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DOUBLY WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS FOR TWO-TERM FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the existence and uniqueness of doubly weighted pseudo almost automorphic mild solutions with Stepanov like doubly weighted pseudo almost automorphic forcing term using two non-equivalent weight functions for two-term fractional differential equations. In this paper, we will use the notion of Weyl fractional derivative, generalized semigroup theory for linear operators and fractional calculus techniques to obtain the main results.

Keywords: Double weighted pseudo Stepanov like almost automorphic functions, generalized semigroup theory, sectorial operator, weighted pseudo almost automorphic and double weighted pseudo almost automorphic functions, Weyl fractional derivative..

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1 Introduction

The notion of almost automorphy was introduced by Bochner in [5] in relation to some aspects of differential geometry, and is known as an important generalization of the classical almost periodicity in the sense of Bohr. Since then, this concept has undergone various natural and useful generalizations. For example, Xiao *et al.* in [25] introduced the concept of pseudo almost automorphy. After that, the concept of weighted pseudo almost automorphic functions was introduced by Blot *et al.* [6]. Further, Chang *et al.* [10] introduced the notion of Stepanov-like weighted pseudo almost

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automorphic functions and established the existence of weighted pseudo almost automorphic solutions with Stepanov-like weighted pseudo almost automorphic forcing term. The existence of almost automorphic, pseudo almost automorphic and weighted pseudo almost automorphic solutions of fractional differential equations is one of the most attractive topic in qualitative theory of differential equations due to a lot of applications and significance in mechanical, physics, control theory, and mathematical biology etc. Therefore, many authors have made important contributions to the theory of pseudo almost automorphic and weighted pseudo almost automorphic solutions to some fractional differential equations (see [1, 3, 6, 10, 12, 22, 26]).

Differential equations with iterated deviating arguments have many applications in automatic control, the problems of long term planning in economics, the theory of self-oscillating systems, and other areas of science. For an admirable introduction to differential equations involving deviating arguments and iterated deviating arguments, we refer to [8, 14, 15, 17, 19]. On the other hand, it is understood that integro-differential equations provide suitable models to many situations arising from science and engineering such as electrical circuit analysis, the activity of interacting inhibitory and excitatory neurons. Fractional integro-differential equations can also be used to model the dynamics system which is slower and faster. The problems considered in this paper may be thought as abstract formulations of general form of fractional relaxation/oscillation partial differential equations.

The fractional order differential equations provide a subject of increasing interest in different context and areas of research (see [1, 3, 8, 9, 12, 13, 21, 22]). We are inspired to study double weighted pseudo almost automorphic mild solutions to the systems (1.1) and (1.2) by recent investigations on this topic. Indeed, in the paper [3], Pardo and Lizama established the existence of weighted pseudo almost automorphic solutions in a special case when $F(t, y(t)) = F(t, y(t), y(w_1(t, y(t))))$, where $w_1(t, y(t)) = b_1(t, y(b_2(t, \cdots, y(b_{m_0}(t, y(t))) \cdots))$, with a single weight function and in [19] Kumar *et al.* obtained the existence results for piecewise continuous mild solutions with iterated deviating arguments. In [12], Diagana introduced the concept of double weighted pseudo almost periodic functions.

In this paper, we will use the concept of double weights introduced by Diagana to establish the existence and uniqueness of weighted pseudo almost automorphic mild solutions for the following classes of two-term time-fractional differential equations with iterated deviating arguments and integral forcing terms

$$D_t^{\mu+1} y(t) + \beta D_t^{\nu} y(t) = Ay(t) + D_t^{\mu} F(t, y(t), y(w_1(t, y(t)))) \tag{1.1}$$

and

$$D_t^{\mu+1} y(t) + \beta D_t^{\nu} y(t) = Ay(t) + D_t^{\mu} F_1\left(t, y(t), \int_{-\infty}^{t} \mathcal{K}(t-s) h(s, y(s))\,\mathrm{d}s\right), \tag{1.2}$$

where $0 < \mu \leq \nu \leq 1$, $\beta \geq 0$, $t \geq 0$, $w_1(t, y(t)) = b_1(t, y(b_2(t, \cdots, y(b_{m_0}(t, y(t))) \cdots)))$, and $A\colon D(A) \subset \mathbb{X} \to \mathbb{X}$ is a sectorial operator of angle $\nu\pi/2$. The forcing terms F, F_1, h and $b_i, i = 1, 2, 3, \ldots, m_0$ are suitable functions satisfying some appropriate conditions in their arguments mentioned later in assumptions. Here, D_t^{η} represents Weyl fractional derivative of order η and $\mathcal{K} \in L^1(\mathbb{R})$ with $|\mathcal{K}(t)| \leq C_{\mathcal{K}} e^{-bt}$, where $b, C_{\mathcal{K}} > 0$.

The rest of this paper is organized as follows. In Section 2, we recall some fundamental results about the notion of weighted pseudo almost automorphic functions. Section 3 is devoted to the main results ensuring the existence and uniqueness of mild solutions for two-term time-fractional differential equations. At last, we will provide an example to show the feasibility of the theory discussed in this paper.

2 Preliminaries

In this section, we recall some definitions and basic results which are used throughout the paper. Let $\mathbb{X}$ be a Banach space. Suppose that $\mathcal{B}(\mathbb{R},\mathbb{X})$, $\mathcal{C}(\mathbb{R},\mathbb{X})$ and $\mathcal{BC}(\mathbb{R},\mathbb{X})$ symbolize the collections of bounded functions, continuous and continuous and bounded functions from $\mathbb{R}$ to $\mathbb{X}$, respectively.

Define $g_\mu(t)$ for $\mu>0$ by

$$g_\mu(t)=\begin{cases}\dfrac{1}{\Gamma(\mu)}t^{\mu-1}, & \text{if } t>0,\\ 0, & \text{if } t<0,\end{cases}$$

where Γ denotes the gamma function. The function g_μ has the property: $g_\alpha * g_\mu = g_{\alpha+\mu}$; here $*$ denotes the convolution defined by $(f*g)(t)=\int_0^t f(t-s)g(s)\,\mathrm{d}s$ for appropriate functions f and g.

Definition 1 *Weyl fractional integral of an appropriate function $f\colon\mathbb{R}\to\mathbb{X}$ of order $\mu>0$ is defined by*

$$J_t^\mu f(t)=\int_{-\infty}^t g_\mu(t-\tau)f(\tau)\,\mathrm{d}\tau,\quad t>0,\ t\in\mathbb{R},$$

when the integral is convergent.

Definition 2 *Weyl fractional derivative of an appropriate function function $f\colon\mathbb{R}\to\mathbb{X}$ of order $\mu>0$ is defined by*

$$D_t^\mu f(t)=\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_{-\infty}^t g_{m-\mu}(t-\tau)f(\tau)\,\mathrm{d}\tau,\quad t>0,\ t\in\mathbb{R},$$

where $m=[\mu]+1$. It is known that $D_t^\mu J_t^\mu f(t)=f(t)$ for any $\mu>0$. For more details see Miller and Ross [21].

Definition 3 *A densely defined closed linear operator A is said to be sectorial of type ω and angle θ if there exist $\theta\in[0,\frac{\pi}{2})$, $\mathcal{M}>0$, $\omega\in\mathbb{R}$ such that its resolvent exists in the sector*

$$\omega+\Sigma_\theta:=\Big\{\omega+\lambda:\lambda\in\mathbb{C},\ |\arg(\lambda)|<\frac{\pi}{2}+\theta\Big\}\setminus\{\omega\}$$

and

$$\|(\lambda-A)^{-1}\|\le\frac{\mathcal{M}}{|\lambda-\omega|},\quad \lambda\in\omega+\Sigma_\theta. \tag{2.1}$$

These operators are generators of analytic semigroups. We can say that A is a sectorial of angle $\frac{\pi}{2}+\theta$ in the case of $\omega=0$. It is not necessary that (2.1) holds in a sector of angle $\frac{\pi}{2}$ in the general theory of sectorial operators. So, we have to make some restrictions that correspond to the class of operators used in this paper. For more details see [16].

Definition 4 ([18]) *Let $\beta \geq 0$ and $0 \leq \mu, \nu \leq 1$. Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $D(A)$. Then, A is said to be the generator of a $(\mu,\nu)_\beta$-regularized family if there exist $\omega \geq 0$ and a strongly continuous function $\mathcal{S}_{\mu,\nu}\colon \mathbb{R}^+ \to \mathcal{B}(\mathbb{X})$ such that $\{\lambda^{\mu+1} + \beta\lambda^\nu : \operatorname{Re}\lambda > \omega\} \subset \varrho(A)$ (the resolvent set of A) and*

$$H(\lambda)y = \lambda^\beta(\lambda^{\mu+1} + \beta\lambda^\nu - A)^{-1}y = \int_0^\infty e^{-\lambda t}\mathcal{S}_{\mu,\nu}(t)y\,\mathrm{d}t, \quad \operatorname{Re}\lambda > \omega,\ y \in \mathbb{X}.$$

By the uniqueness theorem for the Laplace transform, if $\beta = 0$ and $\mu = 0$ this corresponds to the case of a C_0-semigroup, whereas $\beta = 0$ and $\mu = 1$ corresponds to the concept of cosine family. We refer to the monograph [4] for more information about Laplace approach to semigroups and cosine families.

Lemma 1 ([18]) *Let $\beta \geq 0$ and $0 \leq \mu, \nu \leq 1$. There exist Laplace transformable functions $a_{\mu,\nu} \in C^1(\mathbb{R}^+)$ and $k_{\mu,\nu} \in C^1(\mathbb{R}^+)$ such that $\widehat{a}_{\mu,\nu}(\lambda) = \frac{1}{\lambda^{\mu+1}+\beta\lambda^\nu}$ and $\widehat{k}_{\mu,\nu}(\lambda) = \frac{\lambda^\mu}{\lambda^{\mu+1}+\beta\lambda^\nu}$.*

From the above result we note that $a_{\mu,\nu} = g_\mu * k_{\mu,\nu}$. Moreover, $k_{\mu,\nu}(0) = 1$ and $k_{\mu,\nu}$ is a differential function (see [18] for an explicit representation).

Proposition 1 ([18]) *Let $\beta \geq 0$ and $0 \leq \mu, \nu \leq 1$. Let $\mathcal{S}_{\mu,\nu}(t)$ be a $(\mu,\nu)_\beta$-regularized family generated by A on $\mathbb{X}$. Then, the following conditions hold true.*

(i) *$\mathcal{S}_{\mu,\nu}(t)$ is strongly continuous and $\mathcal{S}_{\mu,\nu}(0) = I$.*

(ii) *$\mathcal{S}_{\mu,\nu}(t)y \in D(A)$ and $A\mathcal{S}_{\mu,\nu}(t)y = \mathcal{S}_{\mu,\nu}(t)Ay$ for all $y \in D(A)$ and $t \geq 0$.*

(iii) *Let $y \in \mathbb{X}$ and $t \geq 0$. Then, $\int_0^t a_{\mu,\nu}(t-s)\mathcal{S}_{\mu,\nu}(s)y\,\mathrm{d}s \in D(A)$ and*

$$\mathcal{S}_{\mu,\nu}(t)y = k_{\mu,\nu}(t)y + A\int_0^t (g_\mu * k_{\mu,\nu})(t-s)\mathcal{S}_{\mu,\nu}(s)y\,\mathrm{d}s. \tag{2.2}$$

(iv) *We have $\mathcal{S}_{\mu,\nu}(t)y \in C^1(\mathbb{R}^+, \mathbb{X})$ for some $y \in D(A)$.*

Theorem 1 ([18]) *Let $\beta > 0$, $\omega < 0$ and $0 < \mu \leq \nu \leq 1$. If A is an ω-sectorial operator of angle $\frac{\nu\pi}{2}$, then A generates a $(\mu,\nu)_\beta$-regularized family $\mathcal{S}_{\mu,\nu}(t)$ satisfying the estimate*

$$\|\mathcal{S}_{\mu,\nu}(t)\| \leq \frac{M}{1 + |\omega|(t^{\mu+1} + \beta t^\nu)} = J_{\mu,\nu}(t), \quad t \geq 0, \tag{2.3}$$

where M is a constant depending solely on μ, ν.

Let $\mathbb{U}$ be the collection of all positive and locally integrable functions $\rho\colon \mathbb{R} \to (0,\infty)$. For a given r and for each $\rho \in \mathbb{U}$, we set $w(r,\rho) = \int_{-r}^r \rho(t)\,\mathrm{d}t$. The spaces of weights are given by

$$\mathbb{U}_\infty := \Big\{\rho \in \mathbb{U} : \lim_{r\to\infty} w(r,\rho) = \infty\Big\}$$

and

$$\mathbb{U}_B := \Big\{\rho \in \mathbb{U}_\infty : \rho \text{ is bounded and } \inf_{t\in\mathbb{R}} \rho(t) > 0\Big\}.$$

We can observe that $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$.

Definition 5 *Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. Then, ρ_1 and ρ_2 are said to be equivalent,* i.e. $\rho_1 \sim \rho_2$, *if* $\frac{\rho_1}{\rho_2} \in \mathbb{U}_B$.

It is clear that "$\sim$" is a binary equivalence relation on $\mathbb{U}_\infty$. For a given weight $\rho \in \mathbb{U}_\infty$ let $C_L(\rho)$ be the equivalence class of ρ, that is, $C_L(\rho) := \{\rho^* \in \mathbb{U}_\infty : \rho \sim \rho^*\}$. Moreover, $\mathbb{U}_\infty = \bigcup_{\rho \in \mathbb{U}_\infty} C_L(\rho)$. Further, for $\rho_1 \in \mathbb{U}_\infty$ we define

$$\mathcal{PAA}_{\rho_1}(\mathbb{R}, \mathbb{X}) := \left\{ f \in \mathcal{BC}(\mathbb{R}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{w(r, \rho_1)} \int_{-r}^{r} \|f(t)\| \rho_1(t) \, \mathrm{d}t = 0 \right\}.$$

Particularly, for $\rho_1, \rho_2 \in \mathbb{U}_\infty$, we define

$$\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X}) := \left\{ f \in \mathcal{BC}(\mathbb{R}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{w(r, \rho_1)} \int_{-r}^{r} \|f(t)\| \rho_2(t) \, \mathrm{d}t = 0 \right\}.$$

Similarly,

$$\begin{aligned} &\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \\ &\quad := \left\{ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{X}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{w(r, \rho_1)} \int_{-r}^{r} \|f(t, y)\| \rho_2(t) \, \mathrm{d}t = 0 \, \text{uniformly in } y \in \mathbb{X} \right\}. \end{aligned}$$

For given $\rho \in \mathbb{U}_\infty$ and $\tau \in \mathbb{R}$ define ρ^τ by $\rho^\tau(t) = \rho(\tau + t)$ for $t \in \mathbb{R}$. Set

$$\mathbb{V}_\infty = \{\rho \in \mathbb{U}_\infty : \rho \sim \rho^\tau \text{ for each } t \in \mathbb{R}\}.$$

Remark 1 *It is trivial to see that $\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X}) = \mathcal{PAA}_{\rho_2,\rho_1}(\mathbb{R}, \mathbb{X}) = \mathcal{PAA}_{\rho_1}(\mathbb{R}, \mathbb{X})$, when "$\rho_1 \sim \rho_2$". The space $\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$ is more general and richer than $\mathcal{PAA}_{\rho_1}(\mathbb{R}, \mathbb{X})$ and gives rise to an enlarged space of weighted pseudo almost automorphic functions.*

Definition 6 *A function $f \in \mathcal{C}(\mathbb{R}, \mathbb{X})$ is almost automorphic (in Bochner's sense) if for each sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that $g(t) = \lim_{n \to \infty} f(t + \tau_n)$ is well-defined for each $t \in \mathbb{R}$, and $f(t) = \lim_{n \to \infty} g(t - \tau_n)$.*

The set of all such functions, denoted by $\mathcal{AA}(\mathbb{R}, \mathbb{X})$, constitutes a Banach space when equipped with the supremum norm.

Definition 7 *A function $f \in \mathcal{C}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is called almost automorphic if $f(t, y)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly in $y \in \mathbb{K}$ for a bounded subset $\mathbb{K}$ of $\mathbb{X}$. The set of all such functions is denoted by $\mathcal{AA}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.*

Definition 8 ([11]) *The Bochner transform $f^b(t, s)$, $s \in [0, 1]$, $t \in \mathbb{R}$, of a function $f\colon \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) = f(t + s)$.*

Definition 9 *The space of all Stepanov bounded functions with exponent $p \in [1, \infty)$, denoted by $\mathcal{BS}^p(\mathbb{R}, \mathbb{X})$, consists of all measurable functions $f\colon \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0, 1], \mathbb{X}))$. It constitutes a Banach space with the norm*

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\xi)\|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}}.$$

Definition 10 ([23]) *The space of Stepanov-like almost automorphic functions, denoted by $\mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$, consists of all $f \in \mathcal{BS}^p(\mathbb{R},\mathbb{X})$ such that $f^b \in \mathcal{AA}(\mathbb{R}, L^p([0,1],\mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is a Stepanov-like almost automorphic if its Bochner transform $f^b\colon \mathbb{R} \to L^p([0,1],\mathbb{X})$ is almost automorphic in the sense that for every sequence $\{\tau'_n\}$ of real numbers there exist a subsequence $\{\tau_n\}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that*

$$\lim_{n\to\infty}\left[\int_t^{t+1}\|f(s+\tau_n)-g(s)\|^p\,\mathrm{d}s\right]^{\frac{1}{p}}=0$$

and

$$\lim_{n\to\infty}\left[\int_t^{t+1}\|g(s-\tau_n)-f(s)\|^p\,\mathrm{d}s\right]^{\frac{1}{p}}=0$$

for all $t \in \mathbb{R}$ as $n \to \infty$.

Definition 11 ([23]) *A function $f\colon \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ with $f(.,y) \in L^p(\mathbb{R},\mathbb{X})$ for each $y \in \mathbb{X}$ is said to be Stepanov-like almost automorphic function in $t \in \mathbb{R}$, uniformly for $y \in \mathbb{X}$, if $t \mapsto f(t,y)$ is Stepanov-like almost automorphic for each $y \in \mathbb{X}$; in other words, if for every sequence $\{\tau'_n\}$ of real numbers there exist a subsequence $\{\tau_n\}$ and $g(.,y) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that*

$$\lim_{n\to\infty}\left[\int_t^{t+1}\|f(s+\tau_n,y)-g(s,y)\|^p\,\mathrm{d}s\right]^{\frac{1}{p}}=0$$

and

$$\lim_{n\to\infty}\left[\int_t^{t+1}\|g(s-\tau_n,y)-f(s,y)\|^p\,\mathrm{d}s\right]^{\frac{1}{p}}=0$$

for all $t \in \mathbb{R}$ and $y \in \mathbb{X}$ as $n \to \infty$. The collection of all such functions, denoted by $\mathcal{S}^p\mathcal{AA}(\mathbb{R}\times\mathbb{X},\mathbb{X})$, is a Banach space with the norm $\|.\|_{S^p}$.

Remark 2 ([7]) *It can be observed that if f is almost automorphic, then f is Stepanov-like almost automorphic,* i.e., $\mathcal{AA}(\mathbb{R},\mathbb{X}) \subset \mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$ (*see* [2]). *Moreover, let $1 \le p \le q < \infty$. If $f \in \mathcal{S}^q\mathcal{AA}(\mathbb{R},\mathbb{X})$, then $f \in \mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$.*

Definition 12 *A function $f \in \mathcal{BC}(\mathbb{R},\mathbb{X})$ (respectively, $\mathcal{BC}(\mathbb{R}\times\mathbb{X},\mathbb{X})$) is said to be weighted pseudo almost automorphic if it has a decomposition of the form $f = \phi + \psi$, where $\phi \in \mathcal{AA}(\mathbb{R},\mathbb{X})$ (respectively, $\phi \in \mathcal{AA}(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and $\psi \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ (respectively, $\psi \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}\times\mathbb{X},\mathbb{X})$). The collection of all such functions, denoted by $\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ (respectively, $\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}\times\mathbb{X},\mathbb{X})$), is a Banach space.*

Definition 13 *A function $f \in \mathcal{BS}^p(\mathbb{R},\mathbb{X})$ is called Stepanov-like weighted pseudo almost automorphic if it has a decomposition of the form $f = \phi + \psi$, where $\phi \in \mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$ and $\psi^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$,* i.e.,

$$\lim_{r\to\infty}\frac{1}{w(r,\rho_1)}\int_{-r}^{r}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\rho_2(t)\,\mathrm{d}t=0.$$

In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is Stepanov-like weighted pseudo almost automorphic if its Bochner transform $f^b\colon \mathbb{R} \to L^p([0,1],\mathbb{X})$ is weighted pseudo almost automorphic. We denote the collection of all such functions by $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$.

Definition 14 *A function $f \in \mathcal{BS}^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is called Stepanov-like weighted pseudo almost automorphic if it has a decomposition of the form $f = \phi + \psi$, where $\phi \in \mathcal{S}^p\mathcal{AA}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\psi^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, L^p([0,1],\mathbb{X}))$. We denote the collection of all such functions by $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.*

Lemma 2 ([1]) *Let $\rho_1, \rho_2 \in \mathbb{V}_\infty$. Then, the space $\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$ is translation invariant and the decomposition of a Stepanov-like almost automorphic function is unique. Moreover, the space $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{X})$ is a Banach space when endowed with the norm $\|.\|_{S^p}$.*

Lemma 3 *Let $\rho_1, \rho_2 \in \mathbb{V}_\infty$. Then, we have $\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X}) \subset \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$ and $\mathcal{S}^q\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X}) \subset \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$ for $1 \le p \le q < \infty$.*

Proof. The proof is similar to the proofs of Propositions 4.1 and 4.2 in [7]. So, the details are omitted here. □

Lemma 4 ([9]) *Let $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(\mathbb{X})$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \le \sigma(t)$, $t \in \mathbb{R}^+$, where $\sigma \in L^1(\mathbb{R}^+)$ is non-decreasing. Then, for each $f \in \mathcal{S}^p\mathcal{AA}(\mathbb{R}, \mathbb{X})$, we have*

$$\int_{-\infty}^{t} S(t-s)f(s)\mathrm{d}s \in \mathcal{AA}(\mathbb{R}, \mathbb{X}).$$

Let us make the following assumptions.

(A_1) Let $f = \phi + \psi$ be in $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ with $\phi \in \mathcal{S}^p\mathcal{AA}(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ and $\psi^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, L^p([0,1],\mathbb{X}))$. Moreover, assume that there exist constants $L_f, L_\phi > 0$ such that for each $y_i, z_i \in \mathbb{X}$, $i = 1,2$, $t \in \mathbb{R}$ we have

$$\|f(t,y_1,z_1) - f(t,y_2,z_2)\| \le L_f(\|y_1 - y_2\| + \|z_1 - z_2\|)$$

and

$$\|\phi(t,y_1,z_1) - \phi(t,y_2,z_2)\| \le L_\phi(\|y_1 - y_2\| + \|z_1 - z_2\|).$$

(A_2) The functions $b_i = \phi_i + \psi_i$ contained in $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ are such that $\phi_i \in \mathcal{S}^p\mathcal{AA}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ are uniformly continuous in a bounded set $K \subset \mathbb{X}$ for all $t \in \mathbb{R}$ and $\psi_i^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, L^p([0,1],\mathbb{X}))$. Moreover, there exist positive constants L_{b_i} such that

$$\|b_i(t,y) - b_i(t,z)\| \le L_{b_i}\|y - z\|$$

for all $y, z \in \mathbb{X}$, $t \in \mathbb{R}$ and $i = 1,2,3,\dots,m_0$.

(A_3) Let $h = \phi + \psi$ in $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ be such that $\phi \in \mathcal{S}^p\mathcal{AA}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is uniformly continuous in a bounded set $K \subset \mathbb{X}$ for all $t \in \mathbb{R}$ and $\psi^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, L^p([0,1],\mathbb{X}))$. Moreover, assume that there exists a positive constant L_h such that

$$\|h(t,y) - h(t,z)\| \le L_h\|y - z\|$$

for all $y, z \in \mathbb{X}$, $t \in \mathbb{R}$.

Theorem 2 ([26]) *Let $\rho_1, \rho_2 \in \mathbb{V}_\infty$. Assume that f satisfies* (A_1)*. Moreover, assume that $G_1 = g_1 + h_1$ and $G_2 = g_2 + h_2$ are contained in the set $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$ with $g_1, g_2 \in \mathcal{S}^p\mathcal{AA}(\mathbb{R}, \mathbb{X})$, $h_1^b, h_2^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$ and $\overline{\{g_1(t) : t \in \mathbb{R}\}}$, $\overline{\{g_2(t) : t \in \mathbb{R}\}}$ are compact in $\mathbb{X}$. Then, $f(\cdot, G_1(\cdot), G_2(\cdot)) \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$.*

3 Main results

We consider the two term time-fractional order linear differential equation (linear version of (1.1) or (1.2))

$$D_t^{\mu+1}y(t) + \beta D_t^{\nu}y(t) = Ay(t) + D_t^{\mu}f(t). \tag{3.1}$$

The following theorem established in [3] guarantees the existence of mild solutions of equation (3.1).

Theorem 3 *Let A be a generator of the $(\mu,\nu)_\beta$-regularized family $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0}$. Then, equation* (3.1) *admits a mild solution given by*

$$y(t) = \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{3.2}$$

provided that $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0}$ exists and is integrable.

The following definition is inspired by the above representation of mild solutions for the problem (3.1) established by Alvarez-Pardo and Lizama in [3].

Definition 15 *Let A be a generator of the $(\mu,\nu)_\beta$-regularized family $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0}$. A function $y\colon \mathbb{R} \to \mathbb{X}$ defined by*

$$y(t) = \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)F(s, y(s), y(w_1(s, y(s))))\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{3.3}$$

is called a mild solution of (1.1)*.*

Definition 16 *Let A be a generator of the $(\mu,\nu)_\beta$-regularized family $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0}$. A function $y\colon \mathbb{R} \to \mathbb{X}$ defined by*

$$y(t) = \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)F_1\left(s, y(s), \int_{-\infty}^{s} \mathcal{K}(s-\xi)h(\xi, y(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in \mathbb{R}, \tag{3.4}$$

is called a mild solution of (1.2)*.*

3.1 Results for linear systems

The following theorem is the main result for the linear case.

Theorem 4 *Let $\rho_1, \rho_2 \in \mathbb{V}_\infty$. Let $f = \phi + \psi$ be in $\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$ with $\phi \in \mathcal{S}^p\mathcal{AA}(\mathbb{R}, \mathbb{X})$ and $\psi^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1], \mathbb{X}))$. Then, equation* (3.1) *admits a weighted pseudo almost automorphic mild solution.*

Proof. Define a map

$$\Lambda f(t) := \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s) f(s)\,\mathrm{d}s. \tag{3.5}$$

From Theorem 3 we know that $\Lambda f(t)$ is a mild solution of (3.1). Next, we show that $\Lambda f(t) \in \mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$. Since $f = \phi + \psi \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}\Lambda f(t) =& \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s) f(s)\,\mathrm{d}s = \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)\phi(s)\,\mathrm{d}s + \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)\psi(s)\,\mathrm{d}s\\ =&\Lambda\phi(t) + \Lambda\psi(t),\end{aligned}$$

where

$$\Lambda\phi(t) = \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)\phi(s)\,\mathrm{d}s \quad \text{and} \quad \Lambda\psi(t) = \int_{-\infty}^{t} \mathcal{S}_{\mu,\nu}(t-s)\psi(s)\,\mathrm{d}s.$$

By Theorem 1, we have

$$\|\mathcal{S}_{\mu,\nu}(t)\| \leq \frac{M}{1+|\omega|(t^{\mu+1}+\beta t^{\nu})}, \quad t \geq 0. \tag{3.6}$$

For $0 < \mu \leq \nu \leq 1$ we note that $\sigma(t) := \frac{M}{1+|\omega|(t^{\mu+1}+\beta t^{\nu})} \in L^1(\mathbb{R}^+)$ is non-decreasing. Then, we conclude by Lemma 4 that $\Lambda\phi \in \mathcal{AA}(\mathbb{R},\mathbb{X})$. Next, we show that $\Lambda\psi \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$. We define

$$\Lambda_n\psi(t) = \int_{t-n-1}^{t-n} \mathcal{S}_{\mu,\nu}(t-s)\psi(s)\,\mathrm{d}s,$$

where $n = 0, 1, 2, \ldots$. Then,

$$\begin{aligned}\|\Lambda_n\psi(t)\| \leq& \int_{n}^{n+1} \|\mathcal{S}_{\mu,\nu}(s)\|\|\psi(t-s)\|\,\mathrm{d}s\\ \leq& \int_{n}^{n+1} \frac{M}{1+|\omega|(s^{\mu+1}+\beta s^{\nu})}\|\psi(t-s)\|\,\mathrm{d}s\\ \leq& \frac{M}{1+|\omega|(n^{\mu+1}+\beta n^{\nu})}\left(\int_{n}^{n+1} \|\psi(t-s)\|\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{1}{w(r,\rho_1)}\int_{-r}^{r} \|\Lambda_n\psi(t)\|\rho_2(t)\,\mathrm{d}t\\ &\leq \frac{M}{1+|\omega|(n^{\mu+1}+\beta n^{\nu})}\frac{1}{w(r,\rho_1)}\int_{-r}^{r}\left(\int_{n}^{n+1} \|\psi(t-s)\|\,\mathrm{d}s\right)^{\frac{1}{p}}\rho_2(t)\,\mathrm{d}t.\end{aligned}$$

By the translation invariance property of $\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$, it follows that $\psi^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$. Moreover, the above inequality leads to the conclusion that $\Lambda_n\psi \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ for $n = 0, 1, 2, \ldots$. Further, the last estimation implies that

$$\|\Lambda_n\psi(t)\| \leq \frac{M}{1+|\omega|(n^{\mu+1}+\beta n^{\nu})}\|\psi\|_{S^p}.$$

Since $0 < \mu, \nu \leq 1$, we have $1 < \mu + 1 < 2$, and therefore

$$I_{\mu,\nu} := \int_0^\infty \frac{1}{1 + |\omega|(s^{\mu+1} + \beta s^\nu)} \, \mathrm{d}s < \infty. \tag{3.7}$$

Then,

$$\begin{aligned} \sum_{n=0}^\infty \frac{M}{1 + |\omega|(n^{\mu+1} + \beta n^\nu)} \leq & \Big(M + \sum_{n=1}^\infty \int_{n-1}^n \frac{M}{1 + |\omega|(s^{\mu+1} + \beta s^\nu)} \, \mathrm{d}s \Big) \\ \leq & M + \int_0^\infty \frac{M}{1 + |\omega|(s^{\mu+1} + \beta s^\nu)} \, \mathrm{d}s \\ \leq & M(1 + I_{\mu,\nu}) < \infty. \end{aligned}$$

We conclude by the Weierstrass M-test that the series $\sum_{n=0}^\infty \Lambda_n \psi(t)$ converges uniformly on $\mathbb{R}$. Moreover,

$$\sum_{n=0}^\infty \Lambda_n \psi(t) = \int_{-\infty}^t \mathcal{S}_{\mu,\nu}(t-s)\psi(s) \, \mathrm{d}s = \Lambda\psi(t).$$

We note that

$$\|\Lambda\psi(t)\| \leq \sum_{n=0}^\infty \|\Lambda_n \psi(t)\| \leq M(1 + I_{\mu,\nu})\|\psi\|_{S^p}.$$

Now, for a fixed $n_0 \in \mathbb{N}$ (the set of natural numbers) we have

$$\begin{aligned} & \frac{1}{w(r,\rho_1)} \int_{-r}^r \|\Lambda\psi(t)\|\rho_2(t) \, \mathrm{d}t \\ & \quad \leq \frac{1}{w(r,\rho_1)} \int_{-r}^r \Big\| \Lambda\psi(t) - \sum_{k=0}^{n_0} \Lambda_k \psi(t) \Big\| \rho_2(t) \, \mathrm{d}t + \sum_{k=0}^{n_0} \frac{1}{w(r,\rho_1)} \int_{-r}^r \|\Lambda_k \psi(t)\|\rho_2(t) \, \mathrm{d}t. \end{aligned}$$

Since the series $\sum_{k=0}^\infty \Lambda_k \psi(t)$ converges uniformly to $\Lambda\psi(t)$ and $\Lambda_k \psi(t) \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$, we conclude that

$$\frac{1}{w(r,\rho_1)} \int_{-r}^r \|\Lambda\psi(t)\|\rho_2(t) \, \mathrm{d}t = 0.$$

It implies that $\Lambda\psi(t) = \sum_{n=0}^\infty \Lambda_n \psi(t) \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$. This completes the proof. □

3.2 Results for nonlinear systems

Now, we establish the existence and uniqueness of weighted pseudo almost automorphic mild solutions for (1.1) and (1.2) when the forcing terms F, F_1, h and b_i, where $i = 1, 2, 3, \ldots, m_0$, satisfy Lipschitz type conditions.

Lemma 5 *Let $\rho_1, \rho_2 \in \mathbb{V}_\infty$. Assume that $h \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfies* (A$_3$)*. Then, for $y(t) \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$, we have*

$$\Psi_h(t) := \int_{-\infty}^t \mathcal{K}(t-s)h(s, y(s)) \, \mathrm{d}s \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X}). \tag{3.8}$$

Proof. Since $y \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$, by Theorem 2.18 in [3], we conclude that $g(\cdot) = h(\cdot, y(\cdot)) \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$. Further,

$$\begin{aligned}\Psi_g(t) =& \int_{-\infty}^t \mathcal{K}(t-s)g(s)\,\mathrm{d}s = \int_{-\infty}^t \mathcal{K}(t-s)g_1(s)\,\mathrm{d}s + \int_{-\infty}^t \mathcal{K}(t-s)g_2(s)\,\mathrm{d}s\\ =&\Psi_{g_1}(t) + \Psi_{g_2}(t),\end{aligned}$$

where $g_1 \in \mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$ and $g_2^b \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$. We show that $\Psi_{g_1}(t) \in \mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$ and $\Psi_{g_2}(t) \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$. It easy to check that Ψ_g is bounded and continuous.

For a sequence $\{\tau_n'\}$ chose a subsequence $\{\tau_n\}$ and $g_1' \in L^p_{loc}(\mathbb{R},\mathbb{X})$. Using Fubini's theorem, we have

$$\begin{aligned}&\int_r^{r+1} \|\Psi_{g_1}(t+\tau_n) - \Psi_{g_1'}(t)\|^p\,\mathrm{d}t\\ &\qquad= \int_r^{r+1}\left\| \int_{-\infty}^{t+\tau_n} \mathcal{K}(t+\tau_n-s)g_1(s)\,\mathrm{d}s - \int_{-\infty}^t \mathcal{K}(t-s)g_1'(s)\,\mathrm{d}s\right\|^p\mathrm{d}t\\ &\qquad= \int_r^{r+1}\left\| \int_{-\infty}^{t} \mathcal{K}(t-s)[g_1(s+\tau_n) - g_1'(s)]\,\mathrm{d}s\right\|^p\mathrm{d}t\\ &\qquad\le \int_r^{r+1}\int_0^\infty |\mathcal{K}(s)|^p\|g_1(t+\tau_n-s) - g_1'(t-s)\|^p\,\mathrm{d}s\,\mathrm{d}t\\ &\qquad\le \int_r^{r+1}\|g_1(t+\tau_n-s) - g_1'(t-s)\|^p\,\mathrm{d}t\Big(\int_0^\infty |\mathcal{K}(s)|^p\,\mathrm{d}s\Big)\\ &\qquad\le \frac{C_{\mathcal{K}}^p}{bp}\int_r^{r+1}\|g_1(t+\tau_n-s) - g_1'(t-s)\|^p\,\mathrm{d}t \to 0.\end{aligned}$$

Similarly, we can verify that $\int_r^{r+1} \|\Psi_{g_1'}(t-\tau_n) - \Psi_{g_1}(t)\|^p\,\mathrm{d}t \to 0$. This implies that $\Psi_{g_1}(t) \in \mathcal{S}^p\mathcal{AA}(\mathbb{R},\mathbb{X})$. Next, we show that

$$\lim_{r\to\infty}\frac{1}{w(r,\rho_1)}\int_{-r}^r\left(\int_t^{t+1}\|\Psi_{g_2}(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\rho_2(t)\,\mathrm{d}t = 0.$$

Note that

$$\begin{aligned}\int_t^{t+1}\|\Psi_{g_2}(s)\|^p\,\mathrm{d}s \le& \int_t^{t+1}\int_{-\infty}^{\tau}|\mathcal{K}(\tau-s)|^p\|g_2(s)\|^p\,\mathrm{d}s\,\mathrm{d}\tau\\ \le& \int_t^{t+1}\int_0^\infty|\mathcal{K}(s)|^p\|g_2(\tau-s)\|^p\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

Since $\mathcal{K} \in L^1(0,\infty)$ and $g_2 \in \mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R}, L^p([0,1],\mathbb{X}))$, by Fubini's theorem, we get

$$\int_t^{t+1}\|\Psi_{g_2}(s)\|^p\,\mathrm{d}s \le \int_t^{t+1}\|g_2(\tau-s)\|^p\,\mathrm{d}\tau\Big(\int_0^\infty|\mathcal{K}(s)|^p\,\mathrm{d}s\Big).$$

Now, we conclude that

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{w(r,\rho_1)}&\int_{-r}^{r}\left(\int_t^{t+1}\|\Psi_{g_2}(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\rho_2(t)\,\mathrm{d}t\\ &\le\lim_{r\to\infty}\frac{1}{w(r,\rho_1)}\int_{-r}^{r}\left(\int_t^{t+1}\|g_2(\tau-s)\|^p\frac{C_{\mathcal{K}}^p}{(bp)}\,\mathrm{d}\tau\right)^{\frac{1}{p}}\rho_2(t)\,\mathrm{d}t\\ &=\frac{C_{\mathcal{K}}}{(bp)^{\frac{1}{p}}}\lim_{r\to\infty}\frac{1}{w(r,\rho_1)}\int_{-r}^{r}\left(\int_t^{t+1}\|g_2(\tau-s)\|^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\rho_2(t)\,\mathrm{d}t\to 0,\end{aligned}$$

as $g_2\in\mathcal{PAA}_{\rho_1,\rho_2}(\mathbb{R},L^p([0,1],\mathbb{X}))$. □

Let $L>0$ be a constant. We define

$$\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X}):=\{y\in\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X}):\|y(t)-y(s)\|\le L|t-s| \text{ for all } t,s\in\mathbb{R}\}.$$

It is clear that $\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ is a non-empty closed subspace of $\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$.

Lemma 6 *Let* $\rho_1,\rho_2\in\mathbb{V}_\infty$. *Suppose* $F\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X})$ *and* $b_i\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}\times\mathbb{X},\mathbb{X}), i=1,2,3,\dots,m_0$, *satisfy* (A$_1$) *and* (A$_2$), *respectively. Then, for* $y(t)\in\mathcal{S}^p\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$, *we have*

$$F(t,y(t),y(w_1(t,y(t))))\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X}),$$

where $w_1(t,y(t))=b_1(t,y(b_2(t,\cdots,y(b_{m_0}(t,y(t)))\cdots)))$ *for all* $t\in\mathbb{R}$.

Proof. Using the fact that $y(t)\in\mathcal{S}^p\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ and applying Theorem 2.18 in [3] and the condition (A$_2$), we conclude that $w_1\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$. Further, by Theorem 2 and the condition (A$_1$) imposed on F, we obtain the assertion. □

Theorem 5 *Let* $\rho_1,\rho_2\in\mathbb{V}_\infty$. *Let* A *be a generator of the* $(\mu,\nu)_\beta$*-regularized family* $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\ge0}$. *Suppose* $F\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}\times\mathbb{X}\times\mathbb{X},\mathbb{X})$ *and* $b_i\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}\times\mathbb{X},\mathbb{X}), i=1,2,3,\dots,m_0$, *satisfy the conditions* (A$_1$) *and* (A$_2$), *respectively. Then, the problem* (1.1) *admits a unique weighted pseudo almost automorphic mild solution in the Banach space* $\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ *if*

$$L_F(2+LL_b)I_{\mu,\nu}M<1, \tag{3.9}$$

where $L_b=[L^{m_0-1}L_{b_1}\cdots L_{b_{m_0}}+L^{m_0-2}L_{b_1}\cdots L_{b_{m_0-1}}+\cdots+LL_{b_1}L_{b_2}+L_{b_1}]>0$ *and* $I_{\mu,\nu}$ *is defined in* (3.7).

Proof. Consider the operator $\Lambda_F\colon\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})\to\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ defined by

$$\Lambda_F(y)(t)=\int_{-\infty}^t\mathcal{S}_{\mu,\nu}(t-s)F(s,y(s),y(w_1(s,y(s))))\,\mathrm{d}s,$$

where $w_1(t,y(t))=b_1(t,y(b_2(t,\cdots y(b_m(y(t),t))\cdots)))$. Let $y\in\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$. Now, using Lemma 3 and Lemma 6 we have that $F(t,y(t),y(w_1(t,y(t))))\in\mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\,\mathbb{X})$.

Further, it follows from Theorem 4 that $\Lambda_F(\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X}))\subset\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$. Now, for $t,s\in\mathbb{R}$ such that $t\geq s$, we have

$$\begin{aligned}\|\Lambda_F(y)(t)-\Lambda_F(y)(s)\|\leq&\int_s^t\|\mathcal{S}_{\mu,\nu}(t-\xi)\|\|F(\xi,y(\xi),y(w_1(s,y(\xi))))\|\,\mathrm{d}\xi\\ \leq&M_0\int_s^t J_{\mu,\nu}(t-\xi)\,\mathrm{d}\xi\\ \leq&M_0M_1|t-s|,\end{aligned}$$

where $M_0=\sup_{\xi\in\mathbb{R},y\in\mathbb{X}}\|F(\xi,y(\xi),y(w_1(s,y(\xi))))\|$ and

$$M_1=\sup_{t\geq\xi\geq s}\frac{M}{1+|\omega|((t-\xi)^{\mu+1}+\beta(t-\xi)^{\nu})}.$$

This implies that $\Lambda_F(\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X}))\subset\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$ for $L=M_0M_1$.

For $y,z\in\mathcal{WPAA}^L_{\rho_1,\rho_2}(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}&\|\Lambda_F(y)(t)-\Lambda_F(z)(t)\|\\ &\leq\sup_{t\in\mathbb{R}}\int_{-\infty}^t\|\mathcal{S}_{\mu,\nu}(t-\xi)\|\|F(\xi,y(\xi),y(w_1(\xi,y(\xi))))-F(\xi,z(\xi),z(w_1(\xi,z(\xi))))\|\,\mathrm{d}\xi.\end{aligned}\tag{3.10}$$

Also, we have

$$\begin{aligned}\|F(\xi,y(\xi),&y(w_1(\xi,y(\xi))))-F(\xi,z(\xi),z(w_1(\xi,z(\xi))))\|\\ \leq&L_F\{\|y(\xi)-z(\xi)\|+\|y(w_1(\xi,y(\xi)))-z(w_1(\xi,z(\xi)))\|\}\\ \leq&L_F\{\|y(\xi)-z(\xi)\|+\|y(w_1(\xi,z(\xi)))-z(w_1(\xi,z(\xi)))\|\\ &+\|y(w_1(\xi,y(\xi)))-y(w_1(\xi,z(\xi)))\|\}\\ \leq&L_F\{2\|y(\xi)-z(\xi)\|+\|y(w_1(\xi,y(\xi)))-y(w_1(\xi,z(\xi)))\|\}\\ \leq&L_F\{2\|y-z\|_\infty+L|(w_1(\xi,y(\xi)))-(w_1(\xi,z(\xi)))|\}.\end{aligned}$$

Now, let

$$w_j(\xi,y(\xi))=b_j(\xi,y(b_{j+1}(\xi,\cdots y(\xi,b_{m_0}(\xi,y(\xi)))\cdots))),\ j=1,2,\cdots,m_0,$$

with $w_{m_0+1}(\xi,u(\xi))=\xi$ (for more details we refer the reader to [24, p. 2183]). Thus, we obtain

$$\begin{aligned}|w_1(\xi,y(\xi))-w_1(\xi,z(\xi))|=&|b_1(\xi,y(w_2(\xi,y(\xi))))-b_1(\xi,w(w_2(\xi,w(\xi))))|\\ \leq&L_{b_1}\|y(w_2(\xi,y(\xi)))-z(w_2(\xi,w(\xi)))\|\\ \leq&L_{b_1}[\|y(w_2(\xi,y(\xi)))-y(w_2(\xi,z(\xi)))\|\\ &+\|y(w_2(\xi,z(\xi)))-z(w_2(\xi,z(\xi)))\|]\\ \leq&L_{b_1}[L|b_2(\xi,y(w_3(\xi,y(\xi)))-b_2(\xi,z(w_3(\xi,z(\xi)))|\\ &+\|y-z\|_\infty]\\ &\cdots\\ \leq&[L^{m_0-1}L_{b_1}\cdots L_{b_{m_0}}+L^{m_0-2}L_{b_1}\cdots L_{b_{m_0-1}}+\cdots\\ &+LL_{b_1}L_{b_2}+L_{b_1}]\|y-z\|_\infty.\end{aligned}$$

Therefore, we get

$$\|F(\xi, y(\xi), y(w_1(\xi, y(\xi)))) - F(\xi, z(\xi), z(w_1(\xi, z(\xi))))\|_\infty \leq L_F(2 + LL_b)\|y - z\|_\infty, \quad (3.11)$$

where $L_b = [L^{m_0-1}L_{b_1} \cdots L_{b_{m_0}} + L^{m_0-2}L_{b_1} \cdots L_{b_{m_0-1}} + \cdots + LL_{b_1}L_{b_2} + L_{b_1}] > 0$. Now, from (3.10) and (3.11), we have

$$\begin{aligned}\|\Lambda_F(y) - \Lambda_F(z)\|_\infty \leq & L_F(2 + LL_b)\|y - w\|_\infty \int_0^\infty \frac{M}{1 + |\omega|(\xi^{\mu+1} + \beta\xi^\nu)} \, \mathrm{d}\xi \\ \leq & L_F(2 + LL_b)I_{\mu,\nu}M\|y - z\|_\infty.\end{aligned}$$

This implies that Λ_F is a contraction mapping, so by the Banach contraction principle the equation (1.1) admits a unique weighted pseudo almost automorphic mild solution. □

Theorem 6 *Let $\rho_1, \rho_2 \in \mathbb{V}_\infty$. Let A be a generator of the $(\mu, \nu)_\beta$-regularized family $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0}$. Suppose that $F_1 \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ and $h \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfy the conditions* (A_1) *and* (A_3)*, respectively. Then, the problem* (1.2) *admits a unique weighted pseudo almost automorphic mild solution if*

$$ML_{F_1}I_{\mu,\nu}\left[1 + \frac{C_\mathcal{K}L_h}{b}\right] < 1. \quad (3.12)$$

Proof. Consider the operator $\Lambda_{F_1} : \mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X}) \to \mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$ defined by

$$\Lambda_{F_1}(y)(t) = \int_{-\infty}^t \mathcal{S}_{\mu,\nu}(t - s)F_1(s, y(s), \Psi_h y(s)) \, \mathrm{d}s,$$

where $\Psi_h y(s) = \int_{-\infty}^s \mathcal{K}(s - \xi)h(\xi, y(\xi)) \, \mathrm{d}\xi$. Let $y \in \mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$. Then, we get $y \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$. By Lemma 5 and Theorem 2 we see that $F_1(t, y(t), \int_{-\infty}^t \mathcal{K}(t - s)h(s, y(s)) \, \mathrm{d}s) \in \mathcal{S}^p\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$. Further, it follows from Theorem 4 that $\Lambda_{F_1}(\mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})) \subset \mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$. For $y, z \in \mathcal{WPAA}_{\rho_1,\rho_2}(\mathbb{R}, \mathbb{X})$, we have

$$\begin{aligned}&\|\Lambda_{F_1}y(t) - \Lambda_{F_1}z(t)\|_\infty \\ &\quad \leq \sup_{t\in\mathbb{R}} \int_{-\infty}^t \|\mathcal{S}_{\mu,\nu}(t - s)\|\|F_1(s, y(s), \Psi_h y(s)) - F_1(s, z(s), \Psi_h z(s))\| \, \mathrm{d}s \\ &\quad \leq \sup_{t\in\mathbb{R}} L_{F_1} \int_{-\infty}^t \|\mathcal{S}_{\mu,\nu}(t - s)\| \left[\|y(s) - z(s)\| + \|\Psi_h y(s)) - \Psi_h z(s)\|\right] \mathrm{d}s \\ &\quad \leq \sup_{t\in\mathbb{R}} L_{F_1} \int_{-\infty}^t \|\mathcal{S}_{\mu,\nu}(t - s)\| \left[\|y(s) - z(s)\| + \frac{C_\mathcal{K}L_h}{b}\|y(s) - z(s)\|\right] \mathrm{d}s \\ &\quad \leq L_{F_1}\left[1 + \frac{C_\mathcal{K}L_h}{b}\right]\|y - z\|_\infty \left(\int_0^\infty \|\mathcal{S}_{\mu,\nu}(s)\| \, \mathrm{d}s\right) \\ &\quad \leq ML_{F_1}I_{\mu,\nu}\left[1 + \frac{C_\mathcal{K}L_h}{b}\right]\|y - z\|_\infty.\end{aligned}$$

This implies that Λ_{F_1} is a contraction mapping, so by the Banach contraction principle the equation (1.2) admits a unique weighted pseudo almost automorphic mild solution. □

4 Applications

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let the operator

$$\Pi := \sum_{i,j=1}^{n} b_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}$$

satisfy the condition $\sum_{i,j=1}^{n} b_{ij}(y)\xi_i\xi_j \geq c|\xi|^2$ for all $\xi \in \mathbb{R}^n$, $y \in \Omega$, where $c > 0$ and $0 < \mu \leq \nu \leq 1,\ b_0 > 0$. We consider the following problem

$$\begin{aligned} D_t^{\mu+1}z(t,y(t)) + \beta D_t^{\nu}z(t,y(t)) - \Gamma z(t,y(t)) + b_0 z(t,y(t)) \\ = D_t^{\mu} f(t,y(t),z(t,y(t)),z(w_1(t,z(t,y(t))))), \end{aligned} \tag{4.1}$$

subject to the boundary condition $z|_{\partial\Omega} = 0$.

For convenience, we take $\Omega = [0,\pi] \subset \mathbb{R}^1$, $\mathbb{X} = L^2([0,\pi])$. Let $u(t) = z(t,\cdot)$. Then, (4.1) can be represented in the form (1.1) as

$$D_t^{\mu+1}u(t) + \beta D_t^{\nu}u(t) - Au(t) = D_t^{\mu} f(t,u(t),u(w_1(t,u(t)))), \tag{4.2}$$

where $A := L - b_0$ with the domain $D(A) = \{u \in L^2[0,\pi] : u'' \in L^2[0,\pi], u(0) = u(\pi) = 0\}$. It is known that $A = L - b_0$ is a sectorial operator of angle $\frac{\pi}{2}$ (and hence for $\frac{\nu\pi}{2}$). Thus, A is $\omega = -b_0 < 0$ sectorial operator of angle $\frac{\nu\pi}{2}$.

Case 1: Now, we take

$$\begin{aligned} f(t,u(s),v(s)) =& [\alpha a(t)(\cos u(s) + \cos v(s))] + [\alpha e^{-\frac{|t|}{2}} \sin(\cos u(s) + \cos v(s))] \\ =& \phi(t,u(s),v(s)) + \psi(t,u(s),v(s)) \end{aligned}$$

and

$$v(s) := v(w_1(t,u(s))) = \frac{\|u\|}{4}\cos\left(\frac{1}{2+\sin t+\sin\sqrt{5}t}\right),$$

where for some $\epsilon \in (0,\frac{1}{2})$

$$a(t) := \begin{cases} \cos\left(\frac{1}{2+\sin n+\sin\sqrt{5}n}\right) & \text{for } t \in (n+\epsilon, n-\epsilon),\ n \in \mathbb{Z}, \\ 0, & \text{otherwise;} \end{cases}$$

here $u(s), v(s) \in L^2[0,\pi]$, $t \geq 0, s \in [0,\pi]$ and $\alpha \in \mathbb{R}$. We note that $a(t) \in \mathcal{S}^2\mathcal{AA}(\mathbb{R},\mathbb{R})$ (see Example 2.3 in [23]). Moreover, if $u \in \mathcal{WPAA}_\rho^L(\mathbb{R},\mathbb{X})$, then by the composition theorem $\phi(t,u,v) \in \mathcal{S}^2\mathcal{AA}(\mathbb{R},\mathbb{X})$ and $\psi^b(t,u,v) \in \mathcal{PAA}_\rho(\mathbb{R},L^2([0,1],\mathbb{X}))$ for $\rho_1 = \rho_2 = \rho = e^{\eta t}, \eta > 1$. Consequently, $f \in \mathcal{S}^p\mathcal{WPAA}_\rho(\mathbb{R},\mathbb{X})$ with $\rho = e^{\eta t}$. Then, f and w_1 satisfy (A_1) and (A_2), respectively, and $\|f(t,u_1,v_1) - f(t,u_2,v_2)\|_\infty \leq \alpha(2+\frac{L}{2})\|u_1-u_2\|_\infty$. Assume that $\alpha < \frac{1}{[2+\frac{L}{2}]I_{\mu,\nu}M}$. Then, by Theorem 5, equation (4.2) has a unique weighted pseudo almost automorphic mild solution in the Banach space $\mathcal{WPAA}_\rho^L(\mathbb{R},\mathbb{X})$.

Case 2: Let us consider the nonlinear function

$$f(t,u(s),v(s)) = [\theta a(t)(\cos u(s) + \cos v(s))] + [\theta e^{-\frac{|t|}{2}} \sin(\cos u(s) + \cos v(s))],$$

where

$$v(s) = \frac{\|u\|}{8} \sin\left(\int_{-\infty}^{t} e^{-(t-s)} \cos(s,u(s))\, \mathrm{d}s \right);$$

here $u(s), v(s) \in L^2[0,\pi]$, $t \geq 0$, $s \in [0,\pi]$ and $\theta \in \mathbb{R}$. As in Case 1, we can show that $f \in \mathcal{S}^p\mathcal{WPAA}_\rho(\mathbb{R},\mathbb{X})$ with $\rho = e^{\eta t}$. Further, f and v satisfy (A_1) and (A_2), respectively, and $\|f(t,u_1,v_1) - f(t,u_2,v_2)\|_\infty \leq \frac{9\theta}{4}\|u_1 - u_2\|_\infty$. Assume that $\theta < \frac{4}{9 I_{\mu,\nu} M}$. Then, by Theorem 6, equation (4.2) has a unique weighted pseudo almost automorphic mild solution in the Banach space $\mathcal{WPAA}_\rho(\mathbb{R},\mathbb{X})$.

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References

[1] S. Abbas, V. Kavitha, R. Murugesu, *Stepanov-like weighted pseudo almost automorphic solutions to fractional order abstract integro-differential equations*, Proceedings – Mathematical Sciences **125** (2015), 323–351.

[2] N. A. Abdelkarim, E. Khalil, L. Souden, *Pseudo almost periodic and pseudo almost automorphic mild solutions to nonautonomus partial evolution equation*, Nonautonomous Dynamical Systems **2** (2015), 12–30.

[3] E. Alvarez-Pardo, C. Lizama, *Weighted pseudo almost automorphic mild solutions for two term fractional order differential equations*, Applied Mathematics and Computation **271** (2015), 154–167.

[4] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics 96, Birkhäuser, Basel, 2001.

[5] S. Bouchner, *Continuous mapping of almost automorphic and almost periodic functions*, Proceeding of the National Academy of Science USA 52(1964), 907–910.

[6] J. Blot, G. M. N'Guérékata, G. M. Mophou, D. Pennequin, *Weighted pseudo almost automorphic functions and applications to abstract differential equations*, Nonlinear Analysis: Theory Methods & Applications **71** (2009), no. 3–4, 903–909.

[7] V. Casarino, *Characterizations of almost automorphic groups and semigroups*, Rendiconti Accademia Nazionale delle Scienze detta dei XL, Memorie di Matematica e Applicazioni **24** (2000), 219–235.

[8] A. Chadha, D. N. Pandey, *Mild solutions for non-autonomous impulsive semi-linear differential equations with iterated deviating arguments*, Electronic Journal of Differential Equations **2015** (2015), no. 222, 1–14.

[9] Y. K. Chang, R. Zhang, G. M. N'Guérékata, *Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations*, Computers & Mathematics with Applications **64** (2012), no. 10, 3160–3170.

[10] Y. K. Chang, G. M. N'Guérékata, Z. H. Zhao, *Existence of weighted pseudo almost periodic solutions to some classes of differential equations with S^p-weighted pseudo almost periodic coefficients*, Nonlinear Analysis: Theory Methods & Applications **72** (2010), no. 1, 430–438.

[11] T. Diagana, *Almost automorphic type and almost periodic type functions in abstract spaces*, Springer, New York, 2013.

[12] T. Diagana, *Double-weighted pseudo almost periodic functions*, African Diaspora Journal of Mathematics **12** (2011), 121–136.

[13] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Lecture Notes in Mathematics, vol. 2004, Springer, New York, 2010.

[14] L. E. El'sgol'ts, S. B. Norkin, *Introduction to the theory and application of differential equations with deviating arguments*, Academic Press, New York, 1973.

[15] C. G. Gal, *Almost automorphic mild solutions to some semilinear abstract differential equations with deviated argument in Fréchet spaces*, Electronic Journal of Differential Equations **2006**, no. 16, 1–8.

[16] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel 2006.

[17] R. Haloi, *Existence of weighted pseudo-almost automorphic mild solutions to non-autonomous abstract neutral differential equations with deviating arguments*, Differential Equations and Dynamical Systems **22** (2014), no. 2, 165–179.

[18] V. Keyantuo, C. Lizama, M. Warma, *Asymptotic behavior of fractional order semilinear evolution equations*, Differential and Integral Equations **26** (2013), no. 7-8, 757–780.

[19] P. Kumar, D. N. Pandey, D. Bahuguna, *Existence of piecewise continuous mild solutions for impulsive functional differential equations with iterated deviating arguments*, Electronic Journal of Differential Equations **2013** (2013), no. 241, 1–15.

[20] C. Lizama, *Solutions of two-term time fractional order differential equations with nonlocal initial conditions*, Electronic Journal of Qualitative Theory of Differential Equations **2012** (2012), no. 82, 1–9.

[21] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

[22] G. M. Mophou, *Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations*, Applied Mathematics and Computation **217** (2011), no. 19, 7579–7587.

[23] G. M. N'Guérékata, A. Pankov, *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Analysis: Theory Methods & Applications **68** (2008), no. 9, 2658–2667.

[24] S. Stević, *Globally bounded solution of a system of nonlinear functional differential equations with iterated deviating argument*, Applied Mathematics and Computation **219** (2012), no. 4, 2180–2185.

[25] T.-J. Xiao, J. Liang, J. Zhang, *Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces*, Semigroup Forum **76** (2008), no. 3, 518–524.

[26] Z. Xia, M. Fan, *Weighted Stepanov-like pseudo almost automorphy and applications*, Nonlinear Analysis: Theory Methods & Applications **75** (2012), 2378–2397.