, v) : S \times S \to H$ is measurable, and there exists a constant $L > 0$ such that following inequality holds

$$|c(t\,,\,s\,,\,v_1) - c(t\,,\,s\,,\,v_2)| \le L|v_1 - v_2| \tag{3.1}$$

for a.e. $(t\,,\,s) \in S \times S,$ and for all $v_1\,,\,v_2 \in H$; in addition, $c(t\,,\,s\,,\,0) = 0$ for a.e. $(t\,,\,s) \in S \times S$.

Remark 3.1 *From the condition ($\mathcal{C}$) it follows that for a.e. $(t\,,\,s) \in S \times S$, and for every $v \in H$ the following estimate is valid:*

$$|c(t\,,\,s\,,\,v)| \le L|v|\,. \tag{3.2}$$

Let us consider the evolutionary variational inequality

$$u'(t) + \partial\Phi\big(u(t)\big) + \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,u(s))\,\mathrm{d}s \ni f(t)\,, \quad t \in S\,, \tag{3.3}$$

where $f : S \to V'$ is a given measurable function, $\tau : S \to \mathbb{R}$ is a given bounded continuous function such that $\tau(t) \ge 0$ for all $t \in S$, and $u : S \to V$ is an unknown function.

Definition 3.2 *Let conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{C})$ hold, and $f \in L^2_{\mathrm{loc}}(S; V')$. The* solution *of variational inequality* (3.3) *is a function $u : S \to V$ that satisfies the following conditions:*

1) $u \in W_{2\,,\,\mathrm{loc}}(S; V)$*;*

2) $u(t) \in D(\partial\Phi)$ *for a.e.* $t \in S$*;*

3) *there exists a function $g \in L^2_{\mathrm{loc}}(S; V')$ such that for a.e. $t \in S$ we have $g(t) \in \partial\Phi\big(u(t)\big)$ and*

$$u'(t) + g(t) + \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,u(s))\,\mathrm{d}s = f(t) \quad \textit{in } V'\,.$$

For variational inequality (3.3) consider the problem: find its solution which satisfies the condition

$$\lim_{t\to-\infty} e^{-\gamma\,t}|u(t)| = 0\,, \tag{3.4}$$

where $\gamma \in \mathbb{R}$ is given.

The problem of finding a solution of variational inequality (3.3) for given $\Phi,\ c,\ \tau,\ f$ satisfying the condition (3.4) for given γ, is called the problem without initial conditions for the evolution variational inequality (3.3) or, in short, the problem $\mathbf{P}(\Phi\,,\,c\,,\,\tau\,,\,f\,,\,\gamma)$, and the function u is called its solution.

Additionally, assume that the following conditions hold:

$(\mathcal{A}_3)$ there exists a constant $K_1 > 0$ such that

$$\Phi(v) \ge K_1\|v\|^2 \quad \forall\, v \in \mathrm{dom}(\Phi)\,;$$

moreover, $\Phi(0) = 0$;

$(\mathcal{A}_4)$ there exists a constant $K_2 > 0$ such that

$$(v_1^* - v_2^*\,,\, v_1 - v_2) \geq K_2 |v_1 - v_2|^2 \quad \forall\, [v_1\,,\, v_1^*],\ [v_2\,,\, v_2^*] \in \partial\Phi\,.$$

Remark 3.3 *Condition $(\mathcal{A}_3)$ implies that $\Phi(v) \geq \Phi(0) + (0\,,\, v - 0)$, $\forall v \in V$, hence $[0\,,\, 0] \in \partial\Phi$. From this and condition $(\mathcal{A}_4)$ we have*

$$(v^*\,,\, v) \geq K_2 |v|^2 \quad \forall\, [v\,,\, v^*] \in \partial\Phi\,. \tag{3.5}$$

We denote

$$\tau^+ := \sup_{t \in S} \tau(t)\,, \quad \chi(\gamma) := \begin{cases} \frac{1}{2\gamma}(1 - e^{-2\gamma\tau^+})\,, & \text{if } \gamma \neq 0\,, \\ \tau^+\,, & \text{if } \gamma = 0\,, \end{cases} \tag{3.6}$$

and consider the following inequality

$$K_2 + \gamma - L\sqrt{\tau^+ \chi(\gamma)} > 0\,. \tag{3.7}$$

It is obvious that $L\sqrt{\tau^+\chi(\gamma)} \to 0$ as $\gamma \to +\infty$, because $\chi(\gamma) \to 0$ as $\gamma \to +\infty$. Since $K_2 + \gamma \to +\infty$ as $\gamma \to +\infty$, inequality (3.7) has solutions.

Now we shall formulate the main result.

Theorem 3.4 *Let conditions $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$, $(\mathcal{C})$ hold, and*

$(\mathcal{F})$ $\quad f \in L^2_\gamma(S; H)\,,$

where γ is a solution of inequality (3.7)*. Then the problem $\boldsymbol{P}(\Phi\,,\, c\,,\, \tau\,,\, f\,,\, \gamma)$ has a unique solution, it belongs to the space $L^\infty_\gamma(S; V) \cap L^2_\gamma(S; V) \cap H^1_\gamma(S; H)$ and satisfies the estimate:*

$$\begin{aligned} &e^{-2\gamma\sigma}\|u(\sigma)\|^2 + \int_{-\infty}^{\sigma} e^{-2\gamma t}\|u(t)\|^2\,\mathrm{d}t + \int_{-\infty}^{\sigma} e^{-2\gamma t}|u'(t)|^2\,\mathrm{d}t \\ &+ \int_{-\infty}^{\sigma} e^{-2\gamma t}\Phi(u(t))\,\mathrm{d}t \leq C_1 \int_{-\infty}^{\sigma} e^{-2\gamma t}|f(t)|^2\,\mathrm{d}t\,, \quad \sigma \in S\,, \end{aligned} \tag{3.8}$$

where C_1 is a positive constant depending on K_1, K_2, L, τ^+, λ and γ only.

Remark 3.5 *The problem $\boldsymbol{P}(\Phi\,,\, c\,,\, \tau\,,\, f\,,\, \gamma)$ can be replaced by the following problem. Let K be a convex and closed set in V, $A : V \to V'$ be a monotone, bounded and semi-continuous operator such that $(A(v)\,,\, v) \geq \widetilde{K}_1\|v\|^2 \quad \forall v \in V$, where $\widetilde{K}_1 = const > 0$. The problem is to find a function $u \in W_{2\,,\,\mathrm{loc}}(S; V)$ satisfying the condition* (3.4) *and for a.e. $t \in S$, $u(t) \in K$ and*

$$(u'(t) + A(u(t)) + \int_{t-\tau(t)}^{t} c(t\,,\, s\,,\, u(s))\,\mathrm{d}s\,, v - u(t)) \geq (f(t)\,,\, v - u(t)) \quad \forall\, v \in K\,.$$

4 Proof of the main result

First, we define the functional $\Phi_H : H \to \mathbb{R}_\infty$ by the rule: $\Phi_H(v) := \Phi(v)$, if $v \in V$, and $\Phi_H(v) := +\infty$ otherwise. Note that conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, Lemma IV.5.2 and Proposition IV.5.2 of the monograph [21] imply that Φ_H is a proper, convex, and lower-semi-continuous functional on H, $\mathrm{dom}(\Phi_H) = \mathrm{dom}(\Phi) \subset V$ and $\partial\Phi_H = \partial\Phi \cap (V \times H)$, where $\partial\Phi_H : H \to 2^H$ is the subdifferential of the functional Φ_H. Moreover, condition $(\mathcal{A}_3)$ yields $0 \in \partial\Phi_H(0)$.

The following statements will be used in the sequel.

Lemma 4.1 ([21, Lemma IV.4.3]) *Let $w \in H^1(a\,,\,b;H)$ $(-\infty < a < b < +\infty)$, and there exists $g \in L^2(a\,,\,b;H)$ such that $g(t) \in \partial\Phi_H\big(w(t)\big)$ for a.e. $t \in (a\,,\,b)$. Then the function $\Phi_H\big(w(\cdot)\big)$ is absolutely continuous on the interval $[a\,,\,b]$ and for any function $h : [a\,,\,b] \to H$ such that $h(t) \in \partial\Phi_H\big(w(t)\big)$ the following equality holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_H\big(w(t)\big) = (h(t)\,,\,w'(t)) \quad \textit{for a.e. } t \in (a\,,\,b).$$

Lemma 4.2 ([9, Proposition 3.12], [21, Proposition IV.5.2]) *Let $T > 0$, $\widetilde{f} \in L^2(0\,,\,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in H^1(0\,,\,T;H)$ such that $w(0) = w_0$ and for a.e. $t \in (0\,,\,T]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) \ni \widetilde{f}(t) \quad \textit{in } H\,. \tag{4.1}$$

Lemma 4.3 *Let $t_0 < 0$, $\widetilde{f} \in L^2(t_0\,,\,0;H)$, and $w_0 \in C([\tau_0\,,\,t_0];H)$, $w_0(t) \in \mathrm{dom}(\Phi)$ for every $t \in [\tau_0\,,\,t_0]$, where $\tau_0 := \min\limits_{t\in[t_0\,,\,0]}(t - \tau(t))$ (we assume that $[\tau_0\,,\,t_0] = \{t_0\}$ if $\tau_0 = t_0$). Then there exists a unique function $w \in C([\tau_0\,,\,0];H) \cap H^1(t_0\,,\,0;H)$ such that $w(t) = w_0(t)$ for every $t \in [\tau_0\,,\,t_0]$, and for a.e. $t \in (t_0\,,\,0]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) + \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,w(s))\,\mathrm{d}s \ni \widetilde{f}(t) \quad \textit{in } H\,, \tag{4.2}$$

that is, there exists $\widetilde{g} \in L^2(t_0\,,\,0;H)$ such that for a.e. $t \in (t_0\,,\,0]$ we have $\widetilde{g}(t) \in \partial\Phi_H(w(t))$ and

$$w'(t) + \widetilde{g}(t) + \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,w(s))\,\mathrm{d}s = \widetilde{f}(t) \quad \textit{in } H\,. \tag{4.3}$$

Proof. Let $\alpha > 0$ be an arbitrary fixed number and set

$$M := \{w \in C([\tau_0\,,\,0];H) \mid w(t) = w_0(t)\ \forall t \in [\tau_0\,,\,t_0]\}\,.$$

Consider M with the metric

$$\rho(w_1\,,\,w_2) = \max_{t\in[t_0\,,\,0]}\left[e^{-\alpha(t-t_0)}|w_1(t) - w_2(t)|\right], \quad w_1,\,w_2 \in M\,.$$

It is obvious that the metric space $(M\,,\,\rho)$ is complete. Now let us consider an operator $A : M \to M$ defined as follows: for any given function $\widetilde{w} \in M$, it defines a function $\widehat{w} \in M \cap H^1(t_0\,,\,0;H)$ such that for a.e. $t \in (t_0\,,\,0]$, $\widehat{w}(t) \in D(\partial\Phi_H)$ and

$$\widehat{w}'(t) + \partial\Phi_H(\widehat{w}(t)) \ni \widetilde{f}(t) - \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,\widetilde{w}(s))\,\mathrm{d}s \quad \text{in} \quad H\,. \tag{4.4}$$

Clearly, variational inequality (4.4) coincides with variational inequality (4.1) after replacing $[0\,,\,T]$ by $[t_0\,,\,0]$, $\widetilde{f}$ by $\widetilde{f}-\int_{t-\tau(t)}^{t}c(t\,,\,s\,,\,\widetilde{w}(s))\,\mathrm{d}s$, the condition $w(0)=w_0$ by the condition $\widehat{w}(t_0)=w_0(t_0)$. Thus, using Lemma 4.2 we get that operator A is well-defined. Let us show that the operator A is a contraction for some $\alpha>0$. Indeed, let $\widetilde{w}_1,\widetilde{w}_2$ be arbitrary functions from M and $\widehat{w}_1:=A\widetilde{w}_1$, $\widehat{w}_2=A\widetilde{w}_2$. According to (4.4) there exist functions $\widehat{g}_1$ and $\widehat{g}_2$ from $L^2(t_0\,,\,0;H)$ such that for every $k\in\{1\,,\,2\}$ and for a.e. $t\in(t_0\,,\,0]$ we have $\widehat{g}_k(t)\in\partial\Phi_H(\widehat{w}_k(t))$ and

$$\widehat{w}_k'(t)+\widehat{g}_k(t)=\widetilde{f}(t)-\int_{t-\tau(t)}^{t}c(t\,,\,s\,,\,\widetilde{w}_k(s))\,\mathrm{d}s\,, \tag{4.5}$$

while $\widehat{w}_k(t)=w_0(t)$ for a.e. $t\in[\tau_0\,,\,t_0]$.

Subtracting identity (4.5) for $k=2$ from identity (4.5) for $k=1$, and, for a.e. $t\in(t_0\,,\,0]$, multiplying the obtained identity by $\widehat{w}_1(t)-\widehat{w}_2(t)$, we get

$$\begin{aligned}&\big((\widehat{w}_1(t)-\widehat{w}_2(t))'\,,\,\widehat{w}_1(t)-\widehat{w}_2(t)\big)+(\widehat{g}_1(t)-\widehat{g}_2(t)\,,\,\widehat{w}_1(t)-\widehat{w}_2(t))\\&=-\Big(\int_{t-\tau(t)}^{t}\big(c(t\,,\,s\,,\,\widetilde{w}_1(s))-c(t\,,\,s\,,\,\widetilde{w}_2(s))\big)\,\mathrm{d}s\,,\,\widehat{w}_1(t)-\widehat{w}_2(t)\Big)\quad\text{for a.e. } t\in(t_0\,,\,0]\,, \qquad (4.6)\\&\widehat{w}_1(t)-\widehat{w}_2(t)=0\quad\text{for a.e. } t\in[\tau_0\,,\,t_0]\,. \qquad (4.7)\end{aligned}$$

We integrate equality (4.6) by t from t_0 to $\sigma\in(t_0\,,\,0]$, taking into account that for a.e. $t\in(t_0\,,\,0]$ we have

$$\big((\widehat{w}_1(t)-\widehat{w}_2(t))'\,,\,\widehat{w}_1(t)-\widehat{w}_2(t)\big)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\widehat{w}_1(t)-\widehat{w}_2(t)|^2\,.$$

As a result we get the equality

$$\begin{aligned}&\frac{1}{2}|\widehat{w}_1(\sigma)-\widehat{w}_2(\sigma)|^2+\int_{t_0}^{\sigma}(\widehat{g}_1(t)-\widehat{g}_2(t)\,,\,\widehat{w}_1(t)-\widehat{w}_2(t))\,\mathrm{d}t\\&=-\int_{t_0}^{\sigma}\Big(\int_{t-\tau(t)}^{t}(c(t\,,\,s\,,\,\widetilde{w}_1(s))-c(t\,,\,s\,,\,\widetilde{w}_2(s)))\,\mathrm{d}s\,,\,\widehat{w}_1(t)-\widehat{w}_2(t)\Big)\mathrm{d}t.\end{aligned} \tag{4.8}$$

By condition $(\mathcal{A}_4)$, for a.e. $t\in(t_0\,,\,0]$ we have the inequality

$$(\widehat{g}_1(t)-\widehat{g}_2(t)\,,\,\widehat{w}_1(t)-\widehat{w}_2(t))\geq K_2|\widehat{w}_1(t)-\widehat{w}_2(t))|^2\,. \tag{4.9}$$

From this, taking into account condition $(\mathcal{C})$, the Cauchy inequality, for a.e. $t\in(t_0\,,\,0]$ we obtain

$$\begin{aligned}&\Big|\Big(\int_{t-\tau(t)}^{t}\big(c(t\,,\,s\,,\,\widetilde{w}_1(s))-c(t\,,\,s\,,\,\widetilde{w}_2(s))\big)\,\mathrm{d}s\,,\,\widehat{w}_1(t)-\widehat{w}_2(t)\Big)\Big|\\&\leq\Big|\int_{t-\tau(t)}^{t}(c(t\,,\,s\,,\,\widetilde{w}_1(s))-c(t\,,\,s\,,\,\widetilde{w}_2(s)))\,\mathrm{d}s\Big|\,\big|\widehat{w}_1(t)-\widehat{w}_2(t)\big|\\&\leq L\Big(\int_{t-\tau^+}^{t}|\widetilde{w}_1(s)-\widetilde{w}_2(s)|\,\mathrm{d}s\Big)\cdot|\widehat{w}_1(t)-\widehat{w}_2(t)|\leq\varepsilon|\widehat{w}_1(t)-\widehat{w}_2(t)|^2\\&+\frac{L^2}{4\,\varepsilon}\Big(\int_{t-\tau^+}^{t}|\widetilde{w}_1(s)-\widetilde{w}_2(s)|\,\mathrm{d}s\Big)^2\leq\varepsilon|\widehat{w}_1(t)-\widehat{w}_2(t)|^2+\frac{L^2\tau^+}{4\,\varepsilon}\int_{t-\tau^+}^{t}|\widetilde{w}_1(s)-\widetilde{w}_2(s)|^2\,\mathrm{d}s\,,\end{aligned} \tag{4.10}$$

where $\tau^+:=\sup\limits_{t\in S}\tau(t),\ \widetilde{w}_1(s)-\widetilde{w}_2(s):=0,\ \forall s\leq\tau_0,\ \varepsilon>0$ is arbitrary.

From (4.8), according to (4.9) and (4.10), we have

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + 2(K_2 - \varepsilon) \int_{t_0}^{\sigma} |\widehat{w}_1(t) - \widehat{w}_2(t)|^2 \, dt$$
$$\leq (2\,\varepsilon)^{-1} L^2 \tau^+ \int_{t_0}^{\sigma} \Big(\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|^2 \, ds \Big) dt \, . \tag{4.11}$$

Let us consider the last term from the right-hand side of the inequality above. Taking into account equality $\widetilde{w}_1(s) - \widetilde{w}_2(s) = 0$ for $s \leq t_0$ we obtain

$$\int_{t_0}^{\sigma} \Big(\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|^2 \, ds \Big) dt \leq t_0 \int_{t_0}^{\sigma} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2 \, dt \, . \tag{4.12}$$

From (4.11) and (4.12) we get

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + 2(K_2 - \varepsilon) \int_{t_0}^{\sigma} |\widehat{w}_1(t) - \widehat{w}_2(t)|^2 \, dt \leq (2\,\varepsilon)^{-1} L^2 \tau^+ t_0 \int_{t_0}^{\sigma} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2 \, dt \, . \tag{4.13}$$

Choosing $\varepsilon = 2^{-1} K_2$, from (4.13) we obtain

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \leq C_2 \int_{t_0}^{\sigma} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2 \, dt \, , \quad \sigma \in (t_0 \, , 0] \, , \tag{4.14}$$

where $C_2 > 0$ is a constant.

After multiplying inequality (4.14) by $e^{-2\,\alpha(\sigma - t_0)}$ we obtain

$$e^{-2\,\alpha(\sigma-t_0)} |\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \leq C_2 \, e^{-2\,\alpha(\sigma-t_0)} \int_{t_0}^{\sigma} e^{2\,\alpha(t-t_0)} e^{-2\,\alpha(t-t_0)} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2 \, dt$$
$$\leq C_2 \, e^{-2\,\alpha(\sigma-t_0)} \max_{t \in [t_0 \, , 0]} \big[e^{-\alpha(t-t_0)} |\widetilde{w}_1(t) - \widetilde{w}_2(t)| \big]^2 \int_{t_0}^{\sigma} e^{2\,\alpha(t-t_0)} \, dt$$
$$= \frac{C_2}{2\,\alpha} (1 - e^{-2\,\alpha(\sigma-t_0)}) \big[\rho(\widetilde{w}_1 \, , \widetilde{w}_2) \big]^2 \leq \frac{C_2}{2\,\alpha} \big[\rho(\widetilde{w}_1 \, , \widetilde{w}_2) \big]^2 \, , \quad \sigma \in (t_0 \, , 0] \, . \tag{4.15}$$

From (4.15) it easily follows that

$$\rho(\widehat{w}_1 \, , \widehat{w}_2) \leq \sqrt{C_2/(2\,\alpha)} \rho(\widetilde{w}_1 \, , \widetilde{w}_2) \, .$$

From this, choosing $\alpha > 0$ such that inequality $C_2/(2\,\alpha) < 1$ holds, we obtain that operator A is a contraction. Hence, we may apply the Banach fixed-point theorem, the contraction mapping principle [8, Theorem 5.7] and deduce that there exists a unique function $w \in M$ such that $Aw = w$, *i.e.*, we have proved Lemma 4.3. □

Now let us prove Theorem 3.4. We will prove it in 2 steps. First, we will prove the uniqueness of the solution, and then we will prove the existence of the solution and the correctness of estimate (3.8).

The uniqueness of the solution. Assume the contrary. Let $u_1 \, , u_2$ be two solutions of the problem $\mathbf{P}(\Phi \, , c \, , \tau \, , f \, , \gamma)$. Then for every $i \in \{1 \, , 2\}$ there exists function $g_i \in L^2_{\mathrm{loc}}(S; V')$ such that for a.e. $t \in S, \; g_i(t) \in \partial\Phi\big(u_i(t)\big)$ and

$$u_i'(t) + g_i(t) + \int_{t-\tau(t)}^{t} c(t \, , s \, , u_i(s)) \, ds = f(t) \quad \text{in } V' \, . \tag{4.16}$$

We put $w(t) := u_1(t) - u_2(t)$, $t \in S$. From equalities (4.16) for a.e. $t \in S$ we obtain

$$w'(t) + g_1(t) - g_2(t) + \int_{t-\tau(t)}^{t} \big(c(t\,,\,s\,,\,u_1(s)) - c(t\,,\,s\,,\,u_2(s))\big)\,\mathrm{d}s = 0 \quad \text{in } V'\,. \tag{4.17}$$

From (3.4) it follows that the following condition holds

$$e^{-2\,\gamma\,t}|w(t)|^2 \to 0 \quad \text{as} \quad t \to -\infty\,. \tag{4.18}$$

Let $\sigma_1\,,\,\sigma_2 \in S$ be arbitrary numbers such that $\sigma_1 < \sigma_2$. Multiplying equality (4.17) by $w(t)e^{-2\,\gamma\,t}$, integrating from σ_1 to σ_2 we obtain

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}(w'(t)\,,\,w(t))\,\mathrm{d}t + \int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}(g_1(t) - g_2(t)\,,\,u_1(t) - u_2(t))\,\mathrm{d}t \\ &+ \int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau(t)}^{t} \big(c(t\,,\,s\,,\,u_1(s)) - c(t\,,\,s\,,\,u_2(s))\big)\,\mathrm{d}s\,,\,w(t)\Big)\mathrm{d}t = 0\,.\end{aligned} \tag{4.19}$$

Consider the last term from left-hand side of equality (4.19). Using (3.1), the Fubini Theorem and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\Big|\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau(t)}^{t} \big(c(t\,,\,s\,,\,u_1(s)) - c(t\,,\,s\,,\,u_2(s))\big)\,\mathrm{d}s\,,\,w(t)\Big)\mathrm{d}t\Big| \\ &\le \int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau^+}^{t} \big|c(t\,,\,s\,,\,u_1(s)) - c(t\,,\,s\,,\,u_2(s))\big|\,\mathrm{d}s\Big)|w(t)|\mathrm{d}t \\ &\le L\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau^+}^{t} |w(s)|\,\mathrm{d}s\Big)|w(t)|\mathrm{d}t \\ &\le L\sqrt{\tau^+}\left(\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}|w(t)|^2\,\mathrm{d}t\right)^{1/2}\left(\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau^+}^{t} |w(s)|^2\,\mathrm{d}s\Big)\mathrm{d}t\right)^{1/2}.\end{aligned} \tag{4.20}$$

Assuming $w(t) = 0$ for $t > 0$, and changing the order of integration, we have

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau^+}^{t} |w(s)|^2\,\mathrm{d}s\Big)\mathrm{d}t \le \int_{\sigma_1-\tau^+}^{\sigma_2} |w(s)|^2\,\mathrm{d}s\int_{s}^{s+\tau^+} e^{-2\,\gamma\,t}\,\mathrm{d}t \\ &= \chi(\gamma)\Big(\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,s}|w(s)|^2\,\mathrm{d}s + \int_{\sigma_1-\tau^+}^{\sigma_1} e^{-2\,\gamma\,s}|w(s)|^2\,\mathrm{d}s\Big)\,,\end{aligned} \tag{4.21}$$

where $\chi(\gamma)$ is defined in (3.6).

Substituting in (4.20) the last term from relations chain (4.21) instead of the first one, and using inequalities: $\sqrt{a\,b} \le \varepsilon\,a + (4\,\varepsilon)^{-1}b$, $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ ($a \ge 0$, $b \ge 0$, $\varepsilon > 0$), we obtain

$$\begin{aligned}&\Big|\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}\Big(\int_{t-\tau(t)}^{t} \big(c(t\,,\,s\,,\,u_1(s)) - c(t\,,\,s\,,\,u_2(s))\big)\,\mathrm{d}s\,,\,w(t)\Big)\mathrm{d}t\Big| \\ &\le L\sqrt{\tau^+\chi(\gamma)}\Big((1+\varepsilon)\int_{\sigma_1}^{\sigma_2} e^{-2\,\gamma\,t}|w(x\,,\,t)|^2\,\mathrm{d}t + (4\,\varepsilon)^{-1}\int_{\sigma_1-\tau^+}^{\sigma_1} e^{-2\,\gamma\,t}|w(x\,,\,t)|^2\,\mathrm{d}t\Big)\,,\end{aligned} \tag{4.22}$$

where $\varepsilon > 0$ is arbitrary.

By (2.3), (4.22), $(\mathcal{A}_4)$ and the fact that $g_i(t) \in \partial\Phi(u_i(t))$ $(i = 1, 2)$ for a.e. $t \in S$, from (4.19) we obtain the following inequality

$$\frac{1}{2}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\frac{\mathrm{d}|w(t)|^2}{\mathrm{d}t}\,\mathrm{d}t + \big(K_2 - (1+\varepsilon)L\sqrt{\tau^+\chi(\gamma)}\big)\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|w(x\,,\,t)|^2\,\mathrm{d}t$$
$$- (4\,\varepsilon)^{-1}L\sqrt{\tau^+\chi(\gamma)}\int_{\sigma_1-\tau^+}^{\sigma_1} e^{-2\gamma t}|w(t)|^2\,\mathrm{d}t \le 0\,. \tag{4.23}$$

Using the integration-by-parts formula we have

$$e^{-2\gamma t}|w(t)|^2\Big|_{\sigma_1}^{\sigma_2} + 2\big(K_2 + \gamma - (1+\varepsilon)L\sqrt{\tau^+\chi(\gamma)}\big)\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|w(t)|^2\,\mathrm{d}t$$
$$- (2\,\varepsilon)^{-1}L\sqrt{\tau^+\chi(\gamma)}\int_{\sigma_1-\tau^+}^{\sigma_1} e^{-2\gamma t}|w(t)|^2\,\mathrm{d}t \le 0\,. \tag{4.24}$$

Since condition (3.7) holds, taking $\varepsilon > 0$ such that $K_2 + \gamma - (1+\varepsilon)L\,\tau^+ > 0$, from (4.24) we obtain

$$e^{-2\gamma\sigma_2}|w(\sigma_2)|^2 \le e^{-2\gamma\sigma_1}|w(\sigma_1)|^2 + C_0\int_{\sigma_1-\tau^+}^{\sigma_1} e^{-2\gamma t}|w(t)|^2\,\mathrm{d}t\,, \tag{4.25}$$

where $C_0 > 0$ is a constant independent of σ_1 , σ_2.

Let us fix an arbitrary σ_2 in (4.25), and pass to the limit as $\sigma_1 \to -\infty$. According to condition (4.18), the first term from the right side of inequality (4.25) turns to 0. Obviously, the second term from the right side of inequality (4.25) also turns to 0. Indeed, using (4.18) we obtain

$$0 \le \int_{\sigma_1-\tau^+}^{\sigma_1} e^{-2\gamma t}|w(t)|^2\,\mathrm{d}t \le \tau^+ \max_{t\in[\sigma_1-\tau^+\,,\,\sigma_1]} e^{-2\gamma t}|w(t)|^2 \underset{\sigma_1\to-\infty}{\longrightarrow} 0\,.$$

Thus, we get the equality $e^{-2\gamma\sigma_2}|w(\sigma_2)|^2 = 0$. Since $\sigma_2 \in S$ is an arbitrary number, we have $w(t) = 0$ for a.e. $t \in S$, this contradicts our assumption. Therefore, a solution of the problem $\mathbf{P}(\Phi\,,\,c\,,\,\tau\,,\,f\,,\,\gamma)$ is unique.

The existence of the solution. We divide the proof into three steps.

Step 1 (*solution approximations*). We construct a sequence of functions which, in some sense, approximate the solution of the problem $\mathbf{P}(\Phi\,,\,c\,,\,\tau\,,\,f\,,\,\gamma)$.

Let $\widehat{f}_k(t) := f(t)$ for $t \in S_k := [-k\,,\,0]$, $\tau_k := \min\limits_{t\in S_k}(t - \tau(t)) \le -k$, where $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let us consider the problem of finding a function $\widehat{u}_k \in C([\tau_k\,,\,0];H) \cap H^1(S_k;H)$, where $H^1(S_k;H) := \{w \in L^2(S_k;H) \mid w' \in L^2(S_k;H)\}$, such that for a.e. $t \in S_k$ we have $\widehat{u}_k(t) \in D(\partial\Phi_H)$ and

$$\widehat{u}_k'(t) + \partial\Phi_H\big(\widehat{u}_k(t)\big) + \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,\widehat{u}_k(s))\,\mathrm{d}s \ni \widehat{f}_k(t) \quad \text{in } H\,, \tag{4.26a}$$

$$\widehat{u}_k(t) = 0\,, \quad t \in [\tau_k\,,\,-k]\,. \tag{4.26b}$$

Inclusion (4.26a) means that there exists a function $\widehat{g}_k \in L^2(S_k;H)$ such that for a.e. $t \in S_k$ we have $\widehat{g}_k(t) \in \partial\Phi_H(\widehat{u}_k(t))$ and

$$\widehat{u}_k'(t) + \widehat{g}_k(t) + \int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,\widehat{u}_k(s))\,\mathrm{d}s = \widehat{f}_k(t) \quad \text{in } H\,. \tag{4.27}$$

Since $D(\partial\Phi_H) \subset \mathrm{dom}(\Phi_H) \subset V$, thus $\widehat{u}_k(t) \in V$ for a.e. $t \in S_k$. According to the definition of the subdifferential of a functional and the fact that $\widehat{g}_k(t) \in \partial\Phi(\widehat{u}_k(t))$ for a.e. $t \in S_k$, we have

$$\Phi(0) \geq \Phi(\widehat{u}_k(t)) + (\widehat{g}_k(t)\,,\, 0 - \widehat{u}_k(t)) \quad \text{for a.e.} \quad t \in S_k\,.$$

Using this and condition $(\mathcal{A}_3)$ we obtain

$$(\widehat{g}_k(t)\,,\, \widehat{u}_k(t)) \geq \Phi(\widehat{u}_k(t)) \geq K_1\|\widehat{u}_k(t)\|^2 \quad \text{for a.e.} \quad t \in S_k\,. \tag{4.28}$$

Since the left side of this chain of inequalities belongs to $L^1(S_k)$, $\widehat{u}_k$ belongs to $L^2(S_k;V)$.

For each $k \in \mathbb{N}$ we extend functions $\widehat{f}_k, \widehat{u}_k$ and $\widehat{g}_k$ by zero for the entire interval S, and denote these extensions by f_k, u_k and g_k respectively. From the above it follows that for each $k \in \mathbb{N}$ the function u_k belongs to $L^2(S;V)$, its derivative u_k' belongs to $L^2(S;H)$ and for a.e. $t \in S$ the inclusion $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ and the following equality (see (4.27)) hold

$$u_k'(t) + g_k(t) + \int_{t-\tau(t)}^{t} c(t\,,\, s\,,\, u_k(s))\,\mathrm{d}s = f_k(t) \quad \text{in} \quad H\,. \tag{4.29}$$

In order to show the convergence $\{u_k\}_{k=1}^{+\infty}$ to the solution of the problem $\mathbf{P}(\Phi\,,\, c\,,\, \tau\,,\, f\,,\, \gamma)$ we need some estimates of the functions u_k ($k \in \mathbb{N}$).

Step 2 (*estimates of approximating solutions*). Let $\sigma_1\,,\, \sigma_2 \in S$ be arbitrary numbers such that $\sigma_1 < \sigma_2$. Multiplying identity (4.29), for a.e. $t \in S$, by $e^{-2\gamma t}u_k(t)$ and integrating from σ_1 to σ_2, we obtain

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(u_k'(t)\,,\, u_k(t))\,\mathrm{d}t + \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(g_k(t)\,,\, u_k(t))\,\mathrm{d}t\\ &+ \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} c(t\,,\, s\,,\, u_k(s))\ \mathrm{d}s\,,\, u_k(t)\Big)\mathrm{d}t = \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(f_k(t)\,,\, u_k(t))\,\mathrm{d}t\,.\end{aligned}$$

From this taking into account (2.3) and using the integration-by-parts formula, we obtain

$$\begin{aligned}&e^{-2\gamma t}|u_k(t)|^2\Big|_{\sigma_1}^{\sigma_2} + 2\gamma\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t + 2\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(g_k(t)\,,\, u_k(t))\,\mathrm{d}t\\ &+ 2\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} c(t\,,\, s\,,\, u_k(s))\ \mathrm{d}s\,,\, u_k(t)\Big)\mathrm{d}t = 2\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(f_k(t)\,,\, u_k(t))\,\mathrm{d}t\,.\end{aligned} \tag{4.30}$$

According to the definition of u_k and (4.28), we obtain

$$(g_k(t)\,,\, u_k(t)) \geq \Phi\big(u_k(t)\big) \geq K_1\|u_k(t)\|^2 \quad \text{for a.e. } t \in S\,. \tag{4.31}$$

Let us estimate the third term on the left-hand side of inequality (4.30). From (3.5) and (4.31) for arbitrary $\delta \in (0\,,\, 1)$ we obtain

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(g_k(t)\,,\, u_k(t))\,\mathrm{d}t = (\delta + (1-\delta))\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(g_k(t)\,,\, u_k(t))\,\mathrm{d}t\\ &\geq \delta\, K_2\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t + 2^{-1}(1-\delta)K_1\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\|u_k(t)\|^2\,\mathrm{d}t\\ &+ 2^{-1}(1-\delta)\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Phi\big(u_k(t)\big)\,\mathrm{d}t\,.\end{aligned} \tag{4.32}$$

Now, let us estimate the last item on the left-hand side of inequality (4.30). Using the Cauchy-Schwarz inequality, (3.2) we have

$$\Big|\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,u_k(s))\,\mathrm{d}s\,,\,u_k(t)\Big)\mathrm{d}t\Big| \le \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big|\int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,u_k(s))\,\mathrm{d}s\Big||u_k(t)|\mathrm{d}t$$
$$\le L\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau^+}^{t} |u_k(s)|\,\mathrm{d}s\Big)|u_k(t)|\mathrm{d}t$$
$$\le L\sqrt{\tau^+}\left(\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t\right)^{1/2}\left(\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau^+}^{t} |u_k(s)|^2\,\mathrm{d}s\Big)\mathrm{d}t\right)^{1/2}. \tag{4.33}$$

Consider the last term from the chain of inequalities above. Changing the order of integration we have

$$\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau^+}^{t} |u_k(s)|^2\,\mathrm{d}s\Big)\mathrm{d}t$$
$$\le \int_{\sigma_1-\tau^+}^{\sigma_2} |u_k(s)|^2\,\mathrm{d}s\int_{s}^{s+\tau^+} e^{-2\gamma t}\mathrm{d}t = \chi(\gamma)\int_{\sigma_1-\tau^+}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t\,. \tag{4.34}$$

From (4.33), (4.34) with $\sigma_1 < -k$, and definition of u_k it follows

$$\Big|\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,u_k(s))\,\mathrm{d}s\,,\,u_k(t)\Big)\mathrm{d}t\Big| \le L\sqrt{\tau^+\chi(\gamma)}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k|^2\,\mathrm{d}t\,. \tag{4.35}$$

Using the Cauchy inequality we estimate the right-hand side of (4.30) as follows

$$\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(f_k(t)\,,\,u_k(t))\,\mathrm{d}t \le \varepsilon\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t + (4\,\varepsilon)^{-1}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,, \tag{4.36}$$

where $\varepsilon > 0$ is arbitrary.

From (4.30), taking into account (4.32), (4.35) and (4.36), we obtain

$$e^{-2\gamma t}|u_k(t)|^2\Big|_{\sigma_1}^{\sigma_2} + 2[\delta\,K_2 + \gamma - L\sqrt{\tau^+\chi(\gamma)} - \varepsilon]\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t$$
$$+ (1-\delta)K_1\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\|u_k(t)\|^2\,\mathrm{d}t + (1-\delta)\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Phi\big(u_k(t)\big)\,\mathrm{d}t$$
$$\le (2\,\varepsilon)^{-1}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,,\quad \delta\in(0\,,\,1)\,,\ \varepsilon>0\,. \tag{4.37}$$

Since $K_1 > 0$, γ is a solution of (3.7), we first choose δ from $(0\,,1)$ such that $\delta\,K_2 + \gamma - L\sqrt{\tau^+\chi(\gamma)} > 0$, and then we choose $\varepsilon = 2^{-1}[\delta\,K_2 + \gamma - L\sqrt{\tau^+\chi(\gamma)}] > 0$. As a result, from (4.37) we obtain the estimate

$$e^{-2\gamma t}|u_k(t)|^2\Big|_{\sigma_1}^{\sigma_2} + \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\|u_k(t)\|^2\,\mathrm{d}t + \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Phi\big(u_k(t)\big)\,\mathrm{d}t$$
$$\le C_3\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,, \tag{4.38}$$

where C_3 is a positive constant depending on $K_1\,,\,K_2\,,\,L\,,\,\tau^+$ and γ only.

We take $\sigma_2 = \sigma,\ \sigma \in S$ is arbitrary, and pass to the limit in (4.38) as $\sigma_1 \to -\infty$. Taking into account $(\mathcal{F})$ and the definition of u_k and f_k, we obtain

$$e^{-2\gamma\sigma}|u_k(\sigma)|^2 + \int_{-\infty}^{\sigma} e^{-2\gamma t}\|u_k(t)\|^2\,\mathrm{d}t + \int_{-\infty}^{\sigma} e^{-2\gamma t}\Phi\big(u_k(t)\big)\,\mathrm{d}t \le C_3 \int_{-\infty}^{\sigma} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,, \quad \sigma \in S\,. \tag{4.39}$$

Since $\sigma \in S$ is arbitrary, from (4.39) it follows that

$$\text{the sequence } \{u_k(\cdot)\}_{k=1}^{+\infty} \text{ is bounded in } L^{\infty}_{\gamma}(S;H) \text{ and in } L^2_{\gamma}(S;V)\,, \tag{4.40}$$

$$\text{the sequence } \big\{e^{-2\gamma\cdot}\Phi\big(u_k(\cdot)\big)\big\}_{k=1}^{+\infty} \text{ is bounded in } L^1(S)\,. \tag{4.41}$$

Now let us find estimates of u_k' $(k \in \mathbb{N})$. For almost every $t \in S$ we multiply equality (4.29) by $e^{-2\gamma t}u_k'(t)$ and integrate the resulting equality from σ_1 to σ_2, where $\sigma_1\,,\ \sigma_2 \in S$ are arbitrary numbers, $\sigma_1 < \sigma_2$. From this we obtain

$$\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t + \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(g_k(t)\,,\ u_k'(t))\,\mathrm{d}t = \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(f_k(t)\,,\ u_k'(t))\,\mathrm{d}t - \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} c(t\,,\ s\,,\ u_k(s))\ \mathrm{d}s\,,\ u_k'(t)\Big)\mathrm{d}t\,. \tag{4.42}$$

Since $g_k \in L^2(\sigma_1\,,\ \sigma_2; H)$, Lemma 4.1 implies that the function $\Phi_H\big(u_k(\cdot)\big)$ is absolutely continuous on $[\sigma_1\,,\ \sigma_2]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_H\big(u_k(t)\big) = (g_k(t)\,,\ u_k'(t)) \quad \text{for a.e. } t \in (\sigma_1\,,\ \sigma_2)\,. \tag{4.43}$$

Taking into account (4.43), we can rewrite the second term on the left side of (4.42) as follows

$$\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}(g_k(t)\,,\ u_k'(t))\,\mathrm{d}t = \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_H\big(u_k(t)\big)\,\mathrm{d}t = e^{-2\gamma t}\Phi_H\big(u_k(t)\big)\Big|_{\sigma_1}^{\sigma_2} + 2\gamma\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Phi_H\big(u_k(t)\big)\,\mathrm{d}t\,. \tag{4.44}$$

By the Cauchy inequality and (3.2) and changing the order of integration (see (4.34)) we have

$$\Big|\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\big(f_k(t)\,,\ u_k'(t)\big)\,\mathrm{d}t\Big| \le \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|f_k(t)||u_k'(t)|\,\mathrm{d}t \le \frac{1}{4}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t + \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,, \tag{4.45}$$

$$\begin{aligned}
&\Big|\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} c(t\,,\ s\,,\ u_k(s))\ \mathrm{d}s\,,\ u_k'(t)\Big)\mathrm{d}t\Big| \\
&\le \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big|\int_{t-\tau(t)}^{t} c(t\,,\ s\,,\ u_k(s))\ \mathrm{d}s\Big||u_k'(t)|\mathrm{d}t \\
&\le L\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t} |u_k(s)|\ \mathrm{d}s\Big)|u_k'(t)|\mathrm{d}t \le L^2\tau^+\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau^+}^{t} |u_k(s)|^2\ \mathrm{d}s\Big)\mathrm{d}t \\
&+ \frac{1}{4}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t \le L^2\tau^+\chi(\gamma)\int_{\sigma_1-\tau^+}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t + \frac{1}{4}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t\,.
\end{aligned} \tag{4.46}$$

From (4.42), taking into account (4.44), (4.45), (4.46), we obtain

$$\frac{1}{2}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t + e^{-2\gamma t}\Phi_H\big(u_k(t)\big)\Big|_{\sigma_1}^{\sigma_2} \le L^2\tau^+\chi(\gamma)\int_{\sigma_1-\tau^+}^{\sigma_2} e^{-2\gamma t}|u_k(t)|^2\,\mathrm{d}t$$
$$+\,2|\gamma|\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Phi_H\big(u_k(t)\big)\,\mathrm{d}t + \int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,. \tag{4.47}$$

By the definitions of u_k and f_k we pass to the limit in (4.47) when $\sigma_1 \to -\infty$. From obtained inequality, taking into account condition $(\mathcal{A}_3)$, (2.1) and (4.39), setting $\sigma_2 = \sigma \in S$, we have

$$e^{-2\gamma\sigma}\Phi_H\big(u_k(\sigma)\big) + \int_{-\infty}^{\sigma} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t \le C_4\int_{-\infty}^{\sigma} e^{-2\gamma t}|f_k(t)|^2\,\mathrm{d}t\,, \tag{4.48}$$

where C_4 is a positive constant depending on K_1 , K_2 , γ , λ and L , τ^+ only.

According to the definitions of the functional Φ_H and the function f_k, and condition $(\mathcal{A}_3)$ (recall that $u_k(t) \in V$ for a.e. $t \in S$), from (4.48) we obtain

$$e^{-2\gamma\sigma}\|u_k(\sigma)\|^2 + \int_{-\infty}^{\sigma} e^{-2\gamma t}|u_k'(t)|^2\,\mathrm{d}t \le C_5\int_{-\infty}^{\sigma} e^{-2\gamma t}|f(t)|^2\,\mathrm{d}t\,, \tag{4.49}$$

where $C_5 > 0$ is a constant depending on K_1 , K_2 , γ , λ and L , τ^+ only.

Estimates (4.39), (4.49) imply that

$$\text{the sequence } \big\{u_k\big\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty_\gamma(S;V)\,, \tag{4.50}$$
$$\text{the sequence } \big\{u_k'\big\}_{k=1}^{+\infty} \text{ is bounded in } L^2_\gamma(S;H)\,. \tag{4.51}$$

Let us show that

$$\text{the sequence } \{g_k\}_{k=1}^{+\infty} \text{ is bounded in } L^2_\gamma(S;H)\,. \tag{4.52}$$

Indeed, using (2.1), (3.2) and (4.34) we have

$$\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big|\int_{t-\tau(t)}^{t} c(t\,,\,s\,,\,u_k(s))\,\mathrm{d}s\Big|^2\mathrm{d}t \le L^2\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big|\int_{t-\tau(t)}^{t}|u_k(s)|\,\mathrm{d}s\Big|^2\mathrm{d}t$$
$$\le L^2\,\tau^+\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\Big(\int_{t-\tau(t)}^{t}|u_k(s)|^2\,\mathrm{d}s\Big)\mathrm{d}t \le L^2\,\tau^+\chi(\gamma)\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\,|u_k(t)|^2\,\mathrm{d}t$$
$$\le L^2\,\tau^+\chi(\gamma)\,\lambda^{-1}\int_{\sigma_1}^{\sigma_2} e^{-2\gamma t}\,\|u_k(t)\|^2\,\mathrm{d}t\,. \tag{4.53}$$

Therefore, from (4.29), (4.39), (4.51), $(\mathcal{F})$ and the definition of f_k we obtain (4.52).

Step 3 (*passing to the limit*). Since V and H are Hilbert spaces, and V embeds in H by compact injection, from (4.40), (4.50), (4.51), (4.52) and Lemma 2.4 we have that there exist functions $u \in L^\infty_\gamma(S;V)\cap L^2_\gamma(S;V)\cap H^1_\gamma(S;H)$, $g \in L^2_\gamma(S;H)$ and a subsequence of the sequence $\{u_k\,,\,g_k\}_{k=1}^{+\infty}$ (still denoted by $\{u_k\,,\,g_k\}_{k=1}^{+\infty}$) such that

$$e^{-\gamma\cdot}u_k(\cdot) \underset{k\to\infty}{\longrightarrow} e^{-\gamma\cdot}u(\cdot) \quad *\text{-weakly in } L^\infty(S;V)\,, \tag{4.54}$$
$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{weakly in } L^2_\gamma(S;V) \text{ and weakly in } H^1_\gamma(S;H)\,, \tag{4.55}$$
$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{in } C(S;H)\,, \tag{4.56}$$
$$g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in } L^2_\gamma(S;H)\,. \tag{4.57}$$

Note that (4.55) and (4.57) imply

$$u_k \underset{k\to\infty}{\longrightarrow} u\,, \quad u_k' \underset{k\to\infty}{\longrightarrow} u'\,, \quad g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in} \quad L^2_{\mathrm{loc}}(S;H)\,. \tag{4.58}$$

Using (3.1), the Cauchy-Schwarz inequality, changing the order of integration and (4.56), for each $\sigma \in S$ we obtain

$$\int_\sigma^0 \Big| \int_{t-\tau(t)}^t c(t\,,\, s\,,\, u_k(s))\, \mathrm{d}s - \int_{t-\tau(t)}^t c(t\,,\, s\,,\, u(s))\, \mathrm{d}s \Big|^2 \mathrm{d}t$$
$$\le L^2\tau^+ \int_\sigma^0 \Big(\int_{t-\tau^+}^t |u_k(s)-u(s)|^2\, \mathrm{d}s\Big)\mathrm{d}t \le (L\,\tau^+)^2 \int_{\sigma-\tau^+}^0 |u_k(t)-u(t)|^2\, \mathrm{d}t \underset{k\to\infty}{\longrightarrow} 0\,. \tag{4.59}$$

Thus, we obtain

$$\int_{t-\tau(t)}^t c(t\,,\, s\,,\, u_k(s))\, \mathrm{d}s \underset{m\to\infty}{\longrightarrow} \int_{t-\tau(t)}^t c(t\,,\, s\,,\, u(s))\, \mathrm{d}s \quad \text{strongly in} \quad L^2_{\mathrm{loc}}(S;H)\,. \tag{4.60}$$

Let $v \in H$, $\varphi \in D(-\infty\,,\,0)$ be arbitrary. For a.e. $t \in S$ we multiply equality (4.29) by v, and then we multiply the obtained equality by φ and integrate in t on S. As a result, we obtain the equality

$$\int_S (u_k'(t)\,,\, v\,\varphi(t))\, \mathrm{d}t + \int_S (g_k(t)\,,\, v\,\varphi(t))\, \mathrm{d}t + \int_S \Big(\int_{t-\tau(t)}^t c(t\,,\, s\,,\, u_k(s))\, \mathrm{d}s\,,\, v\,\varphi(t)\Big)\mathrm{d}t$$
$$= \int_S (f_k(t)\,,\, v\,\varphi(t))\, \mathrm{d}t\,, \quad k \in \mathbb{N}\,. \tag{4.61}$$

We pass to the limit in (4.61) as $k \to \infty$, taking into account (4.58), (4.60) and convergence of $\{f_k\}$ to f in $L^2_{\mathrm{loc}}(S;H)$. As a result, since $v \in H$, $\varphi \in D(-\infty\,,\,0)$ are arbitrary, for a.e. $t \in S$ we obtain the equality

$$u'(t) + g(t) + \int_{t-\sigma(t)}^t c(t\,,\, s\,,\, u(s))\, \mathrm{d}s = f(t) \quad \text{in } H\,.$$

In order to complete the proof of the theorem it remains only to show that $u(t) \in D(\partial\Phi)$ and $g(t) \in \partial\Phi\big(u(t)\big)$ for a.e. $t \in S$.

Let $k \in \mathbb{N}$ be an arbitrary number. Since $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ for every $t \in S \setminus \widetilde{S}_k$, where $\widetilde{S}_k \subset S$ is a set of measure zero, applying the monotonicity of the subdifferential $\partial\Phi_H$, we obtain that for every $t \in S \setminus \widetilde{S}_k$ the following equality holds

$$(g_k(t) - v^*\,,\, u_k(t) - v) \ge 0 \quad \forall\, [v\,,\, v^*] \in \partial\Phi_H\,. \tag{4.62}$$

Let $\sigma \in S,\ h > 0$ be arbitrary numbers. We integrate (4.62) on $(\sigma - h\,,\,\sigma)$:

$$\int_{\sigma-h}^\sigma (g_k(t) - v^*\,,\, u_k(t) - v)\, \mathrm{d}t \ge 0 \quad \forall\, [v\,,\, v^*] \in \partial\Phi_H\,. \tag{4.63}$$

Now according to (4.56) and (4.57) we pass to the limit in (4.63) as $k \to \infty$. As a result we obtain

$$\int_{\sigma-h}^\sigma (g(t) - v^*\,,\, u(t) - v)\, \mathrm{d}t \ge 0 \quad \forall\, [v\,,\, v^*] \in \partial\Phi_H\,. \tag{4.64}$$

The monograph [27, Theorem 2, pp. 192] and (4.64) imply that for every $[v\,,\,v^*]\in\partial\Phi_H$ there exists a set $R_{[v\,,\,v_*]}\subset S$ of measure zero such that for all $\sigma\in S\setminus R_{[v\,,\,v_*]}$ we have

$$0\le\lim_{h\to+0}\frac{1}{h}\int_{\sigma-h}^{\sigma}\big(g(t)-v^*\,,\,u(t)-v\big)\,\mathrm{d}t=\big(g(\sigma)-v^*\,,\,u(\sigma)-v\big)\,. \tag{4.65}$$

Let us show that there exists a set of measure zero $R\subset S$ such that

$$\forall\sigma\in S\setminus R:\qquad\big(g(\sigma)-v^*\,,\,u(\sigma)-v\big)\ge 0\quad\forall[v\,,\,v^*]\in\partial\Phi_H\,. \tag{4.66}$$

Since V and H are separable spaces, there exists a countable set $F\subset\partial\Phi_H\subset V\times H$ which is dense in $\partial\Phi_H$. Let us denote $R:=\bigcup\limits_{[v\,,\,v^*]\in F}R_{[v\,,\,v_*]}$. Since the set F is countable, and any countable union of sets of measure zero is a set of measure zero, R is a set of measure zero. Therefore, for any $\sigma\in S\setminus R$ inequality $\big(g(\sigma)-v^*\,,\,u(\sigma)-v\big)\ge 0$ holds for every $[v\,,\,v^*]\in F$. Let $[\widehat{v}\,,\,\widehat{v}^*]$ be an arbitrary element from $\partial\Phi_H$. Then from the density F in $\partial\Phi_H$ we have the existence of a sequence $\{[v_l\,,\,v_l^*]\}_{l=1}^{\infty}$ such that $v_l\to\widehat{v}$ in V, $v_l^*\to\widehat{v}^*$ in H and

$$\forall\sigma\in S\setminus R:\qquad(g(\sigma)-v_l^*\,,\,u(\sigma)-v_l)\ge 0\quad\forall\,l\in\mathbb{N}\,. \tag{4.67}$$

Thus, passing to the limit in this equality as $l\to\infty$, we get $(g(\sigma)-\widehat{v}^*\,,\,u(\sigma)-\widehat{v})\ge 0$, $\forall\sigma\in S\setminus R$. Therefore, inequality (4.66) holds. From this, according to maximal monotonicity of $\partial\Phi_H$, we obtain that $[u(t)\,,\,g(t)]\in\partial\Phi_H$ for a.e. $t\in S$.

Estimate (3.8) of the solution of the problem $\mathbf{P}(\Phi\,,\,c\,,\,\tau\,,\,f\,,\,\gamma)$ follows directly from (4.39), (4.49), (4.54), (4.55) and (4.56), Lemma 2.2, Fatou's Lemma and the fact that Φ_H is lower semi-continuous in H.

From (3.8) according to (2.1) we have

$$e^{-2\gamma\sigma}|u(\sigma)|^2\le C_1\lambda^{-1}\int_{-\infty}^{\sigma}e^{-2\gamma t}|f(t)|^2\,\mathrm{d}t\,.$$

This inequality and condition $(\mathcal{F})$ imply that u satisfies condition (3.4). Thus theorem is proved. □

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