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EXISTENCE AND UNIQUENESS OF MILD SOLUTION AND APPROXIMATE CONTROLLABILITY OF FRACTIONAL EVOLUTION EQUATIONS WITH DEFORMABLE FRACTIONAL DERIVATIVE

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Abstract. This article is concerned with the existence and uniqueness of mild solution, and approximate controllability of evolution equations with deformable fractional derivative. The results are obtained with the help of semigroup theory, Banach fixed point theorem, and Schauder fixed point theorem.

Keywords: Deformable fractional derivative, Banach fixed point theorem, Schauder fixed point theorem, semigroup theory, approximate controllability.

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1 Introduction

In this article, we study the existence and uniqueness of mild solution for the following initial value problem

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + f(t, x(t)) , \quad t \in J , \\ x(0) &= x_0 . \end{aligned} \tag{1.1}$$

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Next, we establish the sufficient condition for the approximate controllability of the following abstract fractional evolution equation

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + f(t, x(t)) + Bu(t)\,, \quad t \in J\,, \\ x(0) &= x_0\,, \end{aligned} \tag{1.2}$$

where D^{α} is the deformable fractional derivative of order $\alpha \in (0,1)$ which is introduced by Zulfeqarr et. al. [1]. $A : D(A) \subset X \to X$ is an infinitesimal generator of a C_0-semigroup $T(t)(t \geqslant 0)$ on a suitable space X , $x_0 \in X$, and $J = [0,b]$, $b > 0$ is a constant. $u \in L^2(J,U),\ U$ is a Hilbert space, $B : U \to X$ is a bounded linear operator. $f : J \times X \to X$ is a given function satisfying certain assumptions.

The rest of the paper is organized as follows. In section 2, we will give some basic definitions, notations and theorems. Section 3 is further subdivided into two subsections. In first part, we will obtain the expression for mild solutions for the system (1.1) and discuss the sufficient conditions for the existence and uniqueness of mild solution. In second part, we will study the existence of mild solutions for the system (1.2), then we show that the control system (1.2) is approximately controllable on J provided that the corresponding linear system is approximately controllable. Finally, in section 4, we will present some examples to illustrate our results.

2 Preliminaries

In this section we will introduce some basic definitions, notations, preliminaries theorems.

Definition 2.1 *([1]) For a function $f : (a,b) \to \mathbb{R}$, the deformable fractional derivative of order $\alpha \in [0,1]$ is defined as*

$$D^{\alpha}f(t) = \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)\, f(t+\epsilon\alpha) - f(t)}{\epsilon}\,,$$

where $\alpha + \beta = 1$.

Remark 2.2 *If $\alpha = 0$, $D^0 f(t) = f(t)$, and if $\alpha = 1$, $Df(t) = f'(t)$.*

Definition 2.3 *([1]) The α-fractional integral of a continuous function defined on $[a,b]$ is defined by*

$$I_a^{\alpha} f(t) = \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_a^t e^{\frac{\beta}{\alpha}s} f(s)\,\mathrm{d}s\,, \quad \text{where} \quad \alpha+\beta = 1\,,\ \alpha \in (0,1]\,.$$

Theorem 2.4 *([1]) A differentiable function f at a point $t \in (a,b)$ is always α-differentiable at that point for any α. Moreover, in this case we have*

$$D^{\alpha}f(t) = \beta f(t) + \alpha\, Df(t)\,,$$

where $Df = \frac{\mathrm{d}}{\mathrm{d}t} f$.

Theorem 2.5 *([1]) Let f be defined in (a,b). For any α, f is α-differentiable if and only if it is differentiable.*

Theorem 2.6 *([1]) The operators D^α and I^α possess the following properties:*

(i) $D^\alpha(af + bg) = a\, D^\alpha f + b\, D^\alpha g$. *(Linearity)*

(ii) $D^{\alpha_1}\, D^{\alpha_2} = D^{\alpha_2}\, D^{\alpha_1}$. *(Commutativity)*

(iii) $D^\alpha k = \beta\, k$, *when k is a constant.*

(iv) $D^\alpha(f \cdot g) = (D^\alpha f) \cdot g + \alpha\, f \cdot Dg$.

(v) $I_a^\alpha(bf + cg) = b\, I_a^\alpha f + c\, I_a^\alpha g$. *(Linearity)*

(vi) $I_a^{\alpha_1}\, I_a^{\alpha_2} = I_a^{\alpha_2}\, I_a^{\alpha_1}$. *(Commutativity)*

Theorem 2.7 (Inverse Property) *([1]) Let f be a continuous function defined on $[a, b]$, then $I_a^\alpha f$ is α-differentiable in (a, b). In fact, we have*

$$D^\alpha(I_a^\alpha f(t)) = f(t).$$

Conversely, suppose g is a continuous anti-α-derivative of f over (a, b), that is $g = D^\alpha f$, then

$$I_a^\alpha(D^\alpha f(t)) = I_a^\alpha(g(t)) = f(t) - e^{\frac{\beta}{\alpha}(a-t)} f(a)\ .$$

Theorem 2.8 (Schauder fixed point theorem) *([4]) If Ω is a closed bounded and convex subset of a Banach space X, and $F : \Omega \to \Omega$ is completely continuous, then F has a fixed point in Ω.*

3 Main Results

In this section, first we will discuss the existence and uniqueness of mild solution for the system (1.1), then we state and prove conditions for the approximate controllability for system (1.2).

3.1 Existence and uniqueness of mild solution

Let X be a Banach space with norm $\|\cdot\|$, and $C(J, X)$ be the Banach space of all continuous functions from J into X endowed with supremum norm $\|x\| = \sup\limits_{t \in J} \|x(t)\|$. Denote $M = \sup_{t \in J} \|T(t)\|_{\mathcal{L}(X)}$, where $\mathcal{L}(X)$ stands for the Banach space of all linear and bounded operators on X, note that $M \geqslant 1$.

Lemma 3.1 *Let A be an infinitesimal generator of a C_0-semigroup $T(t)$ $(t \geqslant 0)$ on X. Then for $x \in D(A)$,*

$$D^\alpha T(t)\, x = (\beta\, I + \alpha\, A)\, T(t) x\ .$$

Proof. By [2], we know that for $x \in D(A)$ we have $T(t)x \in D(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\, T(t)\, x = A\, T(t)\, x = T(t)\, A\, x\ .$$

Now using Theorem 2.4, we get

$$\begin{aligned}D^\alpha T(t)\,x &= \beta\,T(t)\,x + \alpha\,\frac{\mathrm{d}}{\mathrm{d}t}\,T(t)\,x\\ &= \beta\,T(t)\,x + \alpha\,A\,T(t)\,x\\ &= (\beta\,I + \alpha\,A)\,T(t)x\;.\end{aligned}$$

□

Let A be a linear operator from $D(A) \subset X$ into X and $x_0 \in X$. Consider the following linear deformable fractional abstract Cauchy problem

$$\begin{aligned}D^\alpha x(t) &= A\,x(t)\;, \quad t \in J\;,\\ x(0) &= x_0\;. \end{aligned} \tag{3.1}$$

Definition 3.2 *A function x is called a solution to the problem (3.1), if the following hold:*

(i) $x \in C(J, X)$ and $x(t) \in D(A)$ for all $t \in J$.

(ii) $D^\alpha x$ exists and is continuous on J.

(iii) x satisfies (3.1).

Theorem 3.3 *Let A be an infinitesimal generator of a C_0-semigroup $T(t)(t \geqslant 0)$. If $x_0 \in D(A)$, then $e^{\frac{-\beta}{\alpha}t}\,T(\frac{t}{\alpha})x_0$ is a solution to the problem (3.1).*

Proof. Let $x(t) = e^{\frac{-\beta}{\alpha}t}\,T(\frac{t}{\alpha})\,x_0$. Since $x_0 \in D(A)$, $x(t)$ is differentiable. Now using Lemma 3.1, Theorem 2.4, and Theorem 2.6, we get

$$\begin{aligned}D^\alpha x(t) &= \Big[(D^\alpha e^{\frac{-\beta}{\alpha}t})\,T(\frac{t}{\alpha}) + \alpha\,e^{\frac{-\beta}{\alpha}t}\,\frac{\mathrm{d}}{\mathrm{d}t}\,T(\frac{t}{\alpha})\Big]x_0\\ &= \Big[(\beta\,e^{\frac{-\beta}{\alpha}t} + \alpha\,(\frac{-\beta}{\alpha})\,e^{\frac{-\beta}{\alpha}t})\,T(\frac{t}{\alpha}) + e^{\frac{-\beta}{\alpha}t}\,A\,T(\frac{t}{\alpha})\Big]x_0\\ &= A\,e^{\frac{-\beta}{\alpha}t}\,T(\frac{t}{\alpha})\,x_0\\ &= A\,x(t)\;.\end{aligned}$$

□

Now we consider the inhomogeneous initial value problem

$$\begin{aligned}D^\alpha x(t) &= A\,x(t) + f(t)\;, \quad t \in J\;,\\ x(0) &= x_0\;, \end{aligned} \tag{3.2}$$

where A is an infinitesimal generator of a C_0-semigroup, $x_0 \in X$, and $f : J \to X$ is a suitable function.

Theorem 3.4 *Let x be a solution of the problem (3.2) and $f \in L^1(J,X)$, then x satisfies*

$$x(t) = e^{\frac{-\beta}{\alpha} t}\, T(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s.$$

Proof. Since x is a solution of the problem (3.2), we have

$$D^\alpha x(t) = A\, x(t) + f(t). \tag{3.3}$$

Also by Theorem (2.4) we know

$$D^\alpha x(t) = \beta\, x(t) + \alpha\, x'(t). \tag{3.4}$$

From (3.3) and (3.4), we can easily conclude that

$$x'(t) = \frac{1}{\alpha}\left[(A - \beta I)\, x(t) + f(t)\right]. \tag{3.5}$$

Let $h(s) = T(\frac{t-s}{\alpha})\, x(s),\ 0 \leqslant s \leqslant t$, since $x(s) \in D(A)$, therefore h is differentiable and hence α-differentiable by Theorem (2.4). Now using Theorem (2.6), Lemma (3.1), and (3.5), we get

$$\begin{aligned} D^\alpha h(s) &= \left(D^\alpha T(\frac{t-s}{\alpha})\right) x(s) + \alpha\, T(\frac{t-s}{\alpha}) x'(s) \\ &= \left[\beta\, T(\frac{t-s}{\alpha}) + \alpha\, \frac{\mathrm{d}}{\mathrm{d}t} T(\frac{t-s}{\alpha})\right] x(s) + T(\frac{t-s}{\alpha})[(A - \beta\, I)\, x(s) + f(s)] \\ &= \left[\beta\, T(\frac{t-s}{\alpha}) - A\, T(\frac{t-s}{\alpha})\right] x(s) + A\, T(\frac{t-s}{\alpha})\, x(s) - \beta\, T(\frac{t-s}{\alpha})\, x(s) + T(\frac{t-s}{\alpha})\, f(s) \\ &= T(\frac{t-s}{\alpha})\, f(s)\,. \end{aligned} \tag{3.6}$$

By Theorem (2.7), we obtain

$$\begin{aligned} I^\alpha(D^\alpha h(t)) \quad &= h(t) - e^{\frac{-\beta}{\alpha} t}\, h(0) \\ &= x(t) - e^{\frac{-\beta}{\alpha} t}\, T(\tfrac{t}{\alpha})\, x_0\,. \end{aligned} \tag{3.7}$$

Since $f \in L^1(J,X)$, then $T(\frac{t-s}{\alpha})\, fs)$ is integrable. Integrating (3.6), we have

$$I^\alpha(D^\alpha h(t)) = \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s\,. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$x(t) = e^{\frac{-\beta}{\alpha} t}\, T(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s\,.$$

□

Theorem 3.5 *Let A be the infinitesimal generator of a C_0-semigroup and $f \in C(J,X)$. If $f(s) \in D(A)$ for $0 < s < t$ and $Af(s) \in L^1(J,X)$ then for every $x_0 \in D(A)$ the function $x : J \to X$ defined by*

$$x(t) = e^{\frac{-\beta}{\alpha} t}\, T(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s \tag{3.9}$$

is a solution to the initial value problem (3.2).

Proof. Let $u(t) = e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha})\, x_0$ and $v(t) = \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s$, therefore $x(t)$ given in (3.9) can be rewritten as $x(t) = u(t) + v(t)$. Since $x_0 \in D(A)$, $u(t)$ is differentiable and by Theorem (3.3), we know $D^\alpha u(t) = Au(t)$. From the assumptions, it is easy to conclude that $v(t)$ is differentiable, and

$$\begin{aligned} v'(t) &= \frac{1}{\alpha}\, (\frac{-\beta}{\alpha})\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t}\, \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s \\ &= \frac{-\beta}{\alpha^2}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \left[\int_0^t e^{\frac{\beta}{\alpha}s}\, \frac{\partial}{\partial t} T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + e^{\frac{\beta}{\alpha}t}\, f(t) \right] \\ &= \frac{-\beta}{\alpha^2}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \left[\int_0^t e^{\frac{\beta}{\alpha}s}\, \frac{1}{\alpha}\, A\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + e^{\frac{\beta}{\alpha}t}\, f(t) \right] \\ &= \frac{-\beta}{\alpha^2}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + \frac{1}{\alpha^2}\, A\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s)\, \mathrm{d}s + \frac{1}{\alpha}\, f(t) \\ &= \frac{1}{\alpha} \Big(-\beta\, v(t) + A\, v(t) + f(t) \Big)\,. \end{aligned} \tag{3.10}$$

Now by Theorem (2.4) and (3.10), we get

$$\begin{aligned} D^\alpha v(t) &= \beta\, v(t) + \alpha\, v'(t) \\ &= \beta\, v(t) + \Big(-\beta\, v(t) + A\, v(t) + f(t) \Big) \\ &= A\, v(t) + f(t)\,. \end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned} D^\alpha x(t) &= D^\alpha u(t) + D^\alpha v(t) \\ &= A\, u(t) + A\, v(t) + f(t) \\ &= A\, x(t) + f(t)\,, \end{aligned} \tag{3.12}$$

also $x(0) = u(0) + v(0) = x_0$. Thus $x(t)$ given by (3.9) is the solution of inhomogeneous initial value problem (3.2). □

Now we will study the semilinear initial value problem (1.1).

Definition 3.6 *A continuous solution x of the following integral equation*

$$x(t) = e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha})\, x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s, x(s))\, \mathrm{d}s, \tag{3.13}$$

is called a mild solution of the problem (1.1).

We need the following basic assumptions to prove the existence and uniqueness of mild solution to the problem (1.1).

(HA) A is an infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)(t \geqslant 0)$ on X.

(Hf) $f : J \times X \to X$ is continuous and there exists a constant $L > 0$ such that

$$\|f(t,x) - f(t,y)\| \leqslant L\|x-y\|, \quad \forall\, x\,,\, y \in X\,.$$

Theorem 3.7 *Let the hypotheses* $(\mathbf{HA})$ *and* $(\mathbf{Hf})$ *be satisfied, then for every* $x_0 \in X$ *the system (1.1) has a unique mild solution* $x \in C(J,X)$ *provided that* $\frac{ML}{\beta} < 1$.

Proof. For a given $x_0 \in X$, we define a map $F : C(J,X) \to C(J,X)$ by

$$(Fx)(t) = e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha})\, x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s,x(s))\, \mathrm{d}s, \quad t \in J\,.$$

Now we show that F is a contraction. Let $x, y \in C(J,X)$, it follows readily from the definition of F that

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\leqslant \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, \|T(\frac{t-s}{\alpha})\|\, \|f(s,x(s)) - f(s,y(s))\|\, \mathrm{d}s \\
&\leqslant \frac{ML}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \left(\int_0^t e^{\frac{\beta}{\alpha}s}\, \mathrm{d}s \right) \|x-y\| \\
&= \frac{ML}{\beta} e^{\frac{-\beta}{\alpha}t} \left[e^{\frac{\beta}{\alpha}t} - 1 \right] \|x-y\| \\
&= \frac{ML}{\beta} \left[1 - e^{\frac{-\beta}{\alpha}t} \right] \|x-y\| \\
&\leqslant \frac{ML}{\beta} \|x-y\|.
\end{aligned}$$

Hence F is a contraction map, and therefore by Banach contraction principle F has a unique fixed point $x \in C(J,X)$ which is the mild solution of system (1.1). □

3.2 Approximate controllability

Throughout this section we assume that X is a Hilbert space. In this section, we formulate and prove conditions for the approximate controllability of semilinear fractional control differential system (1.2). To do this, we first prove the existence of a fixed point of the operator F_λ defined below by using Schauder fixed point theorem. Next, we show that under certain assumptions the approximate controllability of (1.2) is implied by the approximate controllability of the corresponding linear system.

Definition 3.8 *A function* $x \in C(J,X)$ *is said to be a mild solution of (1.2) if for any* $u \in L^2(J,X)$, *the following integral equation is satisfied*

$$x(t) = e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha}) x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha}) [f(s,x(s)) + Bu(s)]\, \mathrm{d}s, \quad t \in J\,.$$

Let $x_b(x_0,u)$ be the state value of (1.2) at terminal time b corresponding to the control u and the initial value x_0. We introduce the set $\mathcal{R}(b,x_0) = \{x_b(x_0,u) : u \in L^2(J,U)\}$, which is called the reachable set of the system (1.2) at terminal time b, its closure in X is denoted by $\overline{\mathcal{R}(b,x_0)}$.

Definition 3.9 *([4]) The system (1.2) is said to be approximately controllable on J, if $\overline{\mathcal{R}(b,x_0)} = X$, that is, given any $\epsilon > 0$, it is possible to steer from the point x_0 to within a distance ϵ from all points in the state space X at time b.*

Consider the following linear fractional differential system corresponding to (1.2)

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + Bu(t), \quad t \in J\ , \\ x(0) &= x_0\ . \end{aligned} \tag{3.14}$$

Definition 3.10 **(a)** *A controllability map for the system (3.14) on J is a bounded linear map $\mathfrak{B}^b : L^2(J,U) \to X$ which is defined as*

$$\mathfrak{B}^b u := \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, T(\frac{b-s}{\alpha})\, Bu(s)\,\mathrm{d}s\ . \tag{3.15}$$

(b) *The system (3.14) is called approximately controllable on J, if*

$$\overline{ran \mathfrak{B}^b} = X\ .$$

(c) *The controllability gramian of (3.14) on J is defined by*

$$\Gamma_0^b := \mathfrak{B}^b (\mathfrak{B}^b)^*\ . \tag{3.16}$$

Lemma 3.11 *The controllability map and controllability gramian satisfy the following:*

(a) $(\mathfrak{B}^b)^* x(s) = B^* T^*(\frac{b-s}{\alpha})\, x$, *for $s \in J,\ x \in X$.*

(b) $\Gamma_0^b \in \mathcal{L}(X)$, *is symmetric, and has the representation*

$$\Gamma_0^b = \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, T(\frac{b-s}{\alpha})\, B\, B^*\, T^*(\frac{b-s}{\alpha})\,\mathrm{d}s, \tag{3.17}$$

and $\Gamma_0^b \geqslant 0$, where B^* and $T^*(t)$ denote the adjoint of B and $T(t)$ respectively.

Proof. $(\mathbf{a})$: The way of proof is based on [3] (Lemma 4.1.4, page 144). For $x \in X$ and $u \in L^2(J,U)$,

$$\begin{aligned} \langle u, (\mathfrak{B}^b)^* x\rangle &= \langle \mathfrak{B}^b u, x\rangle \\ &= \langle \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, T(\frac{b-s}{\alpha})\, Bu(s)\,\mathrm{d}s, x\rangle \\ &= \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b \langle e^{\frac{\beta}{\alpha} s}\, T(\frac{b-s}{\alpha})\, Bu(s), x\rangle\,\mathrm{d}s \\ &= \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, \langle u(s), B^*\, T^*(\frac{b-s}{\alpha})\, x\rangle\,\mathrm{d}s\ , \end{aligned}$$

and this proves $(\mathbf{a})$.

$(\mathbf{b})$: From (3.16), it is easy to see Γ_0^b is symmetric, and $\Gamma_0^b \geqslant 0$. Equation (3.17) follows easily from (3.15), (3.16) and $(\mathbf{a})$. □

Theorem 3.12 *([3]) The system (3.14) is approximately controllable on J if and only if any one of the following conditions hold:*

(i) Γ_0^b *is a positive operator, that is* $\langle \Gamma_0^b x, x\rangle > 0$ *for all* $0 \neq x \in X$.

(ii) $\ker(\mathfrak{B}^b)^* = \{0\}$.

(iii) $B^* T^*(\frac{b-t}{\alpha})\, x = 0$ *on* $J \implies x = 0$.

Theorem 3.13 *([4]) Let Z be a separable reflexive Banach space and let Z^* stand for its dual space. Assume that $\Gamma : Z^* \to Z$ is a symmetric map, then the following are equivalent:*

(i) $\Gamma : Z^* \to Z$ *is positive.*

(ii) *For all $z \in Z$, $\lambda(\lambda I + \Gamma\mathfrak{J})^{-1}(z)$ strongly converges to zero as $\lambda \to 0^+$. Here $\mathfrak{J}$ is the duality map from $Z \to Z^*$.*

Lemma 3.14 *The linear fractional control system (3.14) is approximately controllable on J if and only if $\lambda R(\lambda, \Gamma_0^b) \to 0$ as $\lambda \to 0^+$ in strong operator topology, where $R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}$.*

Proof. The lemma is straightforward consequence of Theorem 3.12 and Theorem 3.13. □

To investigate the approximate controllability of (1.2), we impose the following assumptions:

(H1) A is an infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)$ $(t \geqslant 0)$ on X, and $T(t)$ $(t > 0)$ is compact.

(H2) For each $t \in J$, the function $f(t, \cdot) : X \to X$ is continuous, and for all $x \in X$, the function $f(\cdot, x) : J \to X$ is Lebesgue measurable.

(H3) There exists a constant $\alpha_1 \in (0, \alpha)$ and a function $\phi \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t,x)\| \leqslant \phi(t)\,, \quad \forall\, x \in X\,;\ t \in J\,.$$

(H4) The linear control system (3.14) is approximately controllable on J.

For convenience we use the following notations:

$$\Omega_r = \{x \in C(J,X) : \|x\| \leqslant r\}, \ \text{ for each finite constant } r > 0,$$

$$M_B = \|B\|,\ \ M_\phi = \|\phi\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)},\ \ q = \frac{1}{1-\alpha_1},\ \ N = \frac{1}{\alpha}\left(\frac{\alpha}{q\,\beta}\right)^{\frac{1}{q}} M_\phi.$$

For an arbitrary function $x \in C(J, X)$, considering the form of a mild solution as defined in Definition 3.8, as well as the controllability gramian and resolvent operator, we choose the feedback control function associated with the nonlinear system (1.2) as follows:

$$u(t) = u_\lambda(t,x) = B^*\, T^*(\frac{b-t}{\alpha})\, R(\lambda, \Gamma_0^b)\, p(x), \tag{3.18}$$

where

$$p(x) = x_b - e^{\frac{-\beta}{\alpha} b}\, T(\frac{b}{\alpha})\, x_0 - \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, T(\frac{b-s}{\alpha})\, f(s, x(s))\, \mathrm{d}s. \tag{3.19}$$

For any $\lambda > 0$, we define the operator $F_\lambda : C(J, X) \to C(J, X)$ as follows:

$$(F_\lambda x)(t) = e^{\frac{-\beta}{\alpha} t}\, T(\frac{t}{\alpha})\, x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, T(\frac{t-s}{\alpha})[f(s, x(s)) + B u_\lambda(s, x)]\, \mathrm{d}s\,. \tag{3.20}$$

Lemma 3.15 *If the assumptions* $(\mathbf{H1}) - (\mathbf{H3})$ *hold, then for any* $t \in J$ *we have*

(i) $\frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, \|T(\frac{t-s}{\alpha})\, f(s, x(s))\|\, \mathrm{d}s \;\leqslant\; MN.$

(ii) $\|u_\lambda(t, x)\| \;\leqslant\; \frac{M M_B}{\lambda}\Big[\|x_b\| + M(\|x_0\| + N)\Big].$

Proof. **(i)**: By using Hölder inequality and $(\mathbf{H3})$, we have

$$\begin{aligned}
\frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, \|T(\frac{t-s}{\alpha})\, f(s, x(s))\|\, \mathrm{d}s \;&\leqslant\; \frac{M}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} \phi(s)\, \mathrm{d}s \\
&\leqslant\; \frac{M}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \bigg(\int_0^t e^{\frac{q\beta s}{\alpha}}\, \mathrm{d}s \bigg)^{\frac{1}{q}} \|\phi\|_{L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)} \\
&\leqslant\; \frac{1}{\alpha} \bigg(\frac{\alpha}{q\beta} \bigg)^{\frac{1}{q}} e^{\frac{-\beta}{\alpha} t}\, ([e^{\frac{q\beta T}{\alpha}} - 1])^{\frac{1}{q}}\, M\, M_\phi \\
&\leqslant\; \frac{1}{\alpha} \bigg(\frac{\alpha}{q\beta} \bigg)^{\frac{1}{q}} e^{\frac{-\beta}{\alpha} t}\, e^{\frac{\beta}{\alpha} t}\, M\, M_\phi \\
&\leqslant\; \frac{1}{\alpha} \bigg(\frac{\alpha}{q\beta} \bigg)^{\frac{1}{q}} M\, M_\phi \;=\; M\, N\,.
\end{aligned}$$

(ii): Using (3.18), (3.19), and (i), we obtain

$$\begin{aligned}
\|u_\lambda(t, x)\| &\leqslant \|B^* T^*(\frac{b-t}{\alpha}) R(\lambda, \Gamma_0^b) p(x)\| \\
&\leqslant \frac{M M_B}{\lambda} \|p(x)\| \\
&\leqslant \frac{M M_B}{\lambda} \bigg[\|x_b\| + M\|x_0\| + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, \|T(\frac{b-s}{\alpha})\, f(s, x(s))\|\, \mathrm{d}s \bigg] \\
&\leqslant \frac{M M_B}{\lambda} \bigg[\|x_b\| + M(\|x_0\| + N) \bigg].
\end{aligned}$$

□

Theorem 3.16 *If the assumptions* $(\mathbf{H1})$-$(\mathbf{H3})$ *hold, then the fractional semilinear control system (1.2) has a mild solution.*

Proof. To prove the fractional semilinear control system (1.2) has a mild solution, we need to prove F_λ has a fixed point. For convenience we divide the proof into the following steps:

Step I: For any $\lambda > 0$, there exists a constant $R = R(\lambda) > 0$, such that $F_\lambda(\Omega_R) \subset \Omega_R$. For any positive constant r and $x \in \Omega_r$, if $t \in J$, then by using Lemma 3.15, we have

$$\begin{aligned}
\|(F_\lambda x)(t)\| &\leqslant e^{\frac{-\beta}{\alpha}t}\|T(\frac{t}{\alpha})x_0\| + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, \|T(\frac{t-s}{\alpha})\,[f(s,x(s)) + Bu_\lambda(s,x)]\|\,\mathrm{d}s \\
&\leqslant M\|x_0\| + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, \|T(\frac{t-s}{\alpha})\, f(s,x(s))\|\,\mathrm{d}s \\
&\quad + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}\, \|T(\frac{t-s}{\alpha})Bu_\lambda(s,x)\|\,\mathrm{d}s \\
&\leqslant M\|x_0\| + MN + \frac{1}{\beta}MM_B e^{\frac{-\beta}{\alpha}t}[e^{\frac{\beta}{\alpha}t} - 1]\|u_\lambda\| \\
&\leqslant M(\|x_0\| + N) + \frac{M^2(M_B)^2}{\lambda\beta}\Big[\|x_b\| + M(\|x_0\| + N)\Big].
\end{aligned}$$

This implies that for large enough $R > 0$, $F_\lambda(\Omega_R) \subset \Omega_R$ holds.

Step II: For any $t \in J$, the set $\{(F_\lambda x)(t) : x \in \Omega_R\}$ is relatively compact in X. In the case $t = 0$, clearly $\{(F_\lambda x)(0) : x \in \Omega_R\} = \{x_0\}$ is compact in X. Let $0 < t \leqslant b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in \Omega_R$, we define

$$\begin{aligned}
(F_\lambda^\varepsilon x)(t) &= e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha})\, x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_0^{t-\varepsilon} e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\,[f(s,x(s)) + Bu_\lambda(s,x)]\,\mathrm{d}s \\
&= e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha})\, x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t}\, T(\frac{\varepsilon}{\alpha}) \int_0^{t-\varepsilon} e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s-\varepsilon}{\alpha})\,[f(s,x(s)) + Bu_\lambda(s,x)]\,\mathrm{d}s \\
&= e^{\frac{-\beta}{\alpha}t}\, T(\frac{t}{\alpha})\, x_0 + \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t}\, T(\frac{\varepsilon}{\alpha})\, y(t,\varepsilon),
\end{aligned}$$

since $T(t)$ is compact for $t > 0$ and $y(t,\varepsilon)$ is bounded on Ω_R, we obtain that the set $\{(F_\lambda^\varepsilon x)(t) : x \in \Omega_R\}$ is relatively compact in X. On the other hand

$$\begin{aligned}
\|(F_\lambda x)(t) - (F_\lambda^\varepsilon x)(t)\| &= \|\tfrac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \textstyle\int_{t-\varepsilon}^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\,[f(s,x(s)) + Bu_\lambda(s,x)]\,\mathrm{d}s\| \\
&\leqslant I_1 + I_2,
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
I_1 &= \|\frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_{t-\varepsilon}^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, f(s,x(s))\,\mathrm{d}s\|, \\
I_2 &= \|\frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha}t} \int_{t-\varepsilon}^t e^{\frac{\beta}{\alpha}s}\, T(\frac{t-s}{\alpha})\, Bu_\lambda(s,x)\,\mathrm{d}s\|.
\end{aligned}$$

Now, proceeding in the same way as Lemma 3.15 yields

$$I_1 \leqslant MN[e^{\frac{\beta q}{\alpha}t} - e^{\frac{\beta q}{\alpha}(t-\varepsilon)}]^{\frac{1}{q}}, \tag{3.22}$$

$$I_2 \leqslant \frac{M^2(M_B)^2}{\lambda\beta}\Big[\|x_b\| + M(\|x_0\| + N)\Big][e^{\frac{\beta}{\alpha}t} - e^{\frac{\beta}{\alpha}(t-\varepsilon)}]\,. \tag{3.23}$$

Therefore, by (3.21), (3.22) and (3.23), we conclude that

$$\|(F_\lambda x)(t) - (F_\lambda^\varepsilon x)(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

This implies that the set $\{(F_\lambda x)(t) : x \in \Omega_R\},\ t \in (0, b]$ is relatively compact in X.

Step III: The family of functions $\{F_\lambda x : x \in \Omega_R\}$ is bounded and equicontinuous on J. Boundedness is obvious. For any $x \in \Omega_R$, and $0 \leqslant t_1 < t_2 \leqslant b$, we have

$$\begin{aligned}
\|(F_\lambda x)(t_2) - (F_\lambda x)(t_1)\| \leqslant & \left\| e^{\frac{-\beta}{\alpha} t_1} [T(\frac{t_2}{\alpha}) - T(\frac{t_1}{\alpha})] x_0 \right\| \\
& + \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_2} \int_{t_1}^{t_2} e^{\frac{\beta}{\alpha} s} T(\frac{t_2 - s}{\alpha}) f(s, x(s)) \, \mathrm{d}s \right\| \\
& + \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_2} \int_{t_1}^{t_2} e^{\frac{\beta}{\alpha} s} T(\frac{t_2 - s}{\alpha}) B u_\lambda(s, x) \, \mathrm{d}s \right\| \\
& + \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_0^{t_1} e^{\frac{\beta}{\alpha} s} [T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha})] f(s, x(s)) \, \mathrm{d}s \right\| \\
& + \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_0^{t_1} e^{\frac{\beta}{\alpha} s} [T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha})] B u_\lambda(s, x) \, \mathrm{d}s \right\| \\
= & J_1 + J_2 + J_3 + J_4 + J_5\,.
\end{aligned}$$

Now, using Lemma 3.15 we get

$$\begin{aligned}
J_1 &\leqslant \left\| T(\frac{t_2}{\alpha}) - T(\frac{t_1}{\alpha}) \right\| \|x_0\|, \\
J_2 &\leqslant \frac{M M_\phi}{\alpha} \left(\frac{\alpha}{\beta q} \right)^{\frac{1}{q}} [e^{\frac{\beta q}{\alpha} t_2} - e^{\frac{\beta q}{\alpha} t_1}]^{\frac{1}{q}}, \\
J_3 &\leqslant \frac{M^2 (M_B)^2}{\lambda \beta} \Big[\|x_b\| + M(\|x_0\| + N) \Big] [e^{\frac{\beta}{\alpha} t_2} - e^{\frac{\beta}{\alpha} t_1}].
\end{aligned}$$

For $t_1 = 0$, it is easy to see that $J_4 = 0$. For $t_1 > 0$ and $\epsilon > 0$ small enough, we obtain

$$\begin{aligned}
J_4 \leqslant & \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_0^{t_1 - \epsilon} e^{\frac{\beta}{\alpha} s} [T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha})] f(s, x(s)) \, \mathrm{d}s \right\| \\
& + \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_{t_1 - \epsilon}^{t_1} e^{\frac{\beta}{\alpha} s} [T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha})] f(s, x(s)) \, \mathrm{d}s \right\| \\
\leqslant & \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_0^{t_1 - \epsilon} e^{\frac{\beta}{\alpha} s} \phi(s) \, \mathrm{d}s \sup_{s \in [0, t_1 - \epsilon]} \left\| T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha}) \right\| \\
& + \frac{2M}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_{t_1 - \epsilon}^{t_1} e^{\frac{\beta}{\alpha} s} \phi(s) \, \mathrm{d}s \\
\leqslant & N e^{\frac{-\beta}{\alpha} \epsilon} \sup_{s \in [0, t_1 - \epsilon]} \left\| T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha}) \right\| + 2\, M\, N\, [e^{\frac{\beta q}{\alpha} t_1} - e^{\frac{\beta q}{\alpha} (t_1 - \epsilon)}]^{\frac{1}{q}}\,.
\end{aligned}$$

Similarly

$$\begin{aligned}
J_5 \leqslant & \frac{M_B}{\beta} [e^{\frac{-\beta}{\alpha} \epsilon} - e^{\frac{-\beta}{\alpha} t_1}] \|u_\lambda\| \sup_{s \in [0, t_1 - \epsilon]} \left\| T(\frac{t_2 - s}{\alpha}) - T(\frac{t_1 - s}{\alpha}) \right\| \\
& + \frac{2 M M_B}{\beta} [e^{\frac{\beta}{\alpha} t_1} - e^{\frac{\beta}{\alpha} (t_1 - \epsilon)}] \|u_\lambda\|\,.
\end{aligned}$$

Using $(\mathbf{H1})$, it is clear that $J_i \to 0$ $(i = 1,2,3,4,5)$ as $t_2 \to t_1,\ \epsilon \to 0$. As a result, $\|(F_\lambda x)(t_2) - (F_\lambda x)(t_1)\| \to 0$ independently of $x \in \Omega_R$ as $t_2 \to t_1$, which means that $F_\lambda : \Omega_R \to \Omega_R$ is equicontinuous.

Thus, combining **Step II** and **Step III**, we conclude that F_λ is compact on Ω_R by Arzela-Ascoli theorem.

Step IV: F_λ is continuous in Ω_R. Let $\{x_n\}$ be a sequence in Ω_R such that $\lim_{n\to\infty} x_n = x$ in Ω_R. By the continuity of nonlinear term f with respect to the second variable, for each $s \in J$, we have

$$\lim_{n\to\infty} f(s, x_n(s)) = f(s, x(s))\ . \tag{3.24}$$

So, we can conclude that

$$\sup_{s\in J} \|f(s, x_n(s)) - f(s, x(s))\| \to 0 \quad \text{as} \quad n \to \infty\ . \tag{3.25}$$

From (3.19) and (3.25), we obtain

$$\begin{aligned}
\|p(x_n) - p(x)\| &= \left\| \frac{1}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s}\, T(\frac{b-s}{\alpha})\, [f(s, x_n(s)) - f(s, x(s))]\, \mathrm{d}s \right\| \\
&\leqslant \frac{M}{\alpha}\, e^{\frac{-\beta}{\alpha} b} \left(\int_0^b e^{\frac{\beta}{\alpha} s}\, \mathrm{d}s \right) \sup_{s\in J} \|f(s, x_n(s)) - f(s, x(s))\| \\
&\leqslant \frac{M}{\beta} [1 - e^{\frac{-\beta}{\alpha} b}] \sup_{s\in J} \|f(s, x_n(s)) - f(s, x(s))\| \\
&\leqslant \frac{M}{\beta} \sup_{s\in J} \|f(s, x_n(s)) - f(s, x(s))\| \\
&\to 0 \quad \text{as} \quad n \to \infty\ ,
\end{aligned} \tag{3.26}$$

therefore, (3.18) and (3.26) imply

$$\|u_\lambda(s, x_n) - u_\lambda(s, x)\| \to 0 \quad \text{as} \quad n \to \infty\ , \tag{3.27}$$

and (3.25), (3.27) yield

$$\begin{aligned}
\|(F_\lambda x_n)(t) - (F_\lambda x)(t)\| &\leqslant \frac{M}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, \|f(s, x_n(s)) - f(s, x(s))\|\, \mathrm{d}s \\
&\quad + \frac{MM_B}{\alpha}\, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\, \|u_\lambda(s, x_n(s)) - u_\lambda(s, x(s))\|\, \mathrm{d}s \\
&\leqslant \frac{M}{\beta} \sup_{s\in J} \|f(s, x_n(s)) - f(s, x(s))\| \\
&\quad + \frac{MM_B}{\beta} \sup_{s\in J} \|u_\lambda(s, x_n(s)) - u_\lambda(s, x(s))\| \\
&\to 0 \quad \text{as} \quad n \to \infty\ ,
\end{aligned} \tag{3.28}$$

which means that F_λ is continuous in Ω_R.

Hence by Theorem 2.8, F_λ has a fixed point, which is a mild solution of (1.2). □

Theorem 3.17 *Assume that the hypotheses* $(\mathbf{H1})$-$(\mathbf{H4})$ *hold. Moreover, assume that the function f is uniformly bounded by a positive constant K, then the semilinear fractional system (1.2) is approximately controllable on J.*

Proof. Let x_λ be a fixed point of F_λ in Ω_R. Any fixed point of F_λ is a mild solution of the problem (1.2) under the control

$$u_\lambda(t, x_\lambda) = B^* \, T^*(\frac{b-t}{\alpha}) \, R(\lambda, \Gamma_0^b) \, p(x_\lambda) \ ,$$

where

$$p(x_\lambda) = x_b - e^{\frac{-\beta}{\alpha} b} \, T(\frac{b}{\alpha}) \, x_0 - \frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s} \, T(\frac{b-s}{\alpha}) \, f(s, x_\lambda(s)) \, \mathrm{d}s \ ,$$

and satisfies the following equality

$$\begin{aligned} x_\lambda(b) &= e^{\frac{-\beta}{\alpha} b} \, T(\frac{b}{\alpha}) \, x_0 + \frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s} \, T(\frac{b-s}{\alpha}) \, [f(s, x_\lambda(s)) + B u_\lambda(s, x_\lambda)] \, \mathrm{d}s \\ &= x_b - p(x_\lambda) + \left(\frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s} \, T(\frac{b-s}{\alpha}) \, B \, B^* \, T^*(\frac{b-s}{\alpha}) \, \mathrm{d}s \right) R(\lambda, \Gamma_0^b) \, p(x_\lambda) \\ &= x_b - p(x_\lambda) + \Gamma_0^b \, R(\lambda, \Gamma_0^b) \, p(x_\lambda) \\ &= x_b - \lambda \, R(\lambda, \Gamma_0^b) \, p(x_\lambda) \ . \end{aligned} \tag{3.29}$$

By the assumption that f is uniformly bounded, we have

$$\int_0^b \|f(s, x_\lambda(s))\|^2 \, \mathrm{d}s \leqslant K^2 \, b \ .$$

Hence the sequence $f(\cdot, x_\lambda(\cdot))$ is bounded in $L^2(J, X)$. Then there exists a subsequence of $\{f(\cdot, x_\lambda(\cdot)) \, : \, \lambda > 0\}$, still denoted by it, converges weakly to some $f(\cdot) \in L^2(J, X)$. Define

$$\omega = x_b - e^{\frac{-\beta}{\alpha} b} \, T(\frac{b}{\alpha}) \, x_0 - \frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s} \, T(\frac{b-s}{\alpha}) \, f(s) \, \mathrm{d}s \ .$$

It follows that

$$\begin{aligned} \|p(x_\lambda) - \omega\| &= \left\| \frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} b} \int_0^b e^{\frac{\beta}{\alpha} s} \, T(\frac{b-s}{\alpha}) [f(s, x_\lambda(s)) - f(s)] \, \mathrm{d}s \right\| \\ &\leqslant \sup_{t \in J} \left\| \frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} \, T(\frac{t-s}{\alpha}) [f(s, x_\lambda(s)) - f(s)] \, \mathrm{d}s \right\| \ . \end{aligned}$$

As in the proof of Theorem 3.16 using Arzela-Ascoli theorem one can show that the operator

$$L^2(J, X) \to C(J, X) \ ; \ \ \ell(\cdot) \ \mapsto \ \frac{1}{\alpha} \, e^{\frac{-\beta}{\alpha} (\cdot)} \int_0^{\cdot} e^{\frac{\beta}{\alpha} s} \, T(\frac{\cdot - s}{\alpha}) \, \ell(s) \, \mathrm{d}s$$

is compact, consequently

$$\|p(x_\lambda) - \omega\| \to 0 \quad \text{as} \quad \lambda \to 0^+ . \tag{3.30}$$

Then, from (3.29), (3.30), and Lemma 3.14, we obtain

$$\begin{aligned} \|x_\lambda(b) - x_b\| &\leqslant \|\lambda R(\lambda, \Gamma_0^b) \, p(x_\lambda)\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^b) \omega\| + \|\lambda \, R(\lambda, \Gamma_0^b)\| \, \|p(x_\lambda) - \omega\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^b) \omega\| + \|p(x_\lambda) - \omega\| \ \to \ 0 \ \text{ as } \ \lambda \to 0^+ \ . \end{aligned}$$

This proves the approximate controllability of (1.2). □

4 Examples

Example 1: Consider the following deformable fractional partial differential equation,

$$\begin{cases} D^{\frac{1}{2}}x(t,z) = \frac{\partial^2}{\partial z^2}x(t,z) + \frac{1}{4}\sin t\frac{|x(t,z)|}{1+|x(t,z)|}, & z\in(0,1),\ t\in(0,1)\ ; \\ x(t,0)=x(t,1)=0\ , & t\in[0,1]\ ; \\ x(0,z)=x_0(z)\ , & z\in[0,1]\ , \end{cases} \tag{4.1}$$

where $X = L^2[0,1]$, $x_0(z) \in X$. Define $Ax = x''$, with

$$D(A) = \{x \in X : x, x' \text{ are absolutely continuous and } x'' \in X,\ x(0) = x(1) = 0\}\ .$$

Then

$$Ax = \sum_{n=1}^{\infty} -n^2\ \langle x, e_n\rangle\ e_n\ ,\ \ x \in D(A)\ , \tag{4.2}$$

where $e_n(z) = \sqrt{\frac{2}{\pi}}\sin(nz),\ 0 \leqslant z \leqslant 1,\ n = 1, 2, \ldots$. It is well known that A generates a C_0-semigroup $T(t)$ $(t \geqslant 0)$, on X and is given by

$$T(t)\,x = \sum_{n=1}^{\infty} e^{-n^2t}\ \langle x, e_n\rangle\ e_n\ ,\ \ x \in X\ , \tag{4.3}$$

with $\|T(t)\| \leqslant 1$, for any $t \geqslant 0$. Put $x(t) = x(t,\cdot)$, that is, $x(t)(z) = x(t,z),\ t, z \in [0,1]$, and

$$f(t, x(t)) = \frac{1}{4}\sin t\frac{|x(t,\cdot)|}{1+|x(t,\cdot)|}\ .$$

Then the system (4.1) can be rewritten into the abstract form of (1.1).

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\leqslant \frac{1}{4}\frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)} \\ &\leqslant \frac{1}{4}\|x-y\|. \end{aligned}$$

Therefore $L = \frac{1}{4}$, also we have $\alpha = \beta = \frac{1}{2}$ and $M = 1$. So $\frac{ML}{\beta} = \frac{1}{2} < 1$. Hence all the required assumptions for Theorem 3.7 are satisfied, and the system (4.1) has a unique mild solution.

Example (2): Consider the following control system governed by a deformable fractional partial differential equation

$$\begin{cases} D^{\frac{1}{2}}x(t,z) = \frac{\partial^2}{\partial z^2}x(t,z) + u(t,z) + \frac{1}{8}\,\frac{e^{-t}}{1+e^t}\,\frac{|x(t,z)|}{1+|x(t,z)|}\ , & z\in(0,1),\ t\in(0,b]\ ; \\ x(t,0)=x(t,1)=0\ , & t\in[0,b]\ ; \\ x(0,z)=x_0(z)\ , & z\in[0,1]\ , \end{cases} \tag{4.4}$$

where $X = U = L^2[0,1]$, $x_0(z) \in X,\ J = [0,b]$. Define $Ax = x''$ with

$$D(A) = \{x \in X : x, x' \text{ are absolutely continuous and } x'' \in X,\ x(0) = x(1) = 0\}\ .$$

A generates a compact semigroup $T(t)$ $(t > 0)$ given by expression (4.3), clearly assumption $(\mathbf{H1})$ holds.

Put $x(t) = x(t,\cdot),$ that is, $x(t)(z) = x(t,z),\ t \in J,\ z \in [0,1]$ and $u(t) = u(t,\cdot)$ is continuous. Let the bounded linear operator $B : U \to X$ be defined as $Bu(t) = u(t,\cdot)$, then the system (4.4) can be rewritten into the abstract form of (1.2). It is easy to verify that the assumptions $(\mathbf{H2})$ and $(\mathbf{H3})$ hold with $\phi(t) = \frac{e^{-t}}{1+e^{t}}$ and $K = \frac{1}{8}$.

By Theorem 3.12, the linear system corresponding to (4.4) is approximately controllable on J if and only if

$$B^* T^*(\frac{b-t}{\alpha})\, x = 0,\ t \in J \implies x = 0\,. \tag{4.5}$$

Using (4.3), we observe that

$$B^* T^*(\frac{b-t}{\alpha})\, x = \sum_{n=1}^{\infty} e^{-n^2(\frac{b-t}{\alpha})}\ \langle x, e_n\rangle\ e_n,\ x \in X,\ t \in J\,.$$

Therefore the condition (4.5) holds, and hence the assumption $(\mathbf{H4})$ holds. Thus by Theorem 3.17 the system (4.4) is approximately controllable on J.

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