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# ANALYTICAL SOLUTION OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNEL BY USING TAYLOR EXPANSION METHOD

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**Abstract.** In this paper, we apply Taylor's approximation and then transform the given $n$-th order weakly singular linear Volterra and Fredholm integro-differential equations with initial conditions into an ordinary linear differential equation. Some different examples are considered, the results of these examples indicated that the procedure of transformation method is simple and effective and could provide an accurate approximate solution or exact solution.

**Keywords:** General Abel integral, integro-differential equations, weakly singular Fredholm integral-equations, weakly singular Volterra integral-equations, Taylor's series approximation.

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## 1 Introduction

In this paper, we consider the following $n$-th order linear-weakly singular Fredholm and Volterra integro-differential equations at the form

$$y^{(n)}(t) + \sum_{m=0}^{n-1} \beta_m(t) y^{(m)}(t) = f(t) + \int_a^b \frac{y^{(p)}(s)}{|t-s|^\alpha} \, \mathrm{d}s, \quad a \le t \le b, \quad 0 < \alpha < 1 \tag{1.1}$$

with initial conditions

$$y(a) = \mu_0, \quad y'(a) = \mu_1, \quad \dots, \quad y^{(n-1)}(a) = \mu_{n-1}, \tag{1.2}$$

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and

$$y^{(n)}(t) + \sum_{m=0}^{n-1} \beta_m(t) y^{(m)}(t) = f(t) + \int_a^t \frac{y^{(p)}(s)}{|t-s|^{\alpha}} \,\mathrm{d}s, \quad a \le t \le b, \quad 0 < \alpha < 1 \tag{1.3}$$

with initial conditions

$$y(a) = \mu_0, \quad y'(a) = \mu_1, \quad \ldots, \quad y^{(n-1)}(a) = \mu_{n-1}, \tag{1.4}$$

where $\mu_i (i = 0,1,\ldots,n-1)$ are real constants; $n$ and $p$ are nonnegative integers and $0 \le p \le n$.

The given functions $f(t)$ and $\beta_k(t)$ $(k = 0,1,\ldots,n-1)$ in (1.1)–(1.4) are assumed to be at least continuous on their domain $a \le t \le b$.

Moreover, suppose the unknown solution $y(t)$, and its derivative $y^{(n)}(t)$ and the right-hand-side $f(t)$ are in $\mathbb{L}^2([a,b])$.

Finding exact solutions of linear or nonlinear integro-differential equations is usually difficult to solve analytically since many physical problems are modeled by integral and integro-differential equations, the numerical solutions of such equations have been thoroughly studied by many authors. The system (1.1)–(1.4) has been studied by different methods including the spline collocation method [2], piecewise polynomials [5], Haar wavelets [12], the homotopy perturbation method HPM [9], the wavelet-Galerkin method [14], Taylor polynomials [15], the Tau method [7], the sinc-collocation method [21], the combined Laplace transform-Adomian decomposition method [19], and the Adomian's asymptotic decomposition method [1] to determine exact and approximate solutions. In recent years there has been an increasing interest in Taylor's series solution of integral and integro-differential equations.

Taylor's expansion approach is a powerful technique used to find the approximate solution to differential, integral and integro-differential equation. Yalcinbas [20], Maleknejad and Mahmoudi [15], Maleknejad and Arzhang [16], Darania and Ebadian [6], Eke and Jackreece [8] develop interest in Taylor's series solution of integral and integro-differential systems. These methods transform the system equation into a matrix equation which corresponds to a system of nonlinear equations.

In the present work, we develop the Taylor-series expansion method to approximate for solving weakly singular Volterra and Fredholm integro-differential equations.

The proposed method is a direct method in which we remove the singularity by using Taylor's approximation and rewrite the weakly-singular integro-differential Fredholm and Volterra equation as either a linear differential equation that can be solved using the analytical method or the known numerical methods.

## 2 Description of the method

In this section, we discuss two cases:

## 2.1 Case 1: Linear Volterra integro-differential equations of $n$-th order

In this case, our study focuses on a class of linear Volterra integro-differential equations of the following form

$$y^{(n)}(t) + \sum_{m=0}^{n-1} \beta_m(t) y^{(m)}(t) = f(t) + \int_a^t \frac{y^{(p)}(s)}{|t-s|^{\alpha}}\,\mathrm{d}s, \quad a \le t \le b, \quad 0 < \alpha < 1 \tag{2.1}$$

with initial conditions

$$y(a) = \mu_0, \quad y'(a) = \mu_1, \quad \ldots, \quad y^{(n-1)}(a) = \mu_{n-1}. \tag{2.2}$$

We use the following Taylor's approximation of degree $n$ of $y^{(p)}(s)$ about $s = t$ as follows

$$\begin{aligned} y^{(p)}(s) &\approx y^{(p)}(t) + \frac{(s-t)}{1!} y^{(p+1)}(t) + \frac{(s-t)^2}{2!} y^{(p+2)}(t) + \cdots + \frac{(s-t)^{n-p}}{(n-p)!} y^{(n)}(t) && (2.3)\\ &= y^{(p)}(t) - \frac{(t-s)}{1!} y^{(p+1)}(t) + \frac{(t-s)^2}{2!} y^{(p+2)}(t) + \cdots + (-1)^{n-p} \frac{(t-s)^{n-p}}{(n-p)!} y^{(n)}(t) \\ &= \sum_{m=0}^{n-p} (-1)^m \frac{(t-s)^m}{m!} y^{(p+m)}(t). && (2.4) \end{aligned}$$

Substituting the approximate relation (2.4) into the right hand side of (2.1) we obtain

$$y^{(n)}(t) + \sum_{m=0}^{n-1} \beta_m(t) y^{(m)}(t) = f(t) + \int_a^t \sum_{m=0}^{n-p} \frac{1}{m!} \frac{(-1)^m}{(t-s)^{\alpha-m}} y^{(p+m)}(t)\,\mathrm{d}s, \tag{2.5}$$

and hence

$$y^{(n)}(t) + \sum_{m=0}^{n-1} \beta_m(t) y^{(m)}(t) = f(t) + \sum_{m=0}^{n-p} \frac{(-1)^m}{m!} y^{(p+m)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s. \tag{2.6}$$

**Lemma 2.1** (Cf. *[1]*) *Suppose that* $m \in \mathbb{N}^*$ *such that* $0 < \alpha - m < 1$. *Then*

*(1) The improper integral* $\int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s$ *is convergent;*

*(2)* $t \mapsto \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s \in \mathbb{L}^2([a,b])$.

*Proof.* 1) Since $(t-s)^{m-\alpha} = \frac{1}{(t-s)^{\alpha-m}}$ is absolute continuous on $[a,t]$, and

$$\int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s = \lim_{x\to t} \int_a^x \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s. \tag{2.7}$$

then by using formula of integration by parts, one has

$$\begin{aligned} \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s &= \lim_{x\to t} \int_a^x \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s \\ &= \frac{-1}{m-\alpha+1} \lim_{x\to t} \left[(t-s)^{m-\alpha+1}\right]_a^x + \frac{1}{m-\alpha+1} \lim_{x\to t} \int_a^x (t-s)^{m-\alpha+1}\,\mathrm{d}s \\ &= \underbrace{\frac{(t-a)^{m-\alpha+1}}{m-\alpha+1}}_{I} + \underbrace{\frac{1}{m-\alpha+1} \int_a^t (t-s)^{m-\alpha+1}\,\mathrm{d}s.}_{II} && (2.8) \end{aligned}$$

The continuity of $I$ implies $\mathbb{L}^2([a,b])$. On the other hand,

$$\frac{1}{m-\alpha+1}\int_a^t (t-s)^{m-\alpha+1}\,\mathrm{d}s \le \frac{1}{m-\alpha+1}\sqrt{\int_a^t (t-s)^{2(m-\alpha+1)}\,\mathrm{d}s} < +\infty. \tag{2.9}$$

Thus, $II$ exists. As a result, the improper integral $\int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s$ is convergent.

Consequently,

$$t \mapsto \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s \in \mathbb{L}^2([a,b]). \tag{2.10}$$

□

**Theorem 2.2** *Consider the linear integro-differential equation* (2.1) *with initial conditions* (2.2). *Then, under the same assumptions as in Lemma 2.1, the problem* (2.1)–(2.2) *has a unique solution, such that* $y(t)$ *is an exact solution of linear differential equation of order* $n$ *of the form*

$$y^{(n)}(t) + \sum_{m=p}^{n-1} Q_m(t)y^{(m)}(t) + \sum_{m=0}^{p-1} p_m(t)y^{(m)}(t) = g(t). \tag{2.11}$$

Indeed, if we substitute equation (2.4) into equation (2.1), we get

$$\begin{aligned} y^{(n)}(t) + \sum_{m=0}^{n-1} \beta_m(t)y^{(m)}(t) &= f(t) + \sum_{m=0}^{n-p} \frac{(-1)^m}{m!} y^{(p+m)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s \\ &= f(t) + \sum_{m=0}^{n-p-1} \frac{(-1)^m}{m!} y^{(p+m)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s \\ &\quad + \frac{(-1)^{n-p}}{(n-p)!} y^{(n)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-n+p}}\,\mathrm{d}s, \end{aligned} \tag{2.12}$$

and hence

$$\begin{aligned} & y^{(n)}(t) + \sum_{m=p}^{n-1} \beta_m(t)y^{(m)}(t) + \sum_{m=0}^{p-1} \beta_m(t)y^{(m)}(t) \\ &= f(t) + \sum_{m=0}^{n-p} \frac{(-1)^m}{m!} y^{(p+m)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-m}}\,\mathrm{d}s \\ &= f(t) + \sum_{m=p}^{n-1} \frac{(-1)^{(m-p)}}{(m-p)!} y^{(m)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-m+p}}\,\mathrm{d}s \\ &\quad + \frac{(-1)^{n-p}}{(n-p)!} y^{(n)}(t) \int_a^t \frac{1}{(t-s)^{\alpha-n+p}}\,\mathrm{d}s, \end{aligned} \tag{2.13}$$

or equivalently

$$\begin{aligned} & \left(1 - \frac{(-1)^{n-p}}{(n-p)!} \int_a^t \frac{1}{(t-s)^{\alpha-n+p}}\,\mathrm{d}s\right) y^{(n)}(t) \\ & \quad + \sum_{m=p}^{n-1} \left(\beta_m(t) - \sum_{m=p}^{n-1} \frac{(-1)^{(m-p)}}{(m-p)!} \int_a^t \frac{1}{(t-s)^{\alpha-m+p}}\,\mathrm{d}s\right) y^{(m)}(t) + \sum_{m=0}^{p-1} \beta_m(t)y^{(m)}(t) \\ &= f(t). \end{aligned} \tag{2.14}$$

The final form is

$$p(t)y^{(n)}(t)+\sum_{m=p}^{n-1}q_m(t)y^{(m)}(t)\ +\sum_{m=0}^{p-1}\beta_m(t)y^{(m)}(t)=f(t), \tag{2.15}$$

where

$$p(t)=1-\frac{(-1)^{n-p}}{(n-p)!}\int_a^t\frac{1}{(t-s)^{\alpha-n+p}}\,\mathrm{d}s, \tag{2.16}$$

and

$$q_m(t)=\beta_m(t)-\sum_{m=p}^{n-1}\frac{(-1)^{(m-p)}}{(m-p)!}\int_a^t\frac{1}{(t-s)^{\alpha-m+p}}\,\mathrm{d}s. \tag{2.17}$$

Therefore, equation (2.1) can be approximated by the following $n$-th order linear differential equation of the form

$$y^{(n)}(t)+\sum_{m=p}^{n-1}Q_m(t)y^{(m)}(t)+\sum_{m=0}^{p-1}p_m(t)y^{(m)}(t)=g(t), \tag{2.18}$$

where $g(t)=\frac{f(t)}{p(t)}$ and

$$Q_m(t)=\frac{q_m(t)}{p(t)},\quad p_m(t)=\frac{\beta_m(t)}{p(t)} \tag{2.19}$$

with initial conditions

$$y(a)=\mu_0,\quad y'(a)=\mu_1,\quad \ldots,\quad y^{(n-1)}(a)=\mu_{n-1}. \tag{2.20}$$

## 2.2 Case 2: Linear Fredholm integro-differential equations of $n$-th order

In this case, our study focuses on a class of linear Fredholm integro-differential equations of the following form

$$y^{(n)}(t)+\sum_{m=0}^{n-1}\beta_m(t)y^{(m)}(t)=f(t)+\int_a^b\frac{y^{(p)}(s)}{|t-s|^\alpha}\,\mathrm{d}s,\quad a\le t\le b,\quad 0<\alpha<1, \tag{2.21}$$

with initial conditions

$$y(a)=\mu_0,\quad y'(a)=\mu_1,\quad \ldots,\quad y^{(n-1)}(a)=\mu_{n-1}. \tag{2.22}$$

We use the following Taylor's approximation of degree $n+1$ of $y^{(p)}(s)$ about $s=t$:

$$\begin{aligned}
&y^{(p)}(s)\\
\approx\ & y^{(p)}(t)+\frac{(s-t)}{1!}y^{(p+1)}(t)+\frac{(s-t)^2}{2!}y^{(p+2)}(t)+\cdots+\frac{(s-t)^{n-p+1}}{(n-p+1)!}y^{(n+1)}(t)\\
=\ & y^{(p)}(t)-\frac{(t-s)}{1!}y^{(p+1)}(t)+\frac{(t-s)^2}{2!}y^{(p+2)}(t)+\cdots+(-1)^{n-p+1}\frac{(t-s)^{n-p+1}}{(n-p+1)!}y^{(n+1)}(t)\\
=\ & \sum_{m=0}^{n-p+1}(-1)^m\frac{(t-s)^m}{m!}y^{(p+m)}(t).
\end{aligned} \tag{2.23}$$

Substituting the approximate relation (2.23) into the right hand side of (2.21) yields

$$y^{(n)}(t)+\sum_{m=0}^{n-1}\beta_m(t)y^{(m)}(t)=f(t)+\sum_{m=0}^{n-p+1}\frac{(-1)^m}{m!}y^{(p+m)}(t)\int_a^b(t-s)^m\,|t-s|^{-\alpha}\,\mathrm{d}s. \quad (2.24)$$

**Lemma 2.3** *(Generalized Abel integral equation [3]) Consider the equation*

$$f(t)=\beta\int_0^t\frac{g(t)}{(t-s)^\mu}\,\mathrm{d}s+\gamma\int_t^1\frac{g(t)}{(s-t)^\mu}\,\mathrm{d}s,\quad 0<\mu<1, \quad (2.25)$$

*where $\beta$ and $\gamma$ are constants.*
*The solution of the equation* (2.25)*, integrable in the interval* $[0,1]$ *is derived as*

$$g(t)=-t^{\gamma+\mu-1}\frac{\sin^2(\pi\gamma)}{\pi^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^1\frac{\mathrm{d}s}{(s-t)^\gamma}\int_0^s\frac{t^{-\gamma}h(t)}{(s-t)^{1-\gamma}}\,\mathrm{d}t+\frac{C}{t^{2-\gamma-\mu}(1-t)^\gamma},\ \lambda<0, \quad (2.26)$$

$$g(t)=-t^{\mu-1}(1-t)^\gamma\frac{\sin^2(\pi\gamma)}{\pi^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{\mathrm{d}s}{(t-s)^\gamma}\int_s^1\frac{(1-t)^{-\gamma}h(t)}{(t-s)^{1-\gamma}}\,\mathrm{d}t+\frac{C}{t^{\gamma+1-\mu}(1-t)^{1-\gamma}},\ \lambda>0, \quad (2.27)$$

*where $C$ is an arbitrary constant. The value of $\lambda$ is defined by the equation*

$$|\lambda|=\pi\cot(\pi\gamma), \quad (2.28)$$

*for $0<\gamma<\frac{1}{2}$, along with*

$$\lambda=-\frac{a\pi}{b\sin(\pi\mu)}+\pi\cot(\pi\mu), \quad (2.29)$$

*and*

$$h(t)=\frac{t^{1-\mu}}{b}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{f(y)}{(t-y)^{1-\mu}}\,\mathrm{d}y. \quad (2.30)$$

*The relation* (2.26) *(respectively* (2.27)*) serves as a new formula (cf. [2]), representing the solution of the generalized Abel integral equation* (2.25)*.*

**Theorem 2.4** *Consider linear integro-differential equation* (2.21) *with initial conditions* (2.22)*. Then, under the same assumptions as in Lemma 2.3, the problem* (2.21)*–*(2.22) *has a unique solution, such that $y(t)$ is an exact solution of linear differential equation of order $n+1$ of the form*

$$y^{(n+1)}(t)+\sum_{m=p}^{n-1}Q_m(t)y^{(m)}(t)+\sum_{m=0}^{p-1}p_m(t)y^{(m)}(t)=g(t). \quad (2.31)$$

Indeed, if we substitute equation (2.23) into equation (2.21), we get

$$y^{(n)}(t)+\sum_{m=0}^{n-1}\beta_m(t)y^{(m)}(t)=f(t)+\sum_{m=0}^{n-p+1}\frac{(-1)^m}{m!}y^{(p+m)}(t)\int_a^b(t-s)^m\,|t-s|^{-\alpha}\,\mathrm{d}s, \quad (2.32)$$

and therefore, using the decomposition (2.25), we obtain

$$\begin{aligned}&\left(1-\frac{(-1)^{n-p+1}}{(n-p+1)!}\left[\int_a^t(t-s)^{n-p-\alpha+1}\,\mathrm{d}s+(-1)^{n-p+1}\int_t^b(s-t)^{n-p-\alpha+1}\,\mathrm{d}s\right]\right)y^{(n+1)}(t)\\&\quad+\sum_{m=0}^{p-1}\beta_m(t)y^{(m)}(t)+\sum_{m=p}^{n-1}\beta_m(t)y^{(m)}(t) \qquad (2.33)\\&=\ f(t)+\sum_{m=p}^{n}\frac{(-1)^{(p+m)}}{(p+m)!}y^{(m)}(t)\left[\int_a^t(t-s)^{p+m-\alpha}\,\mathrm{d}s+(-1)^{p+m}\int_t^b(s-t)^{p+m-\alpha}\,\mathrm{d}s\right],\end{aligned}$$

and hence

$$\begin{aligned}&\left(1-\frac{(-1)^{n-p+1}}{(n-p+1)!}\left[\int_a^t (t-s)^{n-p-\alpha+1}\,\mathrm{d}s+(-1)^{n-p+1}\int_t^b (s-t)^{n-p-\alpha+1}\,\mathrm{d}s\right]\right)y^{(n+1)}(t)\\ &\quad+\sum_{m=p}^{n}\left(\beta_m(t)-\frac{(-1)^{(p+m)}}{(p+m)!}\left[\int_a^t (t-s)^{p+m-\alpha}\,\mathrm{d}s+(-1)^{p+m}\int_t^b (s-t)^{p+m-\alpha}\,\mathrm{d}s\right]\right)y^{(m)}(t)\\ &\quad+\sum_{m=0}^{p-1}\beta_m(t)y^{(m)}(t)\\ &= f(t),\end{aligned} \tag{2.34}$$

or equivalently

$$p(t)y^{(n+1)}(t)+\sum_{m=p}^{n} q_m(t)y^{(m)}(t)+\sum_{m=0}^{p-1}\beta_m(t)y^{(m)}(t)=f(t), \tag{2.35}$$

where

$$p(t)=1-\frac{(-1)^{n-p+1}}{(n-p+1)!}\left[\int_a^t (t-s)^{n-p-\alpha+1}\,\mathrm{d}s+(-1)^{n-p+1}\int_t^b (s-t)^{n-p-\alpha+1}\,\mathrm{d}s\right]$$

and

$$q_m(t)=\beta_m(t)-\frac{(-1)^{(p+m)}}{(p+m)!}\left[\int_a^t (t-s)^{p+m-\alpha}\,\mathrm{d}s+(-1)^{p+m}\int_t^b (s-t)^{p+m-\alpha}\,\mathrm{d}s\right]. \tag{2.36}$$

Therefore, equation (2.21) can be approximated by the following $n$-th order linear differential equation of the form

$$y^{(n+1)}(t)+\sum_{m=p}^{n} Q_m(t)y^{(m)}(t)+\sum_{m=0}^{p-1} p_m(t)y^{(m)}(t)=g(t), \tag{2.37}$$

where $g(t)=\frac{f(t)}{p(t)}$ and

$$Q_m(t)=\frac{q_m(t)}{p(t)},\quad p_m(t)=\frac{\beta_m(t)}{p(t)} \tag{2.38}$$

with initial conditions

$$y(a)=\mu_0,\quad y'(a)=\mu_1,\quad \ldots,\quad y^{(n-1)}(a)=\mu_{n-1}, \tag{2.39}$$

where the initial condition $y^{(n)}(a)$ is obtained directly from the equation (2.37).

## 3 Examples

We illustrate the procedure of solving equations (1.1)–(1.2) and (1.3)–(1.4), which determines the solution of linear weakly-singular Volterra and Fredholm integro-differential equations, by the following examples.

**Example 3.1** *Consider the second order integro-differential equation:*

$$t\,y''(t)-(t+1)\,y=f(t)+\int_0^t \frac{1}{|t-s|^{\frac{1}{2}}}\,y(s)\,\mathrm{d}s,\quad 0\le t\le 1 \tag{3.1}$$

*and*

$$f(t)=(-2\sqrt{t}-\tfrac{2}{3}\,t^{\frac{3}{2}})\,e^t, \tag{3.2}$$

*with the initial conditions $y(0)=1$ and $y'(0)=1$.*

Here the function $f$ is selected such that $y(t)=e^t$ is the exact solution of problem (3.1). Then, after substituting equation (2.4) into equation (3.1) with $p=0$ and $n=2$, we obtain

$$\left(1-\tfrac{2}{3}\,t^{\frac{3}{2}}\right)y''+\left(t-2\sqrt{t}\right)y'-(t+1)\,y=(-2\sqrt{t}-\frac{2}{3}\,t^{\frac{3}{2}})\,e^t, \tag{3.3}$$

which is a linear differential equation of order 2 with the initial conditions $y(0)=1$ and $y'(0)=1$.

The exact solution of this equation is $y(t)=e^t$.

**Example 3.2** *Consider the third order integro-differential equation:*

$$y^{(3)}(t)=f(t)+\int_0^t \frac{1}{|t-s|^{\frac{1}{3}}}\,y(s)\,\mathrm{d}s,\quad 0\le t\le 1 \tag{3.4}$$

*and*

$$f(t)=6-\tfrac{243}{440}\,t^{\frac{11}{3}}, \tag{3.5}$$

*with the initial conditions $y(0)=0$, $y'(0)=0$ and $y''(0)=0$.*
*Here the function $f$ is chosen such that $y(t)=t^3$ is the exact solution of problem* (3.4).
*Then, after substituting equation* (2.4) *into equation* (3.4) *with $p=0$ and $n=3$, we obtain*

$$\left(1+\tfrac{1}{22}\,t^{\frac{11}{3}}\right)y^{(3)}(t)-\tfrac{3}{16}\,t^{\frac{8}{3}}\,y''(t)-\tfrac{3}{5}\,t^{\frac{5}{3}}\,y'(t)-\tfrac{3}{2}\,t^{\frac{2}{3}}\,y(t)=f(t), \tag{3.6}$$

*which is a linear differential equation of order 3 with the initial conditions $y(0)=y'(0)=y''(0)=0$. The exact solution of this equation is $y(t)=t^3$.*

**Example 3.3** *Consider the first order linear integro-differential equation:*

$$y'(t)=f(t)+\int_0^1 \frac{1}{|t-s|^{\frac{1}{2}}}y(s)\,\mathrm{d}s,\quad 0\le t\le 1 \tag{3.7}$$

*with the initial condition $y(0)=0$, and*

$$f(t)=2t-\frac{16}{15}t^2\sqrt{t}-\frac{2}{15}\sqrt{1-t}\left(8t^2+4t+3\right). \tag{3.8}$$

*Here the function $f$ is chosen such that $y(t)=t^2$ is the exact solution of problem* (3.7).

*Then, after substituting equation* (3.7) *into equation* (2.34) *with $p=0$ and $n=1$, we obtain*

$$\frac{1}{5}\left[t^2\sqrt{t}+(1-t)^2\sqrt{1-t}\right]y''+\left[1+\frac{2}{3}\,t\sqrt{t}-\frac{2}{3}\,(1-t)\sqrt{1-t}\right]y'+2\left[\sqrt{t}+\sqrt{1-t}\right]y=f(t), \tag{3.9}$$

*which is a linear differential equation of order 2 with initial conditions $y(0)=y'(0)=0$.*

*The exact solution of this equation is $y(t)=t^2$.*

**Example 3.4** *Consider the second order linear integro-differential equation:*

$$y''(t) + (t+1)y'(t) = f(t) + \int_0^1 \frac{ts}{|t-s|^{\frac{1}{3}}} y(s)\,\mathrm{d}s, \quad 0 \le t \le 1 \tag{3.10}$$

*with initial conditions* $y(0) = 1,\ \ y'(0) = 0$*, and*

$$f(t) = 2t^2 + 2t + 2 - t\left(\tfrac{9}{440}\, t^{\frac{5}{3}}\,(27\, t^2 + 44) + \tfrac{3}{440}(1-t)^{\frac{2}{3}}\,(81\, t^3 + 54\, t^2 + 177\, t + 128)\right). \tag{3.11}$$

*Here the function* $f$ *is chosen such that* $y(t) = t^2 + 1$ *is the exact solution of problem* (3.10).

*Then, after substituting equation* (3.10) *into equation* (2.34) *with* $p = 0$ *and* $n = 2$*, we obtain*

$$\begin{aligned} -\frac{t}{6}\, y^{(3)}(t) + \left[1 - \frac{1}{2}\, t\left(\frac{9}{88}\, t^{\frac{11}{3}} + \frac{3}{88}\,(1-t)^{\frac{8}{3}}(3\, t + 8)\right)\right] y''(t) & \\ + \left[t + 1 - \frac{3}{40}\, t\left(-3t^{\frac{8}{3}} + (1-t)^{\frac{5}{3}}(3\, t + 5)\right)\right] y'(t) & \\ + \left[-\frac{3}{10}\, t\left(3\, t^{\frac{5}{3}} + (1-t)^{\frac{2}{3}}(3t + 2)\right)\right] y(t) & = f(t), \end{aligned} \tag{3.12}$$

*which is a linear differential equation of order 3 with initial conditions* $y(0) = 1$*,* $y'(0) = 0$*,* $y''(0) = 2$*. The exact solution of this equation is* $y(t) = t^2 + 1$.

# 4 Conclusion

We have reduced the solution of a class of linear weakly-singular Volterra and Fredholm integro-differential equations to the one of ordinary differential equations by removing the singularity using an appropriate Taylor's approximation. We have demonstrated that the solution of these ordinary differential equations, is exactly the solution of the original weakly-singular Volterra and Fredholm integro-differential equations proposed.

We have considered several distinct examples to illustrate our new approach and have verified our solutions.

# References

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