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EXISTENCE OF POSITIVE BOUNDED SOLUTIONS TO A CLASS OF NEUTRAL INTEGRAL EQUATIONS WITH PRODUCT PERTURBATIONS

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Abstract. In this paper, we study the existence of positive bounded solutions for the following nonlinear integral equation with product perturbation:

$$x(t) = g(t, x(t)) \left[\alpha(t)\, x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\, \mathrm{d}s + h(t, x(t))\right], \quad t \in \mathbb{R}\,.$$

In addition, a concrete example is given to illustrate our existence theorem.

Keywords: nonlinear integral equation, positive solution, product perturbation.

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1 Introduction

In [6], Cooke and Kaplan initiated the study on the following delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, \mathrm{d}s\,, \tag{1.1}$$

which is a model for the spread of some infectious diseases. Afterwards, Fink and Gatica [10] considered the existence of positive almost periodic solution to equation (1.1). Since the work of Fink and Gatica, there has been of great interest for many mathematicians to investigate the existence of positive almost periodic type solutions to equation (1.1) (see, *e.g.*, [1, 2, 3, 4, 5, 8, 9,

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11, 13, 16, 14, 15] and references therein). Especially, in [1], Ait Dads and Ezzinbi studied the existence of positive almost periodic solutions for the following neutral integral equation,

$$x(t) = \gamma\, x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} f(s, x(s))\,\mathrm{d}s\ . \tag{1.2}$$

Ait Dads and Ezzinbi [2] considered the infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t-s)\, f(s, x(s))\,\mathrm{d}s \tag{1.3}$$

and Ait Dads *et al.* [4] studied the more general infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\,\mathrm{d}s\ . \tag{1.4}$$

In [8], N'Guérékata *et al.* studied the following integral equation

$$x(t) = \alpha(t)\, x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\,\mathrm{d}s + h(t, x(t))\ , \tag{1.5}$$

which unifies (1.1)-(1.4). Very recently, N'Guérékata and the authors of this paper [14] investigated the existence and uniqueness of bounded and continuous solutions to the following nonlinear quadratic integral equation

$$x(t) = g(t, x(t))\int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\,\mathrm{d}s\ , \quad t \in \mathbb{R}\ . \tag{1.6}$$

Stimulated by the above work, this paper aims to do some further research on this direction, *i.e.*, we will study the following nonlinear integral equation with product perturbation and neutral term:

$$x(t) = g(t, x(t))\,[\alpha(t)\, x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\,\mathrm{d}s + h(t, x(t))]\ , \quad t \in \mathbb{R}\ , \tag{1.7}$$

where g , α , β , a , f , h satisfy some conditions in Section 3. It is needed to note that due to the neutral term, we can not follow the organization of the proof in [14]. This leads a weaker main result to some extent than expected extension. In addition, for convenience, we only discuss the existence of positive bounded and continuous solutions for equation (1.7) here. We believe that our approach used in this paper can be applied to study the existence of almost periodic type solutions to equation (1.7).

Also, we note that equation (1.7) and its variants are called quadratic functional integral equation in some recent literature and has attracted great interest for many authors (cf. [12] and references therein).

2 Preliminaries

Let E and F be two metric spaces. We denote by $C(E, F)$ the space of all continuous functions defined on E with values in F, by $BC(E, F)$ the space of continuous and bounded functions defined on E with values in F, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+$ the set of positive real numbers, and by $\mathbb{R}^+$ the set of nonnegative real numbers. In the particular case where $E = \mathbb{R}$ and $F = \mathbb{R}^+$, for every $x\,,\, y \in BC(\mathbb{R}, \mathbb{R}^+)$, we denote the distance between x and y by

$$\|x - y\| = \sup_{t \in \mathbb{R}} |x(t) - y(t)| \ .$$

We denote by $L^1(\mathbb{R}^+)$ the space of all Lebesgue measurable function on $\mathbb{R}^+$ with norm

$$\|x\|_{L^1(\mathbb{R}^+)} = \int_0^{+\infty} |x(t)| \,\mathrm{d}t \ .$$

Next, we need to recall some basic notations about cone (for more details, cf. [7]). Let X be a real Banach space, and θ be the zero element in X. A closed convex set K in X is called a cone if the following conditions are satisfied:

(1) if $x \in K$, then $\lambda\, x \in K$ for any $\lambda \geq 0$,

(2) if $x \in K$ and $-x \in K$, then $x = \theta$.

A cone K induces a partial ordering $\leq$ in X by

$$x \leq y \Leftrightarrow y - x \in K \ .$$

For any given $u\,,\, v \in K$ with $u \leq v$,

$$[u, v] := \{x \in X : u \leq x \leq v\} \ .$$

A cone K is called normal if there exists a constant $k > 0$ such that

$$\theta \leq x \leq y \Rightarrow \|x\| \leq k\|y\| \ ,$$

where $\|\cdot\|$ is the norm on X. We denote by K° the interior of K. A cone K is called a solid cone if $K^\circ \neq \emptyset$.

Lemma 2.1 [4] *Suppose that the function $t \mapsto a(t, \cdot)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}^+))$ and $f \in BC(\mathbb{R}, \mathbb{R})$. Then $F \in BC(\mathbb{R}, \mathbb{R})$, where*

$$F(t) = \int_{-\infty}^{t} a(t, t - s)\, f(s) \,\mathrm{d}s \ , \quad t \in \mathbb{R} \ .$$

The following corollary can be deduced from [8, Theorem 3.1]:

Corollary 2.2 *Let K be a normal solid cone in a real Banach space X, $D : K \to K$ be a linear operator, and $A\,,\, B$ be two operators from $K^\circ \times K^\circ$ to K° with*

$$A(x, z) = B(x, z) + D(x) \ , \quad x\,,\, z \in K^\circ \ .$$

Assume that the following conditions hold:

(C1) for each $x\,,\, z \in K^\circ$, $B(\cdot, z)$ is increasing in K° and $B(x, \cdot)$ is decreasing in K°;

(C2) there exists a function $\varphi : (0,1) \to (0,+\infty)$ *such that for each* x , $z \in K^\circ$ *and* $t \in (0,1)$, $\varphi(t) > t$ *and*

$$B(t\,x, z) \geq \varphi(t)\, B(x,z)\ ;$$

(C3) there exist x_0 , $y_0 \in K^\circ$ *with* $x_0 \leq y_0$ *such that* $A(x_0, x_0) \geq x_0$ *and* $A(y_0, y_0) \leq y_0$;

(C4) there exists a constant $L > 0$ *such that for all* x , z_1 , $z_2 \in K^\circ$ *with* $z_1 \geq z_2$,

$$B(x, z_1) - B(x, z_2) \geq -L(z_1 - z_2)\ .$$

Then A *has a unique fixed point* $x^* \in [x_0, y_0]$*, i.e.,* $A(x^*, x^*) = x^*$*. In addition,* x^* *is the unique fixed point of* A *in* K°*.*

3 Main results

In this section, we will study the following nonlinear integral equation

$$x(t) = g(t, x(t))\,[\alpha(t)\,x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)\,f(s, x(s))\,\mathrm{d}s + h(t, x(t))]\ , \quad t \in \mathbb{R}\ . \tag{3.1}$$

Now we give the list of assumptions which are used in this section.

(H1) $f \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ satisfy that for each $s \in \mathbb{R}$, $f(s,\cdot)$ is increasing in $\mathbb{R}^+$. Moreover, we denote by $M_f = \sup\{|f(s,x)| : s \in \mathbb{R}\,,\ x \in \mathbb{R}^+\}$.

(H2) There exists $\eta \in (0,1)$ such that

$$f(s, \lambda\,x) \geq \lambda^\eta\, f(s,x)$$

for all $s \in \mathbb{R}\,,\ \lambda \in (0,1)$ and $x \geq 0$. In addition, there exists $L_f > 0$ such that

$$|f(s, x_1) - f(s, x_2)| \leq L_f |x_1 - x_2|$$

for all $s \in \mathbb{R}$ and x_1 , $x_2 \in \mathbb{R}^+$.

(H3) a is a function from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}^+$, and the function $t \mapsto a(t,\cdot)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}^+))$. Moreover, we denote by $D = \sup\limits_{t\in\mathbb{R}} \int_0^{+\infty} |a(t,s)|\,\mathrm{d}s$.

(H4) $g \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ satisfy that for each $t \in \mathbb{R}$, $g(t,\cdot)$ is increasing in $\mathbb{R}^+$. In addition, we denote by $M_g = \sup\{|g(t,x)| : t \in \mathbb{R},\ x \in \mathbb{R}^+\}$.

(H5) There exists $L_g > 0$ such that

$$|g(t, x_1) - g(t, x_2)| \leq L_g |x_1 - x_2|$$

for all $t \in \mathbb{R}$ and x_1 , $x_2 \in \mathbb{R}^+$.

(H6) $h \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ satisfy that for all $t \in \mathbb{R}$, $h(t,\cdot)$ is decreasing in $\mathbb{R}^+$. Moreover, we denote by $M_h = \sup\{|h(t,x)| : t \in \mathbb{R}\,,\ x \in \mathbb{R}^+\}$.

(H7) There exists $L_h > 0$ such that

$$h(t, z_1) - h(t, z_2) \geq -L_h(z_1 - z_2)$$

for all $t \in \mathbb{R}$ and $z_1 \geq z_2 \geq 0$.

(H8) The function α is in $BC(\mathbb{R}, \mathbb{R}^+)$, and we denote by $M_\alpha = \sup\{|\alpha(t)| : t \in \mathbb{R}\}$. In addition, $M_g\, M_\alpha < 1$.

(H9) There exists a constant $c > 0$ such that

$$\inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s)\, f(s, c)\, \mathrm{d}s \geq c.$$

Let

$$K = \{x \in BC(\mathbb{R}, \mathbb{R}^+) : x(t) \geq 0\ ,\ \forall t \in \mathbb{R}\}\ .$$

Then

$$K^\circ = \{x \in BC(\mathbb{R}, \mathbb{R}^+) : \text{there exists } \xi > 0 \text{ such that } x(t) \geq \xi\ ,\ \forall t \in \mathbb{R}\}\ .$$

It is easy to verify that K is a normal solid cone in $BC(\mathbb{R}, \mathbb{R}^+)$.

Now, for every $y \in BC(\mathbb{R}, \mathbb{R}^+)$, we define an operator A_y on $K^\circ \times K^\circ$ by

$$A_y(x, z)(t) = B_y(x, z)(t) + D_y(x)(t)\ , \tag{3.2}$$

where

$$B_y(x, z)(t) = g(t, y(t))\, [\int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\, \mathrm{d}s + h(t, z(t))]$$

and

$$D_y(x)(t) = g(t, y(t))\, \alpha(t)\, x(t - \beta)\ ,$$

for all $x\ ,\ z \in K^\circ$ and $t \in \mathbb{R}$.

In order to establish the main result of this paper, we need the following lemma.

Lemma 3.1 *Suppose that the assumptions (H1)-(H9) hold. Then, for every $y \in BC(\mathbb{R}, \mathbb{R}^+)$, the operator A_y has a unique fixed point x_y^* in K°, i.e., $A_y(x_y^*, x_y^*) = x_y^*$.*

Proof. For convenience, we assume that $M_f > 0\,,\ M_g > 0\,,\ M_h > 0$ and $M_\alpha > 0$. Fix $y \in BC(\mathbb{R}, \mathbb{R}^+)$. Next, we divide the proof into five steps.

Step 1. $D_y : K \to K$ is a linear operator, and $A_y\ ,\ B_y$ are two operators from $K^\circ \times K^\circ$ to K°.

By the assumptions, it is not difficult to check that D_y is a linear operator and from K to K. Also, for every fixed $x\ ,\ z \in K^\circ$, one can check that $B_y(x, z) \in BC(\mathbb{R}, \mathbb{R}^+)$. Moreover, there exists $\xi > 0$ such that $x(t) \geq \xi$ for all $t \in \mathbb{R}$. By using (H9), there exists a constant $c > 0$ such that

$$\inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s)\, f(s, c)\, \mathrm{d}s \geq c\ .$$

If $\xi \geq c$, then we have

$$\begin{aligned} B_y(x,z)(t) &= g(t,y(t))\,[\int_{-\infty}^{t} a(t,t-s)\, f(s,x(s))\,\mathrm{d}s + h(t,z(t))] \\ &\geq g(t,0)\int_{-\infty}^{t} a(t,t-s)\, f(s,\xi)\,\mathrm{d}s \\ &\geq g(t,0)\int_{-\infty}^{t} a(t,t-s)\, f(s,c)\,\mathrm{d}s \\ &\geq c > 0 \end{aligned}$$

for all $t \in \mathbb{R}$. If $0 < \xi < c$, we also have

$$\begin{aligned} B_y(x,z)(t) &= g(t,y(t))\,[\int_{-\infty}^{t} a(t,t-s)\, f(s,x(s))\,\mathrm{d}s + h(t,z(t))] \\ &\geq g(t,0)\int_{-\infty}^{t} a(t,t-s)\, f(s,\xi)\,\mathrm{d}s \\ &= g(t,0)\int_{-\infty}^{t} a(t,t-s)\, f(s,\frac{\xi}{c}\cdot c)\,\mathrm{d}s \\ &\geq (\frac{\xi}{c})^{\eta}\, g(t,0)\int_{-\infty}^{t} a(t,t-s)\, f(s,c)\,\mathrm{d}s \\ &\geq \frac{\xi}{c}\cdot c \\ &= \xi > 0 \end{aligned}$$

for all $t \in \mathbb{R}$. Thus, $B_y(x,z) \in K^\circ$. Similarly, one can show that $A_y(x,z) \in K^\circ$ for all $x\,,\ z \in K^\circ$.

Step 2. For every $x\,,\ z \in K^\circ,\ B_y(\cdot,z)$ is increasing in K° and $B_y(x,\cdot)$ is decreasing in K°.

For all $t \in \mathbb{R}\,,\ z \in K^\circ$ and $x_1\,,\ x_2 \in K^\circ$ with $x_1 \leq x_2$, we have

$$\begin{aligned} B_y(x_2,z)(t) - B_y(x_1,z)(t) &= g(t,y(t))\,[\int_{-\infty}^{t} a(t,t-s)\, f(s,x_2(s))\,\mathrm{d}s + h(t,z(t))] \\ &\quad - g(t,y(t))\,[\int_{-\infty}^{t} a(t,t-s)\, f(s,x_1(s))\,\mathrm{d}s + h(t,z(t))] \\ &= g(t,y(t))\int_{-\infty}^{t} a(t,t-s)\,[f(s,x_2(s)) - f(s,x_1(s))]\,\mathrm{d}s \\ &\geq 0\,, \end{aligned}$$

which means that $B_y(x_2,z) \geq B_y(x_1,z)$ and $B_y(\cdot,z)$ is increasing in K°. The fact that $B_y(x,\cdot)$ is decreasing in K° can be analogously proved.

Step 3. There exists a function $\varphi : (0,1) \to (0,+\infty)$ such that for every $x\,,\ z \in K^\circ$ and $\lambda \in (0,1),\ \varphi(\lambda) > \lambda$ and $B_y(\lambda\, x,z) \geq \varphi(\lambda)\, B_y(x,z)$.

In fact, by using (H2), for all $t \in \mathbb{R}\,,\ \lambda \in (0,1)$ and $x\,,\ z \in K^\circ$, we have

$$\begin{aligned}
B_y(\lambda\, x, z)(t) &= g(t, y(t))\,[\int_{-\infty}^{t} a(t, t-s)\, f(s, \lambda\, x(s))\,\mathrm{d}s + h(t, z(t))] \\
&\geq g(t, y(t))\,[\lambda^{\eta} \int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\,\mathrm{d}s + \lambda^{\eta}\, h(t, z(t))] \\
&= \lambda^{\eta}\, g(t, y(t))\,[\int_{-\infty}^{t} a(t, t-s)\, f(s, x(s))\,\mathrm{d}s + h(t, z(t))] \\
&= \lambda^{\eta}\, B_y(x, z)(t) \\
&= \varphi(\lambda)\, B_y(x, z)(t)\ ,
\end{aligned}$$

where $\varphi(\lambda) := \lambda^{\eta}$. Obviously, $\varphi(\lambda) > \lambda$ for every $\lambda \in (0,1)$.

Step 4. There exist $x_0\,,\ y_0 \in K^\circ$ with $x_0 \leq y_0$ such that $A_y(x_0, x_0) \geq x_0$ and $A_y(y_0, y_0) \leq y_0$.

Let $x_0(t) \equiv c$ and $y_0(t) \equiv d$ for all $t \in \mathbb{R}$, where

$$d = \max\left\{c\,,\ \frac{M_g(M_f\, D + M_h)}{1 - M_g\, M_\alpha}\right\}.$$

It is easy to know that $x_0 \leq y_0$. By (3.2), we have

$$\begin{aligned}
A_y(y_0, y_0)(t) &= g(t, y(t))\,[\alpha(t)\, y_0(t-\beta) + \int_{-\infty}^{t} a(t, t-s)\, f(s, y_0(s))\,\mathrm{d}s + h(t, y_0(t))] \\
&= g(t, y(t))\,[\alpha(t)\, d + \int_{-\infty}^{t} a(t, t-s)\, f(s, d)\,\mathrm{d}s + h(t, d)] \\
&\leq M_g\,[M_\alpha\, d + M_f\, D + M_h] \\
&\leq d = y_0(t)
\end{aligned}$$

for all $t \in \mathbb{R}$. Therefore, we obtain $A_y(y_0, y_0) \leq y_0$. We also have

$$\begin{aligned}
A_y(x_0, x_0)(t) &= g(t, y(t))\,[\alpha(t)\, x_0(t-\beta) + \int_{-\infty}^{t} a(t, t-s)\, f(s, x_0(s))\,\mathrm{d}s + h(t, x_0(t))] \\
&= g(t, y(t))\,[\alpha(t)\, c + \int_{-\infty}^{t} a(t, t-s)\, f(s, c)\,\mathrm{d}s + h(t, c)] \\
&\geq g(t, 0) \int_{-\infty}^{t} a(t, t-s)\, f(s, c)\,\mathrm{d}s \\
&\geq c = x_0(t)
\end{aligned}$$

for all $t \in \mathbb{R}$. Thus, we know that $A_y(x_0, x_0) \geq x_0$.

Step 5. There exists a constant $L = L_h\, M_g > 0$ such that

$$\begin{aligned} B_y(x,z_1)(t) - B_y(x,z_2)(t) &= g(t,y(t))\,[\int_{-\infty}^{t} a(t,t-s)\,f(s,x(s))\,\mathrm{d}s + h(t,z_1(t))] \\ &\quad - g(t,y(t))\,[\int_{-\infty}^{t} a(t,t-s)\,f(s,x(s))\,\mathrm{d}s + h(t,z_2(t))] \\ &= g(t,y(t))\,[h(t,z_1(t)) - h(t,z_2(t))] \\ &\ge -L_h\, g(t,y(t))(z_1(t) - z_2(t)) \\ &\ge -L_h\, M_g(z_1(t) - z_2(t)) \\ &= -L(z_1(t) - z_2(t)) \end{aligned}$$

for all $t \in \mathbb{R}$, $x \in K^\circ$ and z_1 , $z_2 \in K^\circ$ with $z_1 \ge z_2$.

Now, all conditions of Corollary 2.2 are verified. Thus, A_y has a unique fixed point $x_y^* \in [x_0, y_0]$, *i.e.*, $A_y(x_y^*, x_y^*) = x_y^*$. In addition, x_y^* is the unique fixed point of A_y in K°. □

Theorem 3.2 *Under the assumptions (H1)-(H9), equation* (3.1) *has a unique solution in* $BC(\mathbb{R}, \mathbb{R}^+)$ *provided that*

$$0 < \frac{L_g(M_\alpha\, d + M_f\, D + M_h)}{1 - M_g(M_\alpha + L_f\, D + L_h)} < 1\ .$$

Proof. Define an operator A on $BC(\mathbb{R}, \mathbb{R}^+)$ by

$$\begin{aligned} A(y)(t) &= x_y^*(t) = A_y(x_y^*, x_y^*)(t) \\ &= g(t,y(t))\,[\alpha(t)\,x_y^*(t-\beta) + \int_{-\infty}^{t} a(t,t-s)\,f(s,x_y^*(s))\,\mathrm{d}s + h(t,x_y^*(t))] \end{aligned} \tag{3.3}$$

for all $t \in \mathbb{R}$ and $y \in BC(\mathbb{R}, \mathbb{R}^+)$, where A_y and x_y^* are the same as in Lemma 3.1.

For all $t \in \mathbb{R}$ and y_1 , $y_2 \in BC(\mathbb{R}, \mathbb{R}^+)$, we have

$$\begin{aligned}
&|A(y_1)(t) - A(y_2)(t)| \\
&= \Big| g(t, y_1(t)) \, [\alpha(t) \, x^*_{y_1}(t-\beta) + \int_{-\infty}^{t} a(t, t-s) \, f(s, x^*_{y_1}(s)) \, \mathrm{d}s + h(t, x^*_{y_1}(t))] \\
&\quad - g(t, y_2(t)) \, [\alpha(t) \, x^*_{y_2}(t-\beta) + \int_{-\infty}^{t} a(t, t-s) \, f(s, x^*_{y_2}(s)) \, \mathrm{d}s + h(t, x^*_{y_2}(t))] \Big| \\
&\le \Big| g(t, y_1(t)) \, [\alpha(t) \, x^*_{y_1}(t-\beta) + \int_{-\infty}^{t} a(t, t-s) \, f(s, x^*_{y_1}(s)) \, \mathrm{d}s + h(t, x^*_{y_1}(t))] \\
&\quad - g(t, y_1(t)) \, [\alpha(t) \, x^*_{y_2}(t-\beta) + \int_{-\infty}^{t} a(t, t-s) \, f(s, x^*_{y_2}(s)) \, \mathrm{d}s + h(t, x^*_{y_2}(t))] \Big| \\
&\quad + \Big| g(t, y_1(t)) \, [\alpha(t) \, x^*_{y_2}(t-\beta) + \int_{-\infty}^{t} a(t, t-s) \, f(s, x^*_{y_2}(s)) \, \mathrm{d}s + h(t, x^*_{y_2}(t))] \\
&\qquad - g(t, y_2(t)) \, [\alpha(t) \, x^*_{y_2}(t-\beta) + \int_{-\infty}^{t} a(t, t-s) \, f(s, x^*_{y_2}(s)) \, \mathrm{d}s + h(t, x^*_{y_2}(t))] \Big| \\
&\le |M_g \, M_\alpha [x^*_{y_1}(t-\beta) - x^*_{y_2}(t-\beta)]| + |M_g \int_{-\infty}^{t} a(t, t-s) \, [f(s, x^*_{y_1}(s)) - f(s, x^*_{y_2}(s))] \, \mathrm{d}s| \\
&\quad + |M_g [h(t, x^*_{y_1}(t)) - h(t, x^*_{y_2}(t))]| + |(M_\alpha \, d + M_f \, D + M_h) \, [g(t, y_1(t)) - g(t, y_2(t))]| \\
&\le M_g \, M_\alpha |x^*_{y_1}(t-\beta) - x^*_{y_2}(t-\beta)| + M_g \, L_f \int_{-\infty}^{t} |a(t, t-s)| |x^*_{y_1}(s) - x^*_{y_2}(s)| \, \mathrm{d}s \\
&\quad + M_g \, L_h |x^*_{y_1}(t) - x^*_{y_2}(t)| + L_g (M_\alpha \, d + M_f \, D + M_h) |y_1(t) - y_2(t)| \\
&\le M_g \, M_\alpha |x^*_{y_1}(t-\beta) - x^*_{y_2}(t-\beta)| + M_g \, L_f \|x^*_{y_1} - x^*_{y_2}\| \int_{-\infty}^{t} |a(t, t-s)| \, \mathrm{d}s \\
&\quad + M_g \, L_h |x^*_{y_1}(t) - x^*_{y_2}(t)| + L_g (M_\alpha \, d + M_f \, D + M_h) |y_1(t) - y_2(t)| \, .
\end{aligned}$$

In view of equation (3.3), we know that

$$|A(y_1)(t) - A(y_2)(t)| = |x^*_{y_1}(t) - x^*_{y_2}(t)| \, .$$

Thus, it follows that

$$\begin{aligned}
&|A(y_1)(t) - A(y_2)(t)| \\
&\le M_g \, M_\alpha |A(y_1)(t-\beta) - A(y_2)(t-\beta)| + M_g \, L_f \|x^*_{y_1} - x^*_{y_2}\| \int_{-\infty}^{t} |a(t, t-s)| \, \mathrm{d}s \\
&\quad + M_g \, L_h |A(y_1)(t) - A(y_2)(t)| + L_g (M_\alpha \, d + M_f \, D + M_h) |y_1(t) - y_2(t)|
\end{aligned}$$

and we know that

$$\begin{aligned}
\|A(y_1) - A(y_2)\| \le {} & M_g \, M_\alpha \|A(y_1) - A(y_2)\| + M_g \, L_f \, D \|A(y_1) - A(y_2)\| \\
& + M_g \, L_h \|A(y_1) - A(y_2)\| + L_g (M_\alpha \, d + M_f \, D + M_h) \|y_1 - y_2\| \, .
\end{aligned}$$

Therefore, we have

$$\|A(y_1) - A(y_2)\| \le \frac{L_g (M_\alpha \, d + M_f \, D + M_h)}{1 - M_g (M_\alpha + L_f \, D + L_h)} \|y_1 - y_2\| \, ,$$

where $0 < \frac{L_g (M_\alpha \, d + M_f \, D + M_h)}{1 - M_g (M_\alpha + L_f \, D + L_h)} < 1$. Noting that A is from $BC(\mathbb{R}, \mathbb{R}^+)$ to $BC(\mathbb{R}, \mathbb{R}^+)$, using the well-known Banach contraction theorem, we know that A has a unique fixed point $y \in BC(\mathbb{R}, \mathbb{R}^+)$, *i.e.*, $A(y) = y$. This yields that equation (3.1) has a unique solution in $BC(\mathbb{R}, \mathbb{R}^+)$. $\square$

4 An example

In this section, we present an example to illustrate how the conditions of Theorem 3.2 can be satisfied.

Example 4.1 Let $\alpha(t) = \frac{\pi}{10}$, $\beta = 1$,

$$g(t,x) = \frac{(\sin t + 2)\,[(x+1)^{\frac{1}{2}} + 2]}{9\,\pi[(x+1)^{\frac{1}{2}} + 3]}$$

and

$$h(t,x) = \frac{\sin^2 t}{1+x}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^+$. Let

$$f(s,x) = \frac{(\sin s + 2)\,[(x+1)^{\frac{1}{3}} + 1]}{(x+1)^{\frac{1}{3}} + 2}$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^+$, and

$$a(t,s) = \frac{1}{1+s^2}$$

for all $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$.

Next, we verify all the assumptions of Theorem 3.2. It is easy to know that $f \in C(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+)$. There exists $M_f = 3 > 0$ such that

$$\begin{aligned} 0 < f(s,x) &= \frac{(\sin s + 2)\,[(x+1)^{\frac{1}{3}} + 1]}{(x+1)^{\frac{1}{3}} + 2} \\ &\le \frac{(\sin s + 2)\,[(x+1)^{\frac{1}{3}} + 2]}{(x+1)^{\frac{1}{3}} + 2} \\ &= \sin s + 2 \\ &\le 3 = M_f \end{aligned}$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^+$. Therefore, $f \in BC(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+)$. In addition, taking $0 \le x_1 \le x_2$, we have

$$\begin{aligned} f(s,x_1) - f(s,x_2) &= \frac{(\sin s + 2)\,[(x_1+1)^{\frac{1}{3}} + 1]}{(x_1+1)^{\frac{1}{3}} + 2} - \frac{(\sin s + 2)\,[(x_2+1)^{\frac{1}{3}} + 1]}{(x_2+1)^{\frac{1}{3}} + 2} \\ &= (\sin s + 2)\left[\frac{(x_1+1)^{\frac{1}{3}} - (x_2+1)^{\frac{1}{3}}}{[(x_1+1)^{\frac{1}{3}} + 2][(x_2+1)^{\frac{1}{3}} + 2]}\right] \\ &\le 0 \end{aligned}$$

for all $s \in \mathbb{R}$. Thus, $f(s,\cdot)$ is increasing in $\mathbb{R}^+$ for all $s \in \mathbb{R}$. We conclude that (H1) holds.

There exists $\eta = \frac{1}{3} \in (0,1)$ such that

$$\begin{aligned}
f(s,\lambda\, x) &= \frac{(\sin s+2)\,[(\lambda\, x+1)^{\frac{1}{3}}+1]}{(\lambda\, x+1)^{\frac{1}{3}}+2} \\
&\geq \frac{(\sin s+2)\,[(\lambda\, x+\lambda)^{\frac{1}{3}}+\lambda^{\frac{1}{3}}]}{(x+1)^{\frac{1}{3}}+2} \\
&= \lambda^{\frac{1}{3}}\,\frac{(\sin s+2)\,[(x+1)^{\frac{1}{3}}+1]}{(x+1)^{\frac{1}{3}}+2} \\
&= \lambda^{\frac{1}{3}}\, f(s,x) \\
&= \lambda^{\eta}\, f(s,x)
\end{aligned}$$

for all $x \geq 0$, $\lambda \in (0,1)$ and $s \in \mathbb{R}$. Moreover, there exists $L_f = 1 > 0$ such that

$$\begin{aligned}
|f(s,x_1)-f(s,x_2)| &= \left|\frac{(\sin s+2)\,[(x_1+1)^{\frac{1}{3}}+1]}{(x_1+1)^{\frac{1}{3}}+2} - \frac{(\sin s+2)\,[(x_2+1)^{\frac{1}{3}}+1]}{(x_2+1)^{\frac{1}{3}}+2}\right| \\
&= \left|(\sin s+2)\left[\frac{(x_1+1)^{\frac{1}{3}}-(x_2+1)^{\frac{1}{3}}}{[(x_1+1)^{\frac{1}{3}}+2][(x_2+1)^{\frac{1}{3}}+2]}\right]\right| \\
&\leq 3\left|(x_1+1)^{\frac{1}{3}}-(x_2+1)^{\frac{1}{3}}\right| \\
&\leq 3\cdot\frac{|x_1-x_2|}{3} \\
&= |x_1-x_2|
\end{aligned}$$

for all $s \in \mathbb{R}$ and x_1 , $x_2 \in \mathbb{R}^+$. Thus, (H2) holds.

For all $t \in \mathbb{R}$, we have

$$\int_0^{+\infty} \frac{1}{1+s^2}\,\mathrm{d}s = \frac{\pi}{2} < +\infty\,.$$

Thus, $a(t,\cdot) \in L^1(\mathbb{R}^+)$. It is easy to know that the function $t \mapsto a(t,\cdot)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}^+))$. Moreover, we know that $D = \sup\limits_{t\in\mathbb{R}} \int_0^{+\infty} \frac{1}{1+s^2}\,\mathrm{d}s = \frac{\pi}{2}$. Thus, (H3) holds.

Obviously, $g \in C(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+)$. There exists $M_g = \frac{1}{3\,\pi} > 0$ such that

$$\begin{aligned}
0 < g(t,x) &= \frac{(\sin t+2)\,[(x+1)^{\frac{1}{2}}+2]}{9\,\pi[(x+1)^{\frac{1}{2}}+3]} \\
&\leq \frac{(\sin t+2)\,[(x+1)^{\frac{1}{2}}+3]}{9\,\pi[(x+1)^{\frac{1}{2}}+3]} \\
&= \frac{\sin t+2}{9\,\pi} \\
&\leq \frac{3}{9\,\pi} = \frac{1}{3\,\pi} = M_g
\end{aligned}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^+$. Thus, $g \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$. In addition, let $0 \leq x_1 \leq x_2$, we know that

$$\begin{aligned} g(t, x_1) - g(t, x_2) &= \frac{(\sin t + 2)\,[(x_1+1)^{\frac{1}{2}} + 2]}{9\,\pi[(x_1+1)^{\frac{1}{2}} + 3]} - \frac{(\sin t + 2)\,[(x_2+1)^{\frac{1}{2}} + 2]}{9\,\pi[(x_2+1)^{\frac{1}{2}} + 3]} \\ &= \frac{\sin t + 2}{9\,\pi} \left[\frac{(x_1+1)^{\frac{1}{2}} - (x_2+1)^{\frac{1}{2}}}{[(x_1+1)^{\frac{1}{2}} + 3][(x_2+1)^{\frac{1}{2}} + 3]} \right] \leq 0 \end{aligned}$$

for all $t \in \mathbb{R}$. Therefore, $g(t, \cdot)$ is increasing in $\mathbb{R}^+$ for all $t \in \mathbb{R}$. So (H4) holds.

There exists $L_g = \frac{1}{3\pi} > 0$ such that

$$\begin{aligned} |g(t, x_1) - g(t, x_2)| &= \left| \frac{(\sin t + 2)\,[(x_1+1)^{\frac{1}{2}} + 2]}{9\,\pi[(x_1+1)^{\frac{1}{2}} + 3]} - \frac{(\sin t + 2)\,[(x_2+1)^{\frac{1}{2}} + 2]}{9\,\pi[(x_2+1)^{\frac{1}{2}} + 3]} \right| \\ &\leq \frac{1}{3\,\pi} \left| \frac{(x_1+1)^{\frac{1}{2}} + 2}{(x_1+1)^{\frac{1}{2}} + 3} - \frac{(x_2+1)^{\frac{1}{2}} + 2}{(x_2+1)^{\frac{1}{2}} + 3} \right| \\ &\leq \frac{1}{3\,\pi} |(x_1+1)^{\frac{1}{2}} - (x_2+1)^{\frac{1}{2}}| \\ &\leq \frac{1}{3\,\pi} |x_1 - x_2| \end{aligned}$$

for all $t \in \mathbb{R}$ and x_1 , $x_2 \in \mathbb{R}^+$. Thus, (H5) holds.

There exists $M_h = 1 > 0$ such that

$$0 \leq h(t, x) = \frac{\sin^2 t}{1 + x} \leq 1$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^+$. we know that $h \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$. Therefore, $h \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$. Moreover, let $0 \leq x_1 \leq x_2$, we have

$$\begin{aligned} h(t, x_1) - h(t, x_2) &= \frac{\sin^2 t}{1 + x_1} - \frac{\sin^2 t}{1 + x_2} \\ &= \sin^2 t \left[\frac{x_2 - x_1}{(1 + x_1) + (1 + x_2)} \right] \\ &\geq 0 \end{aligned}$$

for all $t \in \mathbb{R}$. Thus, $h(t, \cdot)$ is decreasing in $\mathbb{R}^+$ for all $t \in \mathbb{R}$. Thus (H6) holds.

There exists $L_h = 1 > 0$ such that

$$\begin{aligned} h(t, z_1) - h(t, z_2) &= \frac{\sin^2 t}{1 + z_1} - \frac{\sin^2 t}{1 + z_2} \\ &= \sin^2 t \left[\frac{z_2 - z_1}{(1 + z_1)(1 + z_2)} \right] \\ &= -\sin^2 t \left[\frac{z_1 - z_2}{(1 + z_1)(1 + z_2)} \right] \\ &\geq -(z_1 - z_2) \end{aligned}$$

for all $t \in \mathbb{R}$ and $z_1 \geq z_2 \geq 0$, *i.e.*, (H7) holds.

It is easy to know that $\alpha \in BC(\mathbb{R}, \mathbb{R}^+)$ and $M_\alpha = \frac{\pi}{10} > 0$. In addition, $M_g\, M_\alpha = \frac{1}{3\,\pi} \cdot \frac{\pi}{10} = \frac{1}{30} < 1$. Thus, (H8) holds.

There exists a constant $c = \frac{1}{48} \approx 0.020833$ such that

$$
\begin{aligned}
&\inf_{t\in\mathbb{R}} g(t,0) \int_{-\infty}^{t} a(t,t-s)\, f(s,c)\,\mathrm{d}s \\
&= \inf_{t\in\mathbb{R}} \left(\frac{\sin t + 2}{12\,\pi}\right) \int_{-\infty}^{t} \left(\frac{1}{1+(t-s)^2}\right) \left(\frac{(\sin s+2)((c+1)^{\frac{1}{3}}+1)}{(c+1)^{\frac{1}{3}}+2}\right) \mathrm{d}s \\
&= \inf_{t\in\mathbb{R}} \left(\frac{\sin t + 2}{12\,\pi}\right) \left(\frac{(c+1)^{\frac{1}{3}}+1}{(c+1)^{\frac{1}{3}}+2}\right) \int_{0}^{+\infty} \frac{\sin(t-s)+2}{1+s^2}\,\mathrm{d}s \\
&\geq \frac{1}{12\,\pi} \left(\frac{(c+1)^{\frac{1}{3}}+1}{(c+1)^{\frac{1}{3}}+2}\right) \int_{0}^{+\infty} \frac{1}{1+s^2}\,\mathrm{d}s \\
&= \frac{(c+1)^{\frac{1}{3}}+1}{24(c+1)^{\frac{1}{3}}+48} \\
&\approx 0.027810 > c\,,
\end{aligned}
$$

which means that (H9) holds.

In addition, we have

$$
d = \max\{c, \frac{M_g(M_f\, D + M_h)}{1 - M_g\, M_\alpha}\} \approx \max\{0.020833, 0.627003\} = 0.627003
$$

and

$$
0 < \frac{L_g(M_\alpha\, d + M_f\, D + M_h)}{1 - M_g(M_\alpha + L_f\, D + L_h)} \approx 0.903598 < 1\,.
$$

Thus, Theorem 3.2 yields that the following integral equation

$$
\begin{aligned}
x(t) = {} & \frac{(\sin t+2)\,[(x(t)+1)^{\frac{1}{2}}+2]}{9\,\pi[(x(t)+1)^{\frac{1}{2}}+3]} \\
& \times \left[\frac{\pi}{10}\,x(t-1) + \int_{-\infty}^{t} \frac{1}{1+(t-s)^2}\,\frac{(\sin s+2)\,[(x(s)+1)^{\frac{1}{3}}+1]}{(x(s)+1)^{\frac{1}{3}}+2}\,\mathrm{d}s + \frac{\sin^2 t}{1+x(t)}\right]
\end{aligned}
$$

has a unique solution in $BC(\mathbb{R}, \mathbb{R}^+)$.

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References

[1] E. Ait Dads, K. Ezzinbi, *Almost periodic solution for some neutral nonlinear integral equation*, Nonlinear Analysis, Theory, Methods and Applications **28**, (1997), pp. 1479–1489.

[2] E. Ait Dads, K. Ezzinbi, *Existence of positive pseudo almost periodic solution for some nonlinear infinite delay integral equations arising in epidemic problems*, Nonlinear Analysis, Theory, Methods and Applications **41**, (2000), pp. 1–13.

[3] E. Ait Dads, P. Cieutat, L. Lhachimi, *Positive almost automorphic solutions for some nonlinear infinite delay integral equations*, Dynamic Systems and Applications **17**, (2008), pp. 515–538.

[4] E. Ait Dads, P. Cieutat, L. Lhachimi, *Positive pseudo almost periodic solutions for some nonlinear infinite delay integral equations*, Mathematical and Computer Modelling **49**, (2009), pp. 721–739.

[5] A. Bellour, E. Ait Dads, *Periodic solutions for nonlinear neutral delay integro-differential equations*, Electronic Journal of Differential Equations **100**, (2015), pp. 1–9.

[6] K. L. Cooke, J. L. Kaplan, *A periodicity threshold theorem for epidemics and population growth*, Mathematical Biosciences **31**, (1976), pp. 87–104.

[7] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[8] H. S. Ding, Y. Y. Chen, G. M. N'Guérékata, *Existence of positive pseudo almost periodic solutions to a class of neutral integral equations*, Nonlinear Analysis **74**, (2011), pp. 7356–7364.

[9] K. Ezzinbi, M. A. Hachimi, *Existence of positive almost periodic solutions of functional equations via Hilbert's projective metric*, Nonlinear Analysis, Theory, Methods and Applications **26**, (1996), pp. 1169–1176.

[10] A. M. Fink, J. A. Gatica, *Positive almost periodic solutions of some delay integral equations*, Journal of Differential Equations **83**, (1990), pp. 166–178.

[11] W. Long, *Existence of positive almost automorphic solutions to a class of integral equations*, African Diaspora Journal of Mathematics **12**, (2011), pp. 48–56.

[12] M. M. A. Metwali, *On a class of quadratic Urysohn-Hammerstein integral equations of mixed type and initial value problem of fractional order*, Mediterranean Journal of Mathematics **13**, (2016), pp. 2691–2707.

[13] R. Torrejón, *Positive almost periodic solutions of a state-dependent delay nonlinear integral equation*, Nonlinear Analysis, Theory, Methods and Applications **20**, (1993), pp. 1383–1416.

[14] C. H. Wang, H. S. Ding, G. M. N'Guérékata, *Existence of positive solutions for some nonlinear quadratic integral equations*, Electronic Journal of Differential Equations **79**, (2019), pp. 1-11.

[15] J. Q. Zhao, Y. K. Chang, G. M. N'Guérékata, *Existence of asymptotically almost automorphic solutions to nonlinear delay integral equations*, Dynamic Systems and Applications **21**, (2012), pp. 339–349.

[16] J. Y. Zhao, H. S. Ding, G. M. N'Guérékata, *Positive almost periodic solutions to integral equations with superlinear perturbations via a new fixed point theorem in cones*, Electronic Journal of Differential Equations **2**, (2017), pp. 1–10.