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STABILITY PROPERTIES OF SYSTEMS WITH MAXIMUM

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Abstract. In this paper we consider systems of functional differential equations whose dynamics depends on the maximum value of solution over a prehistory time interval. Such kind of systems are infinite-dimensional and nonlinear. We consider controlled systems with maximum and study their input-to-state stability property. As well we compare stability properties of such systems with their linear counterparts and it turns out their global stability properties are in many cases better then for the linear counterparts.

Keywords: infinite-dimensional systems, functional differential equations, input-to-state stability.

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1 Introduction

Stability and robustness of infinite-dimensional dynamical systems play the important roles in applications. The input-to-state stability (ISS) framework, originally introduced for finite-dimensional systems in [28] describes global stability properties of an equilibrium in case of perturbations and seems to be promising for infinite-dimensional systems as well. This property is invariant under nonlinear coordinate transformations and extends the classical global asymptotic stability of an equilibrium to the case of systems that have external perturbing signals. If an unperturbed system possesses a global asymptotically stable equilibrium point (which is by definition the invariant and attracting set), then in case of perturbation entering the ISS property of the system means that instead of the equilibrium point there is an invariant and globally attracting domain. The size of this domain is a nonlinear function of the perturbation norm. This framework was very successful in studying nonlinear interconnected systems and construction of Lyapunov functions for them [11].

During the last five years the ISS framework was actively developed for infinite dimensional systems [23, 10, 25, 24, 18, 27]. The reason of these activities is the success of this framework in many applications in finite-dimensional case. In this paper we are going to consider stability and robustness properties of a special class of functional differential equations, where the dynamics depends on the maximum of the solution taken over a past time interval. This class of systems belongs to nonlinear infinite-dimensional delayed systems with time varying state dependent delay. We are interested in stability properties of such systems and, in particular, in robustness with respect to external perturbations.

Systems of this type appear in modeling of various practical processes, *e.g.*such a model of financial market was discussed in [4], see also [31] for discrete market model including maximization operator; a model of optimal motion for solar panels with a maximization was used in [5, 6]. In [1, pp.166], a model of a hydroelectric power dam which involves a supremum operator was discussed. Other applications can be found in [12] and [17, Example 1.2.2, pp. 7–8].

Systems with maximum without external inputs were extensively studied in [7]. However there is only a few works ([1, 2, 5, 9, 33, 6]) studying such systems perturbed by an external input function. Recall that there are some works devoted to input-to-state-stability property of systems with delays, where the Lyapunov-Krasovskiy functional [26] or the Razumikhin function [32] is applied. However, none of them provides explicit expressions for the ISS gain functions.

In this paper we investigate the behavior of solutions to systems of differential equations with maximum and external perturbing signal. We will derive an explicit expression for the ISS gain of a certain class of systems without usage of any Lyapunov techniques but on the base of trajectory estimations. Due to the *max*-operator in the systems equations, we will see that such systems are nonlinear, that the state-dependent delay is discontinuous in time and that the backward uniqueness property of solutions is not satisfied. In spite of this rather complex dynamics, we will see that the stability properties are in some sense better than for systems with constant time delay. We will observe that dimension of such system may change from infinity to one along its solution *i.e.*, we deal with Multi-Mode Multi Dimensional systems (see [16]) in this paper. We will compare stability properties of one-dimensional differential equations with constant delay and with maximum, and we will see that in many cases for equation with maximum the stability does not depend on the length h of the past time interval which is not true for the delay equations. This work is an essentially extended version of the conference paper [9].

The manuscript is organized as follows. In the next section we introduce all necessary basic notions and notation, study some properties of the max-operator and properties of equations with this operator. Comparison principle for equations with maximum is provided in Section 3. Some further properties of solutions are discussed in Section 4. Stability and input-to-state stability properties are studied in Sections 5 and 6 respectively including several examples. Section 7 concludes the paper.

Nomenclature

$\mathbb{N},\mathbb{R},\mathbb{R}_+$	the set of natural, real, nonnegative real numbers resp.
$\mathbb{R}^n$	the set of n-dimensional vectors
x^T	transposition of a vector $x\in\mathbb{R}^n$
$\lvert x\rvert$	$:=\sqrt{x^Tx}$, the Euclidean norm of $x\in\mathbb{R}^n$
	for an open set $J\subseteq\mathbb{R}$ we define
$C(J;\mathbb{R}^n)$	the set of continuous functions $f:J\to\mathbb{R}^n$
$\lvert\lvert f\rvert\rvert_J$	$:=\sup\limits_{t\in J}\lvert f(t)\rvert$, norm of $f:J\to\mathbb{R}$ (sometimes, we write $\lvert\lvert f\rvert\rvert$ if the set J is clear from the context)
$PC(J;\mathbb{R}^n)$	the set of piecewise continuous functions $f:J\to\mathbb{R}^n$ *i.e.* f is right continuous and has only finitely many discontinuities on any compact subset of $J\subseteq\mathbb{R}$
$AC(J;\mathbb{R}^n)$	the set of absolutely continuous functions $f:J\to\mathbb{R}^n$
$L_\infty(J;\mathbb{R}^n)$	the set of measurable essentially bounded functions $f:J\to\mathbb{R}^n$
$\lvert\lvert A(t)\rvert\rvert$	$:=\max\limits_{i=1,\cdots,n}\sum\limits_{j=1}^{n}\lvert a_{i,j}(t)\rvert$, matrix norm of $A\in C(J;\mathbb{R}^{n\times n})$
$\mathscr{P}$	$:=\{\gamma:\mathbb{R}_+\to\mathbb{R}_+\mid\gamma$ is continuous, $\gamma(0)=0,$ and $\gamma(s)>0$ for $s>0\}$
$\mathscr{K}$	$:=\{\gamma\in\mathscr{P}\mid\gamma$ is strictly increasing$\}$
$\mathscr{K}_\infty$	$:=\{\gamma\in\mathscr{K}\mid\gamma$ is unbounded$\}$
$\mathscr{L}$	$:=\left\{\gamma:\mathbb{R}_+\to\mathbb{R}_+\mid\gamma\text{ is continuous, strictly decreasing and}\lim\limits_{t\to\infty}\gamma(t)=0\right\}$
$\mathscr{K}\mathscr{L}$	$:=\{\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+\mid\beta(\cdot,t)\in\mathscr{K}\ \forall t\geq 0,\ \beta(s,\cdot)\in\mathscr{L}\ \forall s>0\}$

For any $x,y\in\mathbb{R}^n$ we define $x\geq y:\Leftrightarrow x_i\geq y_i,$ for all $i=1,\cdots,n$.

2 Preliminaries

Let $h>0,\ t_0\in\mathbb{R}$ be given and $[t_0-h,T)\subset\mathbb{R},\ T\leq\infty$ be a time interval. For a scalar valued function $g\in C([t_0-h,T);\mathbb{R})$, we introduce the map

$$\max: C([t_0-h,T);\mathbb{R})\to C([t_0,T);\mathbb{R})\ ,\quad g\overset{\max}{\longmapsto}g_h^{\vee}\ , \tag{2.1}$$

where

$$\forall t\in[t_0,T):\ \ g_h^{\vee}(t):=\max_{s\in[t-h,t]}g(s)\in\mathbb{R}\ . \tag{2.2}$$

Similarly for a vector valued function $g\in C([t_0-h,T);\mathbb{R}^n)$, $n\in\mathbb{N}$, we define component-wise

$$\forall t\in[t_0,T):\ \ g_h^{\vee}(t):=\left(g_{1,h}^{\vee}(t),g_{2,h}^{\vee}(t),\dots,g_{n,h}^{\vee}(t)\right)^T\ \in\mathbb{R}^n\ . \tag{2.3}$$

Sometimes, we write $g^{\vee}$ instead of $g_h^{\vee}$ to simplify notation, if the value h is clear from the context. For $\alpha\in C([t_0-h,T);\mathbb{R})$, we introduce the map

$$\arg\max: C([t_0-h,T);\mathbb{R})\to PC([t_0,T);\mathbb{R})\ ,\quad\alpha\overset{\arg\max}{\longmapsto}\arg\max\alpha\ ,$$

$$\forall t \in [t_0\,,T): \quad (\arg\max \alpha)(t) := \sup\left\{\tau \in [t_0\,,T): \max_{s\in[t-h\,,t]} \alpha(s) = \alpha(\tau)\right\} \in \mathbb{R}\,, \tag{2.4}$$

similarly, for a vector valued function $\alpha \in C([t_0-h\,,T);\mathbb{R}^n)$, we define

$$\forall t \in [t_0\,,T): \quad (\arg\max \alpha)(t) := ((\arg\max \alpha_1)(t)\,,\cdots,(\arg\max \alpha_n)(t))^T \in \mathbb{R}^n\,. \tag{2.5}$$

Using the definition (2.5), we can rewrite the definition (2.3) as follows

$$g_h^\vee(t) = \Big(g_1\Big((\arg\max g_1)(t)\Big)\,,\cdots,g_n\Big((\arg\max g_n)(t)\Big)\Big)^T. \tag{2.6}$$

From (2.4) it follows that $\arg\max \alpha$ is piecewise continuous function satisfying

$$\forall t \in [t_0\,,\infty): \quad (\arg\max \alpha)(t) \le t\,.$$

To illustrate the above definitions, consider the following example.

Example 2.1 *Let $g(t) = \sin t$ on $[-1\,,6]$ and $h = 1$. Then we have*

$$g_h^\vee(t) = \begin{cases} \sin t\,, & t \in [0\,,\frac{\pi}{2}]\,,\\ 1\,, & t \in [\frac{\pi}{2}\,,\frac{\pi}{2}+1]\,,\\ \sin(t-1)\,, & t \in [\frac{\pi}{2}+1\,,t^*)\,,\\ \sin t\,, & t \in [t^*\,,6]\,, \end{cases} \qquad (\arg\max \sin)(t) = \begin{cases} t\,, & t \in [0\,,\frac{\pi}{2}]\,,\\ \frac{\pi}{2}\,, & t \in [\frac{\pi}{2}\,,\frac{\pi}{2}+1]\,,\\ t-1\,, & t \in [\frac{\pi}{2}+1\,,t^*)\,,\\ t\,, & t \in [t^*\,,6]\,, \end{cases}$$

where $t^* := \frac{1}{2}(3\pi+1)$ is the unique solution of $\sin t^* = \sin(t^*-1)$ on $[\frac{\pi}{2}+1\,,6]$(Figures 1, 2).

The *max*-operator has the following properties:

1. *Sublinearity.*
   For continuous functions $g\,,\eta : [t_0-h\,,T) \to \mathbb{R}^n$, $T \le \infty$, the following holds
   $$\forall t \in [t_0\,,T): (g+\eta)^\vee(t) \le g^\vee(t) + \eta^\vee(t)\,. \tag{2.7}$$
   Indeed, fix any $t \in [t_0\,,T)$. Fix any $i \in \{1,\cdots,n\}$. Let $t_1\,,t_2\,,t_3 \in [t_0\,,T)$ be such that
   $$g_i(t_1) + \eta_i(t_1) = (g_i+\eta_i)^\vee(t)\,,\ \ g_i(t_2) = {g_i}^\vee(t)\,,\ \ \eta_i(t_3) = {\eta_i}^\vee(t)\,.$$
   Then we have
   $$(g_i+\eta_i)^\vee(t) = g_i(t_1) + \eta_i(t_1) \le {g_i}^\vee(t) + {\eta_i}^\vee(t) = g_i(t_2) + \eta_i(t_3)\,. \tag{2.8}$$
   Inequality (2.8) holds for any $i \in \{1,\cdots,n\}$, hence (2.7) is proved.
   Moreover, for any number $\alpha \ge 0$, the map defined by (2.1) satisfies
   $$(\alpha+g)_h^\vee = \alpha + g_h^\vee\,, \qquad (\alpha g)_h^\vee = \alpha g_h^\vee\,.$$

2. *Monotonicity.*
   Let $\xi\,,g \in C([t_0-h\,,T);\mathbb{R}^n)$, $T \le \infty$. If $\xi(t) \ge g(t)$ for all $t \in [t_0-h\,,T)$, then $\xi_h^\vee(t) \ge g_h^\vee(t)$ for all $t \in [t_0\,,T)$. Moreover, if for any $i = 1\,,2\,,\cdots,n$, function g_i is nondecreasing (nonincreasing) for all $[t_0-h\,,T)$, then function $g_i^\vee$ is nondecreasing (nonincreasing) for all $t \in [t_0\,,T)$. In this case for any $t_1 \ge t_2$, it follows that $g^\vee(t_1) \ge g^\vee(t_2)$ (resp. $g^\vee(t_1) \le g^\vee(t_2)$).

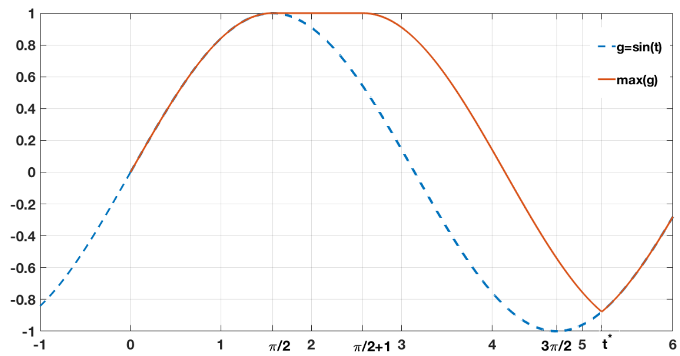

Figure 1: *The graphs of* $g(t) = \sin t\,,\ g_1^{\vee}(t) = \max\limits_{s\in[t-1\,,t]} g(s)$.

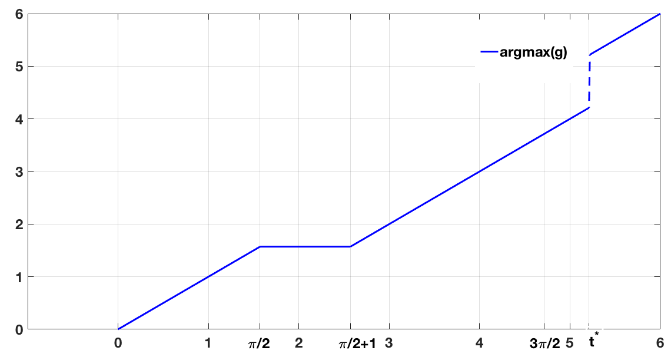

Figure 2: *The graph of* $(\arg\max \sin)(t)$.

3. *Periodicity*
   Let $g \in C([t_0 - h\,,\infty);\mathbb{R})$ be periodic with period $T' > 0$. Then for any $h > 0$, the function $g_h^{\vee}$ is periodic on $[t_0\,,\infty)$ with the same period T'. Indeed, for all $t \in [t_0\,,\infty)$, we obtain

$$g_h^{\vee}(t) := \max_{s\in[t-h,t]} g(s) = \max_{s\in[t-h,t]} g(s+T') = \max_{s\in[t-h+T',t+T']} g(s) =: g_h^{\vee}(t+T)\ .$$

4. *Useful inequality.*
   The next Lemma provides useful estimation for the work with *max*-operator.

Lemma 2.2 *Let* $h > 0\,,\ t_0 \in \mathbb{R},\ T \le +\infty$, *and* $g \in C([t_0 - h\,,T);\mathbb{R}^n)$. *Then for all* $t \in [t_0\,,T)$

$$n \max_{s\in[t-h,t]} |g(s)| =: n|g^{\vee}|(t) \ge |g^{\vee}(t)| \ge |g^{\vee}|(t)\ . \tag{2.9}$$

Proof. For any fixed $t \in [t_0\,,T)$, we have

$$\begin{aligned}
|g^{\vee}(t)| &= |(g_1(\arg\max_{s\in[t-h,t]} g_1(s))\,,\ldots,g_n(\arg\max_{s\in[t-h,t]} g_n(s)))^T| \\
&= \sqrt{\sum_{i=1}^{n} g_i^2(\arg\max_{s\in[t-h,t]} g_i(s))} \ge \sqrt{\sum_{i=1}^{n} g_i^2(\arg\max_{s\in[t-h,t]} |g(s)|)} \\
&= |g(\arg\max_{s\in[t-h,t]} |g(s)|)| = |g^{\vee}|(t)\ .
\end{aligned}$$

This implies the second inequality in (2.9). To prove the first one observe that for any $i = 1,\dots,n$, the following holds

$$\max_{s\in[t-h,t]} |g(s)| = \max_{s\in[t-h,t]} \sqrt{\sum_{k=1}^{n} g_k^2(s)} \ge \max_{s\in[t-h,t]} |g_i(s)| \ .$$

Taking the sum left and right for all $i = 1,\dots,n$, and using that $\sum_{i=1}^{n} c_i \ge \sqrt{\sum_{i=1}^{n} c_i^2}$ holds for any $c_i \ge 0$, $i = 1,\dots,n$, and $|g_i(s)| \ge g_i(s)$ for any s, we obtain

$$\begin{aligned} n \max_{s\in[t-h,t]} |g(s)| &\ge \sum_{i=1}^{n} \max_{s\in[t-h,t]} |g_i(s)| \ge \sqrt{\sum_{i=1}^{n} (\max_{s\in[t-h,t]} |g_i(s)|)^2} \\ &\ge \sqrt{\sum_{i=1}^{n} (\max_{s\in[t-h,t]} g_i(s))^2} = \left| \max_{s\in[t-h,t]} g(s) \right| = \left| g^{\vee}(t) \right| \ . \end{aligned} \tag{2.10}$$

□

5. *Lipschitz continuity.*

Definition 2.3 *Let $(X\,, ||\cdot||_X)$ and $(Y\,, ||\cdot||_Y)$ be two normed spaces. A map $\chi : X \mapsto Y$ is called* Lipschitz continuous *if there exists a Lipschitz constant $L > 0$ such that for all $x_1\,, x_2 \in X$ the following holds*

$$||\chi(x_1) - \chi(x_2)||_Y \le L||x_1 - x_2||_X \ .$$

Lemma 2.4 *Let $h > 0,\ t_0 \in \mathbb{R},\ T \le \infty,\ n \in \mathbb{N}$. Max-operator is Lipschitz continuous with Lipschitz constant $L = n$ i.e.*

$$\forall v\,, w \in C([t_0 - h\,, T);\mathbb{R}^n) : \ \ ||v_h^{\vee} - w_h^{\vee}||_{[t_0\,,T)} \le n||v - w||_{[t_0-h\,,T)} \ . \tag{2.11}$$

Proof. For short we denote $v_h^{\vee} =: v^{\vee}$. We obtain

$$\begin{aligned} &||v^{\vee} - w^{\vee}||_{[t_0\,,T)} := \sup_{t\in[t_0\,,T)} |v^{\vee}(t) - w^{\vee}(t)| = \sup_{t\in[t_0\,,T)} |(v - w + w)^{\vee}(t) - w^{\vee}(t)| \\ &\overset{(2.7)}{\le} \sup_{t\in[t_0\,,T)} \left| \max_{s\in[t-h,t]} (v(s) - w(s)) \right| \overset{(2.10)}{\le} n \sup_{t\in[t_0-h\,,T)} \max_{s\in[t-h,t]} |v(s) - w(s)| \\ &= n \sup_{t\in[t_0-h\,,T)} |v(t) - w(t)| =: n||v - w||_{[t_0-h\,,T)} \ . \end{aligned}$$

□

Note that *max*-operator does not preserve smoothness as already seen from the Example 2.1, where $g_h^{\vee}$ is not differentiable at t^*.

Let $t_0 = 0$, $h > 0$ and consider the system of differential equations with *max*-operator and input signal $u \in L_\infty([0\,,\infty);\mathbb{R}^m)$ of the form

$$\dot{x}(t) = f(t\,, x(t)\,, x_h^{\vee}(t)\,, u(t)) \ , \quad t \ge 0 \ , \tag{2.12}$$

where $f \in C([0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ with initial condition

$$x(t) = \varphi(t)\,, \quad t \in [-h\,,0]\,, \tag{2.13}$$

where $\varphi \in C([-h\,,0];\mathbb{R}^n)$. By $\dot{x}(t)$ we mean the right-hand side derivative of x at t. Note that (2.12) is a system of functional differential equations with state-dependent piecewise continuous delay. The state space of this system is $C([-h\,,0];\mathbb{R}^n)$, therefore we deal with an infinite-dimensional dynamical system.

Remark 2.5 *Since* max *is sublinear operator, system* (2.12) *is, in general, nonlinear i.e., it can happen that the sum of two solutions of the problem* (2.12)*,* (2.13) *is not a solution to* (2.12)*,* (2.13)*.*

Definition 2.6 *For an input $u \in L_\infty([0,\infty);\mathbb{R}^m)$ and an initial function $\varphi \in C([-h\,,0];\mathbb{R}^n)$, a function $x(\cdot) = x(\cdot;\varphi\,,h\,,u)$ is called a solution to the problem* (2.12)*,* (2.13)*, if there is a $T_f \in (0\,,\infty]$ such that $x \in C([-h\,,T_f);\mathbb{R}^n)$, $x(t) = \varphi(t)$ for all $t \in [-h\,,0]$, x is absolutely continuous on $[0\,,T_f)$ and satisfies differential equation* (2.12) *almost everywhere on $[0\,,T_f)$; x is a maximal solution if it has no right extension that is also a solution.*

For each input $u \in L_\infty([0,\infty);\mathbb{R}^m)$, the function $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by $F(t\,,\xi_1\,,\xi_2) := f(t\,,\xi_1\,,\xi_2\,,u(t))$.

Suppose that Ω is an open subset of $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. The function $F : \Omega \to \mathbb{R}^n$ is said to satisfy the *Carathéodory conditions* on Ω if

(i) for each fixed $t \in \mathbb{R}_+$, function $F(t\,,\cdot,\cdot)$ is continuous;

(ii) for each fixed $\xi_1\,,\xi_2 \in \mathbb{R}^n$, function $F(\cdot,\xi_1\,,\xi_2)$ is measurable;

(iii) for any fixed $(t\,,\xi_1\,,\xi_2) \in \Omega$, there is a neighborhood $W(t\,,\xi_1\,,\xi_2)$ and a Lebesgue integrable function $m : [0,\infty) \to \mathbb{R}_+$ such that

$$|F(t\,,\xi_1'\,,\xi_2')| \le m(t) \quad \text{for all } \ (t\,,\xi_1'\,,\xi_2') \in W(t\,,\xi_1\,,\xi_2)\,;$$

(iv) for each compact subset $\overline{\Omega} \subseteq \Omega$, there is a locally integrable function $l : [0,\infty) \to \mathbb{R}_+$ such that

$$|F(t\,,\xi_1\,,\xi_2) - F(t\,,\overline{\xi_1}\,,\overline{\xi_2})| \le l(t)\Big(||\xi_1 - \xi_2|| + ||\overline{\xi_1} - \overline{\xi_2}||\Big)$$

for all $\ \xi_1\,,\xi_2\,,\overline{\xi_1}\,,\overline{\xi_2} \in \overline{\Omega}\,,$ and for almost all $\ t \in [0,\infty)\,.$

By Lemma 2.4, *max*-operator is Lipschitz continuous, hence under Carathéodory conditions imposed on $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, existence of unique maximal solution $x(\cdot) = x(\cdot;\varphi\,,h)$ of the problem

$$\begin{array}{ll} \dot{x}(t) = F(t\,,x(t)\,,x_h^\vee(t))\,, & t \ge 0\,, \\ x(t) = \varphi(t)\,, & t \in [-h\,,0]\,, \end{array} \tag{2.14}$$

where $x(t) \in \mathbb{R}^n\,,\ h > 0,\ \varphi \in C([-h\,,0];\mathbb{R}^n)$, follows from Theorems 2.1, 2.3, 3.1, 3.2 in [14]. Therefore we can formulate the following theorem.

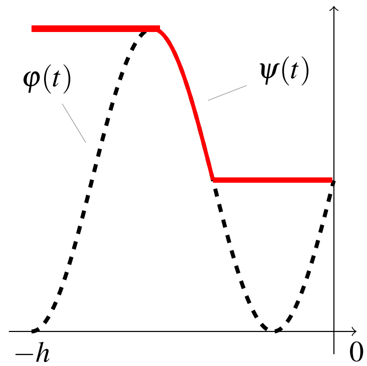

Figure 3: Graph of functions ψ and φ.

Theorem 2.7 *Let $h>0$, $\varphi\in C([-h,0];\mathbb{R}^n)$. Assume $F:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the Carathéodory conditions. Then there exists a unique solution $x:[-h,T_f)\to\mathbb{R}^n$, $T_f\in(0,\infty]$ of the problem* (2.14)*, and every solution can be extended to a maximal solution; moreover, if x is bounded for all $t\in[0,T_f)$, then $T_f=\infty$.*

For any function $\varphi\in C([-h,0];\mathbb{R}^n)$, we define the function $\psi\in C([-h,0];\mathbb{R}^n)$ by

$$\psi(t):=\max_{s\in[t,0]}\varphi(s)\,,\quad t\in[-h,0]\,. \tag{2.15}$$

Here *max* is taken component-wise as in (2.3). Note that each component of the function $\psi(\cdot)$ defined by (2.15) is a nonincreasing function (see Figure 3). Indeed, let $t_1,t_2\in[-h,0]$ such that $t_1<t_2$. Then we have

$$\psi(t_1)=\max_{s\in[t_1,0]}\varphi(s)\geq\max_{s\in[t_2,0]}\varphi(s)=\psi(t_2)\,,$$

and

$$\forall t\in[-h,0]:\ \max_{s\in[t,0]}\varphi(s)=\psi(t)=\max_{s\in[t,0]}\psi(s)\,, \tag{2.16}$$

holds. Also,

$$\forall t\in[-h,0]:\ \ \psi(t)\geq\varphi(t)\,,$$

and

$$\psi(0)=\varphi(0)\,.$$

Consider the equation with maximum (2.12) with the initial condition

$$x(t)=\psi(t)\,,\quad t\in[-h,0]\,, \tag{2.17}$$

where the function ψ is defined by (2.16).

Assumption 2.8 *For any initial function $\varphi\in C([-h,0];\mathbb{R}^n)$, $h>0$ and any input $u\in L_\infty([0,\infty);\mathbb{R}^m)$, there exists a unique solution $x(\cdot)=x(\cdot;\varphi,h,u)$ of the problem* (2.12)*,* (2.13) *defined on $[0,\infty)$.*

Lemma 2.9 *Let Assumption 2.8 be satisfied. Then for any $\varphi\in C([-h,0];\mathbb{R}^n)$ and any input $u\in L_\infty([0,\infty);\mathbb{R}^m)$, the solution of the problem* (2.12)*,* (2.13) *coincides with the solution of the problem* (2.12)*,* (2.17) *on $[0,\infty)$.*

Proof. Let $h > 0$, $\varphi \in C([-h, 0];\mathbb{R}^n)$ and $u \in L_\infty([0, \infty);\mathbb{R}^m)$. By the Assumption 2.8, for any initial function solution to the problem (2.12), (2.13) exists on $[0, \infty)$, therefore denote by $x_1(\cdot) = x(\cdot;\varphi, h, u)$ and $x_2(\cdot) = x(\cdot;\psi, h, u)$ the solution to the Cauchy problem (2.12), (2.13) and the solution to the Cauchy problem (2.12), (2.17) on $[0, \infty)$, respectively. Let $t \in [0, h]$. Then we have

$$\dot{x}_1(t) = f(t, x_1(t), x^{\vee}_{1,h}(t), u) = f(t, x_1(t), \max\{\max_{s\in[t-h,0]} \varphi(s), \max_{s\in[0,t]} x_1(s)\}, u)$$
$$\overset{(2.16)}{=} f(t, x_1(t), \max\{\max_{s\in[t-h,0]} \psi(s), \max_{s\in[0,t]} x_1(s)\}, u) .$$

Therefore, $x_1(\cdot) = x(\cdot;\varphi, h, u)$ is a solution to the Cauchy problem (2.12), (2.17) on $[0, h]$. By the Assumption 2.8, there exists a unique solution of (2.12) with initial condition (2.17), hence

$$\forall t \in [0, h]: \ \ x_1(t) = x_2(t) . \tag{2.18}$$

Let $t \geq h$. Consider (2.12) for $t \geq h$ with the initial condition

$$\forall t \in [0, h]: \ \ x(t) = x_1(t) . \tag{2.19}$$

By the Assumption 2.8, there exists a unique solution $x_1(\cdot)$ to the problem (2.12), (2.19), and in conjunction with (2.18), we conclude that $x_1(\cdot) = x_2(\cdot)$ on $[h, \infty)$, where $x_2(\cdot)$ is the solution to (2.12) with initial condition $x(t) = x_2(t)$ for all $t \in [0, h]$. Therefore, $x_1(t) = x_2(t)$ on $[0, \infty)$. This proves the lemma. □

Remark 2.10 *Lemma 2.9 shows that without loss of generality we can restrict our consideration to the case of the nonincreasing initial functions. Also from Lemma 2.9 it follows that problem* (2.12), (2.13) *has no backward uniqueness property.*

The following notion of stability, originally introduced for ordinary differential equations (ODEs) in [28], is used in this paper.

Definition 2.11 *System* (2.12) *is called input-to-state stable from u to x if there exist a function γ of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$ such that for each input $u \in L_\infty([0, \infty);\mathbb{R}^m)$ and each initial function $\varphi \in C([-h, 0];\mathbb{R}^n)$, the unique solution $x(\cdot) = x(\cdot;\varphi, h, u)$ to* (2.12), (2.13) *exists for all $t \in [0, \infty)$ and furthermore it satisfies*

$$|x(t)| \leq \beta(||\varphi||, t) + \gamma(||u||) , \quad t \geq 0 . \tag{2.20}$$

The function γ is called ISS-gain.

Definition 2.12 *If in Definition 2.11 function $\beta(s, t) = M e^{-\lambda t} s$ for any $s, t \in \mathbb{R}_+$ and some $M > 0$, $\lambda > 0$, then system* (2.12) *is called exponentially ISS (eISS).*

Assume that $f(t, 0, 0, 0) = 0$ for all $t \geq 0$, so that $x \equiv 0$ is an equilibrium of (2.12).

Definition 2.13 *The solution $x \equiv 0$ of system* (2.12) *with $u \equiv 0$ is said to be zero stable if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $||\varphi|| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \in [0, \infty)$. It is said to be zero asymptotically stable if it is zero stable, and for some $\delta > 0$, $||\varphi|| < \delta$ implies $\lim_{t\to\infty} x(t) = 0$. It is zero globally asymptotically stable (0-GAS) if it is zero asymptotically stable with $\delta = \infty$. It is zero globally exponentially stable (0-GES) if there exist $c > 0$, $k > 0$ such that for any $\varphi \in C([-h, 0];\mathbb{R}^n)$ the solution satisfies $|x(t)| \leq c\, e^{-kt} ||\varphi||$ for all $t \in [0, \infty)$.*

In the special case when f has no input signal

$$\dot{x}(t) = f(t\,, x(t)\,, x_h^{\vee}(t))\,, \quad t \geq 0\,, \tag{2.21}$$

instead of 0-GAS (0-GES) of $x \equiv 0$ we say that $x \equiv 0$ is globally asymptotically stable (GAS) (globally exponentially stable (GES)).

3 Comparison lemmas

In many problems of mathematical control theory one needs to compute bounds of a solution $x(\cdot;u)$ to equation $\dot{x} = f(t\,, x\,, u)$ without knowing the solution itself. One of the powerful tools which is widely used for this purpose is the comparison principle (see [19, Lemma 3.4, pp.102] and *e.g.*[29, 30] for its applications).

In the literature there are some comparison results for delay equation in the form

$$\dot{x}(t) = -\sum_{i=1}^{n} q_i(t)x(t-\tau_i(t))\,, \quad t \geq 0\,, \tag{3.1}$$

with initial condition

$$x(t) = \varphi(t)\,, \quad t \in [\min_{i=1,\cdots,n}\inf_{s\geq 0}(s-\tau_i(s))\,, 0]\,, \tag{3.2}$$

where $\tau_i \in C([0\,,\infty);\mathbb{R}_+),\ i = 1,\cdots,n,\ \varphi \in C([\min_{i=1,\cdots,n}\inf_{s\geq 0}(s-\tau_i(s))\,, 0];\mathbb{R}_+)$. For example, in [22] the equation (3.1) in case $\tau_i(t) = \tau$ is studied. The authors consider two different initial conditions and compare corresponding solutions. In [21] the case of continuous variable delays is considered and the following comparison result is obtained:

Theorem 3.1 *Let $\tau_i \in C([0\,,\infty);\mathbb{R}_+),\ i = 1,\cdots,n,\ \varphi \in C([\min_{i=1,\cdots,n}\inf_{s\geq 0}(s-\tau_i(s))\,, 0];\mathbb{R}_+), L \in (0\,,\infty]$. Assume x satisfies* (3.1) *and* (3.2) *for all $t \in [0\,, L)$. Suppose $y \in C([\min_{i=1,\cdots,n}\inf_{s\geq 0}(s-\tau_i(s))\,,\infty);\mathbb{R})$ and $y \in C^1([0\,,\infty);\mathbb{R})$ satisfies*

$$\begin{aligned} &\dot{y}(t) \geq -\sum_{i=1}^{n} q_i(t)y(t-\tau_i(t))\,, && t \geq 0\,, \\ &y(t) \leq \varphi(t)\,, && t \in [\min_{i=1,\cdots,n}\inf_{s\geq 0}(s-\tau_i(s))\,, 0]\,, \end{aligned}$$

with $y(0) = \varphi(0) > 0$. Then $y(t) \geq x(t)$ for all $t \in [0\,, L)$, where the interval can be closed if L is finite.

To the best of our knowledge, comparison results for general state-dependent delay differential equation are not available in the literature.

Consider the differential equation with maximum in the form

$$\begin{aligned} &\dot{x}(t) = f(t\,, x(t)\,, x_h^{\vee}(t))\,, && t \geq 0\,, \\ &x(t) = \varphi(t)\,, && t \in [-h\,, 0]\,, \end{aligned} \tag{3.3}$$

where $x(t) \in \mathbb{R},\ h > 0,\ \varphi \in C([-h\,, 0];\mathbb{R})$.

Assumption 3.2 *$f \in C([0\,,\infty)\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ satisfies the implication*

$$y_1 \le y_2 \;\Rightarrow\; f(t\,,x\,,y_1) \le f(t\,,x\,,y_2) \;\text{ for all }\; t \ge 0\,, \quad x \in \mathbb{R}\,.$$

Lemma 3.3 *Let $h > 0$, $\varphi \in C([-h\,,0];\mathbb{R})$. Assume*

- *the Assumption 3.2 holds;*
- *there exists $x(\cdot) = x(\cdot;\varphi\,,h)$ solution to (3.3) on $[-h\,,T)$, $0 < T \le \infty$;*
- *there is $y \in C([-h\,,\infty);\mathbb{R})$ such that y is differentiable almost everywhere on $[0\,,\infty)$, satisfies*

$$\dot{y}(t) < f(t\,,y(t)\,,y_h^{\vee}(t))\,,\;\; t \in [0\,,T)\,, \tag{3.4}$$

and $y(t) < \varphi(t)$ for all $t \in [-h\,,0]$. Then $y(t) < x(t)$ for all $[0\,,T)$.

Proof. For $t = 0$ we have $y(0) < x(0)$ and $y_h^{\vee}(0) < x_h^{\vee}(0)$. Consider the function $r(t) := x(t) - y(t)$ on $[0\,,T)$. Note that $r(0) > 0$. Assume there exists $t^* \in (0\,,T)$ such that $r(t^*) = 0$ and $r(\cdot) > 0$ for all $t \in [0\,,t^*)$. Since $x(\cdot)$ is right hand side differentiable on $[0\,,T)$ as a solution to the problem (3.3), and by the assertion of the lemma y is differentiable almost everywhere on $[0\,,T)$, then function $r(\cdot)$ is right hand side differentiable on $[0\,,T)$. This in particular means that $\dot{r}(t^*) \le 0$. On the other hand, by Assumption 3.2

$$\dot{y}(t^*) < f(t^*\,,y(t^*)\,,y_h^{\vee}(t^*)) \le f(t^*\,,x(t^*)\,,x_h^{\vee}(t^*)) = \dot{x}(t^*)\,. \tag{3.5}$$

It follows from (3.5) that $\dot{r}(t^*) > 0$ which contradicts $\dot{r}(t^*) \le 0$. Thus, such t^* does not exist. This proves the lemma. □

4 Scalar differential equations with maximum and input

Consider the scalar problem with *max*-operator and input in the linear form

$$\begin{aligned} &\dot{x}(t) = a\,x(t) + b\,x_h^{\vee}(t) + u(t)\,, && t \ge 0\,,\\ &x(t) = \varphi(t)\,, && t \in [-h\,,0]\,, \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}$, $h > 0$, $a\,,b \in \mathbb{R}$, $u \in L_\infty([0\,,\infty);\mathbb{R})$, $\varphi \in C([-h\,,0];\mathbb{R})$. Recall that (4.1) defined an infinite-dimensional dynamical system. Since the right-hand side of (4.1) satisfies Carathéodory conditions, Theorem 2.7 insures that for every input $u \in L_\infty([0\,,\infty);\mathbb{R})$ and every initial function $\varphi \in C([-h\,,0];\mathbb{R})$, there exists a unique maximal solution $x(\cdot) := x(\cdot;\varphi\,,h\,,u)$ on $[0\,,T_f)$, $T_f \in (0\,,\infty]$. Recall that inputs are essentially bounded, then solution $x(\cdot)$ is bounded on $[0\,,T_f)$, and therefore $T_f = +\infty$. We define, for this $x(\cdot)$

$$\begin{aligned} D_{x(\cdot)} &= \left\{ t \in [0\,,\infty) \mid x(t) < x_h^{\vee}(t) \right\}\,,\\ G_{x(\cdot)} &= \left\{ t \in [0\,,\infty) \mid x(t) = x_h^{\vee}(t) \le 0 \right\}\,. \end{aligned}$$

Note that by this definition for any solution $x(\cdot)$ to (4.1), it holds $D_{x(\cdot)} \cap G_{x(\cdot)} = \varnothing$.

Theorem 4.1 *Let $h > 0$, $u \in L_\infty([0,\infty);\mathbb{R})$ with $u(t) \leq 0$ for all $t \in [0,\infty)$ and $a, b \in \mathbb{R}$ be such that*

$$a + b < 0. \tag{4.2}$$

Let $x(\cdot)$ be the unique solution of the problem (4.1) *on $[0,\infty)$. Then*

$$[0,\infty) = D_{x(\cdot)} \cup G_{x(\cdot)}\ . \tag{4.3}$$

Proof. By the definition of sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$, the following holds

$$[0,\infty) \supset D_{x(\cdot)} \cup G_{x(\cdot)}\ .$$

We will prove that

$$[0,\infty) \subset D_{x(\cdot)} \cup G_{x(\cdot)}\ . \tag{4.4}$$

Assume there exists $t_1 \in [0,\infty)$ such that $t_1 \notin D_{x(\cdot)} \cup G_{x(\cdot)}$ which indicates that

$$x(t_1) = x_h^{\vee}(t_1) > 0\ . \tag{4.5}$$

Then

$$\dot{x}(t_1) = a\,x(t_1) + b\,x_h^{\vee}(t_1) + u(t_1) = x(t_1)(a+b) + u(t_1)\ .$$

Since $u(t) \leq 0$ for any $t \geq 0$ and taking into account (4.2), it follows that there exists $\varepsilon > 0$ such that $\dot{x}(t) < 0$ almost everywhere on $(t_1 - \varepsilon, t_1 + \varepsilon)$. Therefore, x is decreasing on $(t_1 - \varepsilon, t_1 + \varepsilon) \Rightarrow x(t_1) < x_h^{\vee}(t_1)$. This contradicts (4.5). Hence inclusion (4.4) is obtained. $\square$

Remark 4.2 *From the proof of Theorem 4.1 it follows that if $D_{x(\cdot)}$ is bounded then there exists $t^* \in [0,\infty)$ such that the solution $x(\cdot)$ satisfies the ODE $\dot{x}(t) = (a+b)x(t) + u(t)$ for all $t \leq t^*$. In this sense the infinite-dimensional system* (4.1) *reduces to a one-dimensional after t^*.*

Example 4.3 *Consider the following problem*

$$\begin{array}{ll} \dot{x}(t) = \frac{1}{2}x(t) - 2x_h^{\vee}(t) + u(t)\ , & t \geq 0\ , \\ x(t) = 0\ , & t \in [-h, 0]\ , \end{array} \tag{4.6}$$

with $h = 2$, $u \in L_\infty([0,\infty);\mathbb{R})$. The condition (4.2) *holds and let $u(t) := -e^{-t} - 1 < 0$ for all $t \in [0,\infty)$. The solution to* (4.6) *is*

$$x(t) = \begin{cases} -\frac{8}{3}e^{\frac{1}{2}t} + \frac{2}{3}e^{-t} + 2\ , & t \in [0,2]\ , \\ (-e^2 - 2\frac{59}{125})e^{\frac{1}{2}t} + 5\frac{1}{3}e^{\frac{1}{2}t-1} + 7\frac{23}{98}e^{-t} + 10\ , & t \in [2, 3\frac{97}{100}]\ , \\ -1727\frac{1}{10}e^{-\frac{3t}{2}} - 2e^{-t} - \frac{2}{3}\ , & t \in [3\frac{97}{100}, \infty)\ . \end{cases}$$

The point $t^* = 3\frac{97}{100}$ satisfies $x(t^*) = x(t^* - h)$ and $x(t^*) = x_h^{\vee}(t^*)$ i.e., $x(t) < x_h^{\vee}(t)$ for all $t < t^*$ which indicates that the equation (4.6) *becomes an ODE for all $t^* \geq 3\frac{97}{100}$ i.e., we have $D_{x(\cdot)} = [0, 3\frac{97}{100})$ and $G_{x(\cdot)} = [3\frac{97}{100}, \infty)$ (see Figure 4).*

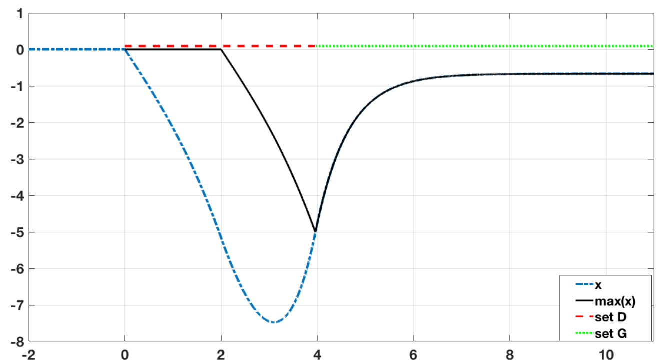

Figure 4: *Sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ for the solution to the problem* (4.6).

In the considered example differential equation reduces an ODE at t^* and remains ODE for all $t \geq t^*$. However, in general, dynamic of system with maximum and input may change back from one to infinite. For example, let us consider the following Cauchy problem

$$\begin{aligned} &\dot{x}(t) = -x_2^{\vee}(t) - u(t)\ , && t \in [0\,,\,\infty)\ , \\ &x(t) = 0\ , && t \in [-2\,,\,0]\ . \end{aligned} \tag{4.7}$$

Choose $u(t) = -|\sin t| < 0$ for all $t \geq 0$. By Theorem 2.7, there exists a unique solution $x(\cdot)$ for (4.7) for which, by Theorem 4.1, we have

$$[0\,,\,\infty) = D_{x(\cdot)} \cup G_{x(\cdot)}\ .$$

The sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ are shown in the Figure 5.

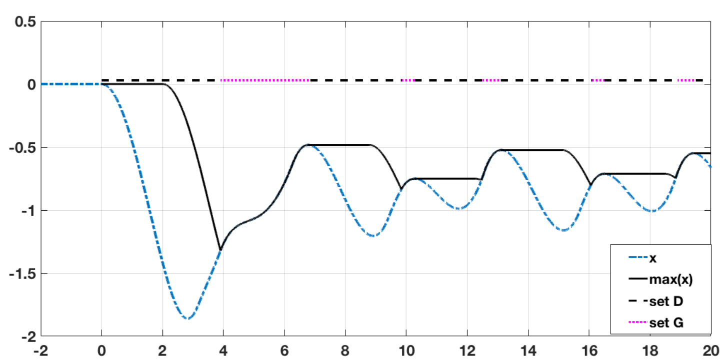

Figure 5: *Sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ for the solution to* (4.7).

Lemma 4.4 *Let $a\,,\,b \in \mathbb{R}$, $t^* \geq 0$, $h > 0$, $\varphi \in C([-h\,,\,0];\mathbb{R})$ be given and function $x \in C([-h\,,\,\infty);\mathbb{R})$ be such that*

$$\begin{aligned} &\dot{x}(t) = a\,x(t) + b\,x_h^{\vee}(t)\ , && t \geq t^*\ , \\ &x(t) = \varphi(t)\ , && t \in [t^* - h\,,\,t^*]\ . \end{aligned} \tag{4.8}$$

Assume (4.2) *holds and*

$$\varphi_h^{\vee}(t^*) = \varphi(t^*) \leq 0\ . \tag{4.9}$$

Then for all $t \in [t^*, \infty)$ the solution to the problem (4.8) *coincides with the solution of*

$$\begin{aligned} &\dot{x}(t) = (a+b)x(t) , \quad t \geq t^* , \\ &x(t^*) = \varphi(t^*) \leq 0 . \end{aligned} \tag{4.10}$$

Proof. It follows from (4.9) that there exists $\varepsilon > 0$ such that the solution to the problem (4.8) coincides with the solution to the problem (4.10) for all $t \in [t^*, t^* + \varepsilon]$. By (4.2) and continuity of $\dot{x}$, we obtain $\dot{x}(t) \geq 0$ for all $t \in [t^*, t^* + \varepsilon]$, then x is nondecreasing on $[t^*, t^* + \varepsilon]$ and, therefore, maximum is attained at the right end of $[t-h, t]$ for $t \in [t^*, t^* + \varepsilon]$. Since the solution to (4.10), $x(t) = \varphi(t^*)e^{(a+b)(t-t^*)}$ is a nondecreasing function for all $t \geq t^*$, maximum is taken at the right end of $[t-h, t]$ for all $t \geq t^*$, thus (4.8) is equivalent to (4.10) for all $t \geq t^*$. □

Remark 4.5 *By definition of $G_{x(\cdot)}$, it follows that $t^* = \inf_{t \geq 0} G_{x(\cdot)}$. Hence* (4.8) *reduces to an ODE for all $t \geq t^*$.*

5 Stability of scalar equations with *max*-operator

Here we consider scalar equations without input. In [35], it was proved that the trivial solution to the delay equation

$$\dot{x}(t) = -x(t-h) , \quad t \in [0, \infty) ,$$

is GES if and only if $h \in [0, \frac{\pi}{2})$. The following theorem shows that GES of the trivial solution to the corresponding equation with maximum does not depend on the value of h.

Proposition 5.1 *Let $h,\ b \geq 0$, $\varphi \in C([-h, 0]; \mathbb{R})$. Suppose $x(\cdot) = x(\cdot; \varphi, h)$ is the solution to the Cauchy problem*

$$\begin{aligned} &\dot{x}(t) = -b x_h^{\vee}(t) , \quad t \geq 0 , \\ &x(t) = \varphi(t) , \qquad [-h, 0] . \end{aligned} \tag{5.1}$$

Then the trivial solution to (5.1) *is GES.*

Proof. By [34, Theorem 3.1], it follows that (5.1) is asymptotically stable. Notice that (5.1) is a particular case of (4.1) for which Theorem 4.1 holds. Assume that $D_{x(\cdot)}$ is bounded, then for some $t^* \geq 0$ by Remark 4.2 and Lemma 4.4, it follows that the solution to problem 5.1 coincides with the solution to

$$\begin{aligned} &\dot{x}(t) = -b x(t) , \quad t \geq t^* , \\ &x(t^*) = x_h^{\vee}(t^*) . \end{aligned}$$

Then $|x(t)| \leq |x(t^*)| e^{-b(t-t^*)}$ for all $t \geq t^*$ and therefore, there exists $c = c(\varphi, h, t^*) > 0$ such that $|x(t)| \leq c e^{-bt} ||\varphi||$ for all $t \geq 0$. Next, assume $D_{x(\cdot)}$ is unbounded, then by definition of $D_{x(\cdot)}$ for all $t \geq 0$, the solution to problem (5.1) coincides with the solution to the delay differential equation

$$\begin{aligned} &\dot{x}(t) = -b x(t-h) , \quad t \geq 0 , \\ &x(t) = \varphi(t) , \qquad t \in [-h, 0] . \end{aligned} \tag{5.2}$$

By [20, Remark 2.2], asymptotic stability of (5.2) is equivalent to GES. Thus theorem is proved. □

Proposition 5.2 *Let $h\,,\ b \geq 0$, $\varphi \in C([-h\,,0];\mathbb{R})$, $c \in \mathbb{R}$. Suppose $x(\cdot) = x(\cdot;\varphi\,,h)$ is the solution to*

$$\begin{aligned} &\dot{x}(t) = -b x_h^{\vee}(t) + c\,, && t \geq 0\,,\\ &x(t) = \varphi(t)\,, && t \in [-h\,,0]\,. \end{aligned} \tag{5.3}$$

Then the trivial solution to (5.3) *is GES.*

Proof. Denote $z(t) := x(t) - \frac{c}{b}$ for all $t \in [0\,,\infty)$. Then

$$\begin{aligned} &\dot{z}(t) = -b z_h^{\vee}(t)\,, && t \geq 0\,,\\ &z(t) = \varphi(t) - \tfrac{c}{b}\,, && t \in [-h\,,0]\,. \end{aligned} \tag{5.4}$$

Applying Proposition 5.1 to the problem (5.4), we obtain the GES of (5.4) and, therefore, of (5.3). □

Stability of the problem

$$\begin{aligned} &\dot{x}(t) = -a x(t) - b x_h^{\vee}(t)\,, && t \geq 0\,,\\ &x(t) = \varphi(t)\,, && t \in [-h\,,0]\,, \end{aligned} \tag{5.5}$$

where $a\,,b \in \mathbb{R}$, $h > 0$, $\varphi \in C([-h\,,0];\mathbb{R}^n)$, is studied in [34]. There it is shown that the trivial solution of the problem (5.5) is GAS if and only if $a\,,b$ satisfy

$$\begin{aligned} &b > -a\,, && \text{for} \quad a \geq -1/h\,,\\ &b > \tfrac{1}{h} e^{-ah-1}\,, && \text{for} \quad a \leq -1/h\,, \end{aligned} \tag{5.6}$$

see Figure 6.

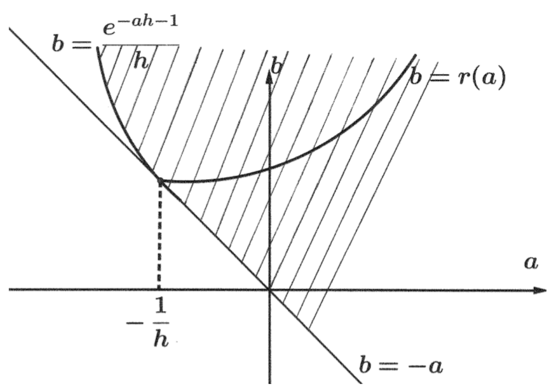

Figure 6: *Shaded region stands for the set of parameters for which the trivial solution to the problem* (5.5) *is GAS.*

Recall that the trivial solution to the delay differential equation in the form

$$\begin{aligned} &\dot{x}(t) = -a x(t) - b x(t-h)\,, && t \geq 0\,,\\ &x(t) = \varphi(t)\,, && t \in [-h\,,0]\,, \end{aligned} \tag{5.7}$$

where $a\,,b \in \mathbb{R}$, $h > 0$, $\varphi \in C([-h\,,0];\mathbb{R}^n)$, is GAS if and only if $a\,,b$ satisfy

$$a > -\frac{1}{h}\,, \quad -a < b < r(a)\,,$$

where $r(a) = \lambda(h\sin\lambda)^{-1}$, λ is the unique root of $a = -\lambda\, h^{-1}\cot\lambda$ in the interval $(0\,,\pi)$ [3], [13, pp.108–109], see Figure 7.

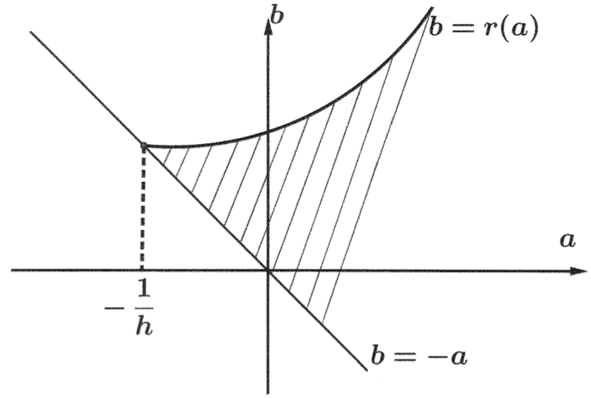

Figure 7: *Shaded region stands for the set of parameters for which the trivial solution to the problem* (5.7) *is GAS.*

6 Input-to-state stability of systems with maximum

Consider the following system of differential equations with maximum in the linear form

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)x_h^{\vee}(t) + u(t) \ , \quad && t \geq 0 \ , \\ &x(t) = \varphi(t) \ , && t \in [-h\,,0] \ , \end{aligned} \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$, $A\,,B \in C([0\,,\infty);\mathbb{R}^{n\times n})$, initial function $\varphi \in C([-h\,,0];\mathbb{R}^n)$, input $u \in L_\infty([0\,,\infty);\mathbb{R}^n)$. The right-hand side of (6.1) satisfies Carathéodory conditions, thus by Theorem 2.7, for every input $u \in L_\infty([0\,,\infty);\mathbb{R}^n)$ and every initial function $\varphi \in C([-h\,,0];\mathbb{R}^n)$, there exists a unique maximal solution $x(\cdot) := x(\cdot;\varphi\,,h\,,u)$ to problem (6.1) on $[0\,,T_f)$, $T_f \in (0\,,\infty]$. Since inputs are essentially bounded, solution $x(\cdot)$ is bounded on $[0\,,T_f)$, and therefore $T_f = +\infty$. The scalar case of (6.1) with a periodic input $u(\cdot)$ is studied in [15, 8], where several stability properties are obtained.

Now we investigate whether the system (6.1) is ISS and what is the corresponding gain function.

First, derive the following auxiliary result for the scalar case. Consider the differential inequality

$$\dot{x}(t) \leq -a(t)x(t) + b(t)x_h^{\vee}(t) + w(t) \ , \quad t \geq 0 \ , \tag{6.2}$$

with a non-negative initial function

$$x(t) = \varphi(t) \ , \quad t \in [-h\,,0] \ , \tag{6.3}$$

where $x(t) \in \mathbb{R}$, $\varphi \in C([-h\,,0];\mathbb{R}_+)$, $h > 0$, $a\,,b \in C([0\,,\infty);(0\,,\infty))$, input $w \in L_\infty([0\,,\infty);\mathbb{R})$.

Lemma 6.1 *Let $x \in AC([0\,,\infty);\mathbb{R}_+)$ be a solution to* (6.2)*,* (6.3)*. Assume $a\,,b \in C([0\,,\infty);(0\,,\infty))$ such that*

$$\forall t \in [0\,,\infty): \ \ a(t) - b(t) \geq \delta > 0 \ ,$$

where $\delta := \inf\limits_{t\in[0\,,\infty)} (a(t) - b(t))$. Then there exists $\lambda > 0$ such that

$$x(t) \leq \max\{1\,,h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{\|w\|}{\delta} \ , \ \ t \geq 0 \ . \tag{6.4}$$

Proof. Define the function H as follows:

$$H(t\,,\lambda^*) := \lambda^* - a(t) + b(t)e^{\lambda^* h}\,, \quad \lambda^* \in \mathbb{R}_+\,, \quad t \in [0\,,\infty)\,. \tag{6.5}$$

Consider

$$F(\lambda^*) := \sup_{t\in[0\,,\infty)} H(t\,,\lambda^*)\,, \quad \lambda^* \in \mathbb{R}_+\,. \tag{6.6}$$

Observe that function F is continuous for all $\lambda^* \in \mathbb{R}_+$, then

$$\begin{aligned} F(0) &= \sup_{t\in[0\,,\infty)} H(t\,,0) = \sup_{t\in[0\,,\infty)} (-a(t)+b(t)) \\ &= -(-\sup_{t\in[0\,,\infty)} (-[a(t)-b(t)])) = -\inf_{t\in[0\,,\infty)} (a(t)-b(t)) = -\delta < 0\,. \end{aligned} \tag{6.7}$$

Write

$$\begin{aligned} \underline{a} &:= \inf_{t\in[0\,,\infty)} a(t)\,, \quad \underline{b} := \inf_{t\in[0\,,\infty)} b(t)\,, \\ \overline{a} &:= \sup_{t\in[0\,,\infty)} a(t)\,, \quad \overline{b} := \sup_{t\in[0\,,\infty)} b(t)\,, \end{aligned}$$

since functions $a\,,b$ are bounded on $[0\,,\infty)$ by assumption $\underline{a}\,,\overline{a}\,,\underline{b}\,,\overline{b} \in \mathbb{R}_+$ are finite numbers. Set $\overline{F}(\lambda^*) := \lambda^* - \underline{a} + \overline{b}\,e^{\lambda^* h}$, $\underline{F}(\lambda^*) := \lambda^* - \overline{a} + \underline{b}\,e^{\lambda^* h}$ then

$$\underline{F}(\lambda^*) \le F(\lambda^*) \le \overline{F}(\lambda^*)\,, \quad \lambda^* \in \mathbb{R}_+\,.$$

Since $\underline{F}(\lambda^*) \to \infty$, $\overline{F}(\lambda^*) \to \infty$, when $\lambda^* \to \infty$, one obtains

$$F(\lambda^*) \to \infty\,, \text{ as } \lambda^* \to \infty\,. \tag{6.8}$$

Let δ^* be such that $0 < \delta^* < \delta$. Then from (6.7), (6.8) and by the continuity of F it follows that there exists $\lambda > 0$ such that,

$$F(\lambda) = \sup_{t\in[0\,,\infty)} H(t\,,\lambda) = \sup_{t\in[0\,,\infty)} (\lambda - a(t) + b(t)e^{\lambda h}) = -\delta^* < 0\,.$$

Hence

$$\forall t \in [0\,,\infty): \ \lambda - a(t) + b(t)e^{\lambda h} \le -\delta^* < 0\,. \tag{6.9}$$

For any $\varepsilon > 0$, define the function $q \in C([-h\,,\infty);\mathbb{R}_+)$ by

$$q(t) := [\max\{1\,,h\}\,\|\varphi\| + \varepsilon]\,e^{-\lambda t} + \frac{\|w\|}{\delta}\,.$$

Observe that $q(t) > x(t)$ for all $t \in [-h\,,0]$ by this definition. We prove that $q(t) > x(t)$ for all $t \in (0\,,\infty)$. Suppose this is not true, *i.e.*, there exists $t_1 > 0$ such that $q(t_1) = x(t_1)$ and $q(t) > x(t)$ for $t \in (0\,,t_1)$. Therefore, for the left side derivative the following holds

$$\dot{q}(t_1-) \le \dot{x}(t_1-)\,.$$

At the same time

$$\begin{aligned} \dot{q}(t_1-) &= -\lambda\,e^{-\lambda t_1}\,[\max\{1\,,h\}\,\|\varphi\| + \varepsilon] > (-a(t_1) + b(t_1)e^{\lambda h})e^{-\lambda t_1}\,[\max\{1\,,h\}\,\|\varphi\| + \varepsilon] \\ &= -a(t_1)e^{-\lambda t_1}\,[\max\{1\,,h\}\,\|\varphi\| + \varepsilon] + b(t_1)e^{\lambda h}e^{-\lambda t_1}\,[\max\{1\,,h\}\,\|\varphi\| + \varepsilon] \\ &= -a(t_1)\left[q(t_1) - \frac{\|w\|}{\delta}\right] b(t_1)\left[q(t_1-h) - \frac{\|w\|}{\delta}\right] \\ &= -a(t_1)x(t_1) + b(t_1)q(t_1-h) - (-a(t_1)+b(t_1))\frac{\|w\|}{\delta}\,. \end{aligned}$$

Let $t' \in [t_1 - h, t_1]$ be such that $x_h^{\vee}(t_1) = x(t')$.

In case $t_1 \geq h$ we have $t' > 0$ and from the choice of the point t_1 and monotonicity (decreasing) of the function q on $[0, \infty)$ we obtain $x(t') \leq q(t') < q(t_1 - h)$.

In case $t_1 < h$ if $t' < 0$ then $x(t') \leq \|\varphi\| < q(t_1 - h)$. Therefore,

$$\dot{q}(t_1-) > -a(t_1)x(t_1) + b(t_1)x_h^{\vee}(t_1) + \|w\| > \dot{x}(t_1-) .$$

The obtained contradiction proves the inequality $q(t) > x(t)$ for all $t \in (0, \infty)$ and the claim of lemma. □

Coming back to the system (6.1) we assume that the following assumptions hold for (6.1):

Assumption 6.2

(i) Let $\tilde{b} \in \mathbb{R}_+$, $B \in C([0, \infty); \mathbb{R}^{n\times n})$ in (6.1) *be such that $\|B(t)\| \leq \tilde{b}$ for all $t \in [0, \infty)$.*

(ii) Let there exist a function

$$V : \mathbb{R}^m \to \mathbb{R}\ , \ \ y \mapsto V(y) = y^T P y\ ,$$

where $P \in \mathbb{R}^{n\times n}$ is a positive definite, symmetric matrix $P \in \mathbb{R}^{n\times n}$, i.e. for some constants $\alpha_2 \geq \alpha_1 > 0$

$$\alpha_1 |y|^2 \leq V(y) \leq \alpha_2 |y|^2 \quad \textit{for all } y \in \mathbb{R}^n\ . \tag{6.10}$$

(iii) the inequality

$$y^T(PA(t) + A^T(t)P)y \leq -\alpha_3 |y|^2 \textit{ for all } t \geq 0\ , \ \ y \in \mathbb{R}^n\ ,$$

holds with

$$\alpha_3 > 2n\tilde{b}m\frac{\alpha_2}{\alpha}\ , \tag{6.11}$$

where $\alpha = \min\{1, \alpha_1\}$, $m = ||P||$, $||B(t)|| \leq \tilde{b} \in \mathbb{R}_+$ for all $t \in [0, \infty)$.

Theorem 6.3 *Let $\tilde{b} \in \mathbb{R}_+$, $n \in \mathbb{N}$, h, λ, α, m, α_2, $\alpha_3 > 0$, Assumption 6.2 hold, $u \in L_\infty([0, \infty); \mathbb{R}^n)$ and $\varphi \in C([-h, 0]; \mathbb{R}^n)$. Then the following estimate for the solution to the problem* (6.1) *holds*

$$|x(t)| \leq \frac{1}{\sqrt{\alpha_1}} \max\{1, h\} \|\varphi\| e^{-\lambda t} + \frac{2\alpha_2 m}{\alpha\alpha_3 - 2\alpha_2 n\tilde{b}m} \|u\|\ , \quad t \geq 0\ .$$

Proof. Compute the derivative $\dot{V}(x(t))$ of the Lyapunov function along the solution x

$$\begin{aligned} \dot{V}(x(t)) &= x^T(t)\left(PA(t) + A^T(t)P\right)x(t) \\ &+ \left(B^T(t)x_h^{\vee T}(t) + u^T(t)\right)Px(t) + x^T(t)P\left(B(t)x_h^{\vee}(t) + u(t)\right) \\ &\leq -\alpha_3 |x(t)|^2 + 2|x(t)| \|B(t)P\| |x_h^{\vee}(t)| + 2|x(t)| \|P\| \|u\| . \end{aligned} \tag{6.12}$$

Let $\varepsilon > 0$ be an arbitrary small number and define the function $R : [-h, \infty) \to (0, \infty)$ s.t.

$$R^2(t) := \begin{cases} V(x(t)) + \varepsilon\ , & t \geq 0\ , \\ |\varphi(t)|^2 + \varepsilon\ , & t \in [-h, 0)\ . \end{cases} \tag{6.13}$$

From (6.10) it follows that for all $t \in [0\,,\infty)$

$$|x(t)| \le \sqrt{\frac{V(x(t))}{\alpha_1}} < \sqrt{\frac{V(x(t))+\varepsilon}{\alpha_1}} = \sqrt{\frac{R^2(t)}{\alpha_1}} = \frac{R(t)}{\sqrt{\alpha_1}} \le \frac{R(t)}{\sqrt{\alpha}}\ .$$

Let $\xi \in [-h\,,\infty)$ be such that $\max\limits_{s\in[t-h\,,t]}|x(s)| = |x(\xi)|$. Assume $t \in (h\,,\infty)$. Then $\xi > 0$ and by (2.9), we obtain

$$\begin{aligned} |x_h^\vee(t)| &\le n \max_{s\in[t-h\,,t]}|x(s)| = n|x(\xi)| \le n\frac{\sqrt{V(x(\xi))}}{\sqrt{\alpha_1}} \\ &< n\frac{\sqrt{V(x(\xi))+\varepsilon}}{\sqrt{\alpha_1}} = n\frac{R(\xi)}{\sqrt{\alpha_1}} \le n\frac{R_h^\vee(t)}{\sqrt{\alpha_1}} \le n\frac{R_h^\vee(t)}{\sqrt{\alpha}}\ . \end{aligned} \tag{6.14}$$

Now, let $t \in [0\,,h]$. If $\xi \in (0\,,\infty)$ then inequality (6.14) holds. If $\xi \in [-h\,,0]$ then

$$|x_h^\vee(t)| \le n \max_{s\in[t-h\,,t]}|x(s)| = n|x(\xi)| = n|\varphi(\xi)| < nR(\xi) \le n\frac{R(\xi)}{\sqrt{\alpha}} \le n\frac{R_h^\vee(t)}{\sqrt{\alpha}}\ .$$

Then for $t \ge 0$ we obtain

$$2R(t)\dot{R}(t) = \dot{V}(x(t)) \le -\alpha_3\frac{R^2(t)}{\alpha_2} + 2R(t)\frac{n\tilde{b}m}{\alpha}R_h^\vee(t) + \frac{2R(t)}{\sqrt{\alpha}}m\,\|u\|\ ,$$

or

$$\dot{R}(t) \le -\alpha_3\frac{R(t)}{2\,\alpha_2} + \frac{n\tilde{b}m}{\alpha}R_h^\vee(t) + \frac{m\,\|u\|}{\sqrt{\alpha_1}}\ .$$

By Lemma 6.1 with

$$a(t) = \frac{\alpha_3}{2\,\alpha_2}\ , \qquad b(t) = \frac{n\tilde{b}m}{\alpha}\ , \qquad w(t) = \frac{m\,\|u\|}{\sqrt{\alpha_1}}\ , \tag{6.15}$$

$$\exists \lambda > 0:\ R(t) \le \max\{1\,,h\}\,\|\varphi\|\,e^{-\lambda\,t} + \frac{2\,\alpha_2\sqrt{\alpha_1}\,m}{\alpha\,\alpha_3 - 2\,\alpha_2\,n\,\tilde{b}\,m}\,\|u\|\ .$$

Since $\varepsilon > 0$ is an arbitrary number, we can take the limit for $\varepsilon \to 0$ and obtain

$$\sqrt{V(x(t))} \le \max\{1\,,h\}\,\|\varphi\|\,e^{-\lambda\,t} + \frac{2\,\alpha_2\sqrt{\alpha_1}\,m}{\alpha\,\alpha_3 - 2\,\alpha_2\,n\,\tilde{b}\,m}\,\|u\|\ .$$

Therefore from $|x(t)| \le \frac{\sqrt{V(x(t))}}{\sqrt{\alpha_1}}$ for any $t \ge 0$, we get

$$|x(t)| \le \frac{1}{\sqrt{\alpha_1}}\max\{1\,,h\}\,\|\varphi\|_h\,e^{-\lambda\,t} + \frac{2\,\alpha_2\,m}{\alpha\,\alpha_3 - 2\,\alpha_2\,n\,\tilde{b}\,m}\,\|u\|\ .$$

□

Remark 6.4 *Note that from the proof of the Theorem 6.3 it follows that* (6.1) *is not only ISS but even eISS.*

The next statement follows immediately from the Theorem 6.3.

Corollary 6.5 *Let system* (6.1) *be eISS. Then system* (6.1) *is 0-GES.*

6.1 Examples

Example 6.6 *Let $h > 0$, $a > b > 0$, $u \in L_\infty([0,\infty);\mathbb{R})$, $\varphi \in C([-h,0];\mathbb{R})$. Consider a scalar system with input*

$$\begin{aligned} \dot{x}(t) &= -a\,x(t) + b\,x_h^\vee(t) + u\,, & t &\geq 0\,,\\ x(t) &= \varphi(t)\,, & t &\in [-h,0]\,. \end{aligned}$$

Applying Theorem 6.3 with $A(t) = -a$, $B(t) = b$, $P = m = 1$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 2a > 2b$, we get $\gamma = \frac{1}{a-b}$ and

$$|x(t)| \leq \max\{1\,,h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{1}{a-b}\,\|u\|\;, \quad t \geq 0\,,$$

for some $\lambda > 0$ i.e., the considered problem is eISS.

Example 6.7 *Consider*

$$\begin{aligned} \dot{x}(t) &= -2x(t) + \sin t\, x_h^\vee(t) + u\,, & t &\geq 0\,,\\ x(t) &= 1\,, & t &\in [-h,0]\,. \end{aligned} \tag{6.16}$$

for fixed $h = 2$. Check the eISS property. Apply Theorem 6.3 in case $A(t) = -2$, $B(t) = \sin t$, $|\sin t| \leq 1 =: \tilde{b}$, for all $t \in [0,\infty)$, and check the conditions of the Theorem 6.3:

(i) choose $V(x) = x^2$ then $P = 1 = m$ and $\alpha_1 = \alpha_2 = \alpha = 1$;

(ii) $A(t) = -2$ then $2x(-2x) = -4x^2$ and the inequality $\alpha_3 = 4 > 2$ holds.

Since the conditions of Theorem 6.3 hold, one can calculate

$$\gamma(\|u\|) = \frac{2\,\alpha_2\,m}{\alpha\,\alpha_3 - 2\,\alpha_2\,n\,\tilde{b}\,m}\,\|u\| = \|u\|\;,$$

$$\beta(\|\varphi\|\,,t) = \frac{1}{\sqrt{\alpha_1}}\max\{1\,,h\}\,\|\varphi\|\,e^{-\lambda t} = 2\,e^{-\lambda t}\;, \quad t \geq 0\,.$$

From Lemma 6.1 it follows that there exists $\lambda > 0$ such that

$$|x(t)| \leq 2\,e^{-\lambda t} + ||u||\;,\;\; t \geq 0\,.$$

One can check that $\lambda = 0.27$ satisfies $\lambda - 2 + e^{2\lambda} < 0$. The Figure 8 illustrates the behavior of solution to the problem (6.16) *and its input-to-state estimate in case $u(t) = \cos t$ for all $t \in [0,\infty)$.*

Example 6.8 *Consider the following system*

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B(t)x_h^\vee(t) + u(t)\,, & t &\geq 0\,,\\ x(t) &= \varphi(t)\,, & t &\in [-h,0]\,, \end{aligned} \tag{6.17}$$

where

$$A = \begin{pmatrix} -3 & 1 \\ -\frac{1}{4} & -2 \end{pmatrix},\;\; B(t) = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{3}\cos t & 0 \end{pmatrix},\;\; u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

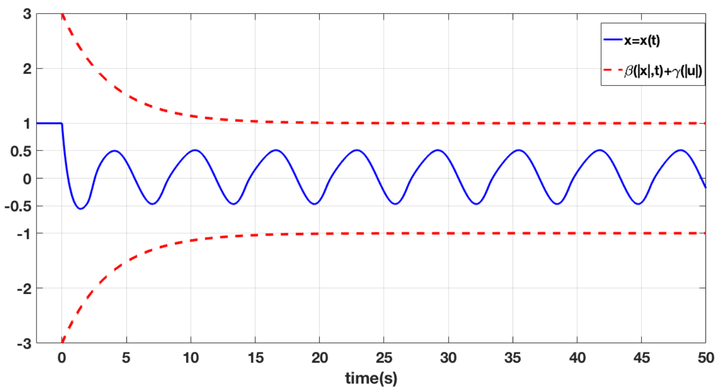

Figure 8: *Graph of the solution to* (6.16) *and its boundedness.*

$h = 1$, $\varphi \in C([-1,0];\mathbb{R}^2)$, $u \in L_\infty([0,\infty);\mathbb{R}^2)$. *Find the ISS estimate for the solution of the problem* (6.17) *using Theorem 6.3. Let* $\tilde{b} := \frac{1}{3} = ||B(t)||$, $t \in [0,\infty)$. *Check Assumption 6.2 (ii), (iii). Define*

$$P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

then $||P|| = m = 2$ *and Lyapunov function*

$$V(x) = x^T \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} x = 2x_1^2 + 2x_2^2 \text{ for all } x \in \mathbb{R}^2 .$$

Then the estimate (6.10) *holds for* $\alpha_1 = \alpha_2 = 2$. *Find* α_3 *such that the following inequality holds*

$$\begin{aligned} x^T(PA + A^T P)x &= x^T \left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -\frac{1}{4} & -2 \end{pmatrix} + \begin{pmatrix} -3 & -\frac{1}{4} \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] x \\ &= x^T \begin{pmatrix} -12 & \frac{3}{2} \\ \frac{3}{2} & -8 \end{pmatrix} x = -x^T Q x \le -\tilde{\lambda}_{min}(Q)|x|^2 = -\alpha_3 |x|^2 , \end{aligned} \tag{6.18}$$

where

$$Q = \begin{pmatrix} 12 & -\frac{3}{2} \\ -\frac{3}{2} & 8 \end{pmatrix},$$

and $\alpha_3 = \tilde{\lambda}_{min}(Q) = \min\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$. *Here* $\tilde{\lambda}_1 = \frac{25}{2}$, $\tilde{\lambda}_2 = \frac{15}{2}$ *are eigenvalues of* Q.

Notice that Assumption 6.2 (ii), (iii) hold with $\alpha_3 = \frac{15}{2} > \frac{16}{3}$. *Hence, the conditions of the Theorem 6.3 hold and one obtains the following estimate for the solution of the problem* (6.17)

$$|x(t)| \le \frac{1}{\sqrt{2}} ||\varphi|| e^{-\lambda t} + \frac{48}{13} ||u|| , \quad t \ge 0 ,$$

where $\lambda > 0$ *satisfies* $\lambda - \frac{15}{8} + \frac{4}{3} e^{\lambda} < 0$.

7 Conclusions

This work presents several interesting properties of solutions to differential equations with *max*-operator subjected to external perturbations. In particular the ISS property including explicit expression for the ISS gain function is derived. In our future work, characterizations of ISS by means of superpositions of other stability properties like asymptotic gain, global stability and 0-GAS property will be investigated for systems of more general type, then it is considered in Theorem 6.3.

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