Journal of Nonlinear Evolution Equations and Applications
Volume 2017, Number 6, pp. 81–93 (September 2018)

ISSN 2161-3680
http://www.jneea.com

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FRACTIONAL ORDER NONAUTONOMOUS NEUTRAL DIFFERENTIAL EQUATIONS WITH DEVIATED ARGUMENTS

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Received on June 29, 2016

Accepted on April 12, 2017

Communicated by Gaston M. N'Guérékata

Abstract. In this paper, we discuss the existence and uniqueness of solutions for a fractional order neutral differential equation with deviated arguments. We use the method of resolvent operators for integral equations suggested by Hernández et al. (*On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal. **73** (2010), 3462–3471) for fractional differential equations. At the end, an example is given to illustrate our analytical findings.

Keywords: Abstract differential equation, fractional derivative, resolvent operator.

2010 Mathematics Subject Classification: 34K30, 34K37, 47A10.

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1 Introduction

The theory of fractional calculus started with a correspondence between L'Hospital and Leibniz in 1695. Presently, lots of literature is available on theoretical as well as numerical work on this topic. It has applications in numerous fields, for example, control theory, signal and image processing, aerodynamics and biophysics, etc. For the fundamental concepts of fractional calculus, we refer the readers to a few excellent books by Kilbas et al. [11], Miller and Ross [13], Hilfer [9], Podlubny [16]

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and Abbas et al. [2]. The existence and uniqueness of solutions for fractional order differential systems has been studied by several authors; see, for example, [1, 3, 4, 5, 6, 10, 12, 14] and reference therein.

In this work, we consider the following nonautonomous neutral differential equation of fractional order $\alpha \in (0, 1]$ with deviated arguments in a Banach space X [7]:

$$\begin{gathered} D_t^\alpha(u(t) + g(t, u(a(t)))) = A(t)u(t) + f(t, u(t), u(h(t, u(t)))), \\ u(0) = u_0, \quad a(t) \leq t, \ \forall\, t > 0, \end{gathered} \tag{1.1}$$

where D_t^α is the $\alpha-$fractional derivative in the Caputo sense. We denote $A = A(0)$ and assume that $0 \in \rho(A)$. In order to establish our results, we introduce some notations. $L(X)$ denotes the space of bounded linear operators from the Banach space X into X endowed with the operator norm $\|\cdot\|_{L(X)}$. $D(A)$ is the domain of the operator A endowed with the graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$. The space of all the continuous functions from $[0, T]$ into X, endowed with the sup-norm $\|\cdot\|_{C([0,T],X)}$, is denoted by $C([0, T], X)$; similarly, the space of all the continuous functions from $[0, T]$ into $\mathbb{R}^+$, endowed with the sup-norm $|\cdot|_{C([0,T],\mathbb{R}^+)}$, is denoted by $C([0, T], \mathbb{R}^+)$. The notation $C^\gamma([0, T], X)$, $\gamma \in (0, 1)$, represents the space formed of all the γ-Hölder X-valued continuous functions from $[0, T]$ into X endowed with the norm $\|u\|_{C^\gamma([0,T],X)} = \|u\|_{C([0,T],X)} + [|u|]_{C^\gamma([0,T],X)}$, where

$$[|u|]_{C^\gamma([0,T],X)} = \sup_{\substack{t,s\in[0,T] \\ t\neq s}} \frac{\|u(t) - u(s)\|_X}{(t - s)^\gamma}.$$

For simplicity, instead of $|\cdot|_{C([0,T],\mathbb{R}^+)}$, $\|\cdot\|_{C([0,T],X)}$ and $\|\cdot\|_{C^\gamma([0,T],X)}$ we will write $|\cdot|_C$, $\|\cdot\|_C$ and $\|\cdot\|_{C^\gamma}$, respectively.

In [8] Hernández et al. showed that the approach used by several authors to define mild solutions of abstract differential equations with fractional derivatives is not appropriate. In order to overcome the shortcoming, Hernández et al. used another approach to treat abstract differential equations with fractional derivatives based on the well developed theory of resolvent operators for integral equations. Motivated by the work of Hernández et al. [8], we investigate the existence and uniqueness of mild solutions for the problem (1.1). We use their approach to solve a nonautonomous problem of fractional order.

In this paper, Section 2 provides some basic definitions and notations mainly concerned with analytic resolvent operators. In Section 3, we establish sufficient conditions for the existence and uniqueness of mild solutions for the problem (1.1). In the last section, Section 4, we give an example to show an application of our results.

2 Preliminaries and assumptions

We assume that the integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} Ax(s)\, \mathrm{d}s, \ t \geq 0,$$

has an associated resolvent operator $(\mathcal{S}(t))_{t\geq 0}$ on X.

Definition 1 *A family of bounded linear operators $(\mathcal{S}(t))_{t\geq 0}$ on X is said to be a resolvent operator for the above integral equation if the following conditions hold*:

- $\mathcal{S}(\cdot)u \in C([0,\infty), X)$ *and* $\mathcal{S}(0)u = u$ *for all* $u \in X$;
- $\mathcal{S}(t)D(A) \subset D(A), A\mathcal{S}(t)u = \mathcal{S}(t)Au$ *and*

$$\mathcal{S}(t)u = u + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A\mathcal{S}(s)u\,\mathrm{d}s \quad \textit{for all } u \in D(A),\ t \geq 0.$$

Throughout the paper we assume that $(\mathcal{S}(t))_{t\geq 0}$ is an analytic resolvent for the integral equation. Therefore, we assume that there exist a function $\phi_A \in L^1_{\text{loc}}([0,\infty), \mathbb{R}^+)$ and positive constants M_i, $i = 0,1,2$, such that $\|\mathcal{S}'(t)x\| \leq \phi_A(t)\|x\|_{D(A)}$ for all $t > 0$, $x \in D(A)$, $\|\mathcal{S}(t)\|_{L(X)} \leq M_0$ and $\|\mathcal{S}^i(t)\|_{L(X)} \leq \frac{M_i}{t^i}$ for $i = 1,2$; for more details we refer to [17].

Now, we consider the following integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} Ax(s)\,\mathrm{d}s + h(t),\ t \in [0,T]. \tag{2.1}$$

Definition 2 ([8, Definition 1.2]) *A continuous function x defined on $[0,T_0]$, $T_0 \leq T$, is called a mild solution of* (2.1) *if* $\int_0^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s \in D(A)$ *and*

$$x(t) = \frac{1}{\Gamma(\alpha)} A\int_0^t (t-s)^{\alpha-1} x(s)\,\mathrm{d}s + h(t),\ t \in [0,T_0].$$

Definition 3 *A continuous function x defined on $[0,T_0]$, $T_0 \leq T$, is called a strong solution of* (2.1) *if $x \in C([0,T_0], D)$ and the integral equation is satisfied for each $t \in [0,T_0]$.*

Let $C([0,T_0],X)$ be the space of all continuous functions $x\colon J = [0,T_0] \to X$; endowed with the sup-norm it is a Banach space. We define another set:

$$C_L(J,X) = \{x \in C(J,X) : \|x(t) - x(s)\| \leq L|t-s|, \forall\, t,s \in J\},$$

where L is a suitable positive constant.

In order to prove the existence of solution of the problem (1.1), we need the following assumptions:

(A1) $f\colon J \times X \times X \to X$ is a continuous function and there exists $L_f \in C([0,T], \mathbb{R}^+)$ such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_f(t)(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for every x_1, x_2, y_1 and $y_2 \in X$;

(A2) $g\colon J \times X \to X$ is a continuous function and there exists $L_g \in C([0,T], \mathbb{R}^+)$ such that

$$\|g(t,x_1) - g(s,x_2)\| \leq L_g(t)(|t-s| + \|x_1 - x_2\|)$$

for every $x_1, x_2 \in X$;

(A3) $h\colon X \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Lipschitz condition

$$|h(s,x_1) - h(s,x_2)| \le L_h \|x_1 - x_2\|;$$

(A4) $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition

$$|a(t) - a(s)| \le L_a |t - s|.$$

Lemma 1 ([8, Lemma 1.1]) *The following properties are valid*:

(i) *if* $x(.)$ *is a mild solution of* (2.1) *on* $[0, T_0]$, *then the function* $t \mapsto \int_0^t \mathcal{S}(t-s)h(s)\,\mathrm{d}s$ *is continuously differentiable on* $[0, T_0]$ *and*

$$x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathcal{S}(t-s)h(s)\,\mathrm{d}s \quad \textit{for every } t \in [0, T_0];$$

(ii) *if* $h \in C^\beta([0, T_0], X)$ *for some* $\beta \in (0,1)$, *then the function defined by*

$$x(t) = \mathcal{S}(t)(h(t) - h(0)) + \int_0^t \mathcal{S}'(t-s)(h(s) - h(t))\,\mathrm{d}s + \mathcal{S}(t)h(0),\ t \in [0, T_0],$$

is a mild solution of the equation (2.1) *on* $[0, T_0]$;

(iii) *if* $h \in C([0, T_0], [D(A)])$ *for some* $0 \le T_0 \le T$, *then the function* $x\colon [0, T_0] \to X$ *defined by*

$$x(t) = \int_0^t \mathcal{S}'(t-s)h(s)\,\mathrm{d}s + h(t),\ t \in [0, T_0],$$

is a mild solution of the equation (2.1) *on* $[0, T_0]$.

3 Existence of solutions

Now, we do some setting in order to get the form of solution of the problem (1.1). Define $A_v^*\colon [0, T_0] \to X$ by $A_v^*(t) = A(t)v(t) - Av(t)$ (see [15]). Assume that there exists $L_A \in C([0,T], \mathbb{R}^+)$ such that

$$\|A_u^*(t) - A_v^*(t)\| \le L_A(t)\|u - v\| \quad \text{for all } u, v \in X.$$

For our convenience we take

$$\widehat{A}_v(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A_v^*(s)\,\mathrm{d}s.$$

We also define

$$F_u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, u(s), u(h(s, u(s))))\,\mathrm{d}s.$$

By applying the Riemann–Liouville fractional integral of order α to both sides of the problem (1.1), we obtain

$$
\begin{aligned}
&u(t) + g(t, u(a(t))) \\
&\quad = u_0 + g(0, u(a(0))) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) u(s)\, \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u(h(s, u(s))))\, \mathrm{d}s \\
&\quad = u_0 + g(0, u(a(0))) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (A_u^*(s) + Au(s))\, \mathrm{d}s + F_u(t) \\
&\quad = u_0 + g(0, u(a(0))) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A_u^*(s)\, \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, \mathrm{d}s + F_u(t) \\
&\quad = u_0 + g(0, u(a(0))) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, \mathrm{d}s + \widehat{A}_u(t) + F_u(t)
\end{aligned}
$$

for each $t \in [0, T_0]$. Hence, we get

$$
\begin{aligned}
u(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, \mathrm{d}s \\
&+ \widehat{A}_u(t) + F_u(t) - g(t, u(a(t))) + u_0 + g(0, u(a(0))).
\end{aligned}
$$

For our convenience we use the notation G_u for the function $G_u\colon [0, T_0] \to X$ given by $G_u(t) = u_0 + g(0, u(a(0))) - g(t, u(a(t)))$. Incorporating these notations, we get the following form

$$
u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, \mathrm{d}s + \widehat{A}_u(t) + F_u(t) + G_u(t).
$$

Definition 4 *A continuous function $u \in C([0, T_0], X)$ is called a mild solution of the equation* (1.1) *on $[0, T_0]$, $0 < T_0 \le T$, if $\int_0^t (t-s)^{\alpha-1} u(s)\, \mathrm{d}s \in D(A)$ for each $t \in [0, T_0]$ and u satisfies the integral equation*

$$
u(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} u(s)\, \mathrm{d}s + \widehat{A}_u(t) + F_u(t) + G_u(t). \tag{3.1}
$$

Theorem 1 *Assume that $g \in C([0, T] \times X, [D(A)])$ and the function $L_g \in C([0, T], \mathbb{R}^+)$ are such that $L_g(0) < 1$ and $\|g(t, x) - g(t, y)\|_{[D(A)]} \le L_g(t)\|x - y\|$ for all $t \in [0, T]$, $x, y \in X$. Then there exists a unique mild solution of* (1.1) *on $[0, T_0]$ for some $0 < T_0 \le T$, if assumptions* (**A1**)–(**A4**) *are satisfied.*

Proof. Since $L_g(0) < 1$, $|L_g|_{C([0,T_1],\mathbb{R}^+)} \to L_g(0)$ and $\|\phi_A\|_{L^1([0,T_1],\mathbb{R}^+)} \to 0$ as $T_1 \to 0$, there exists $0 < T_0 \le T$ such that

$$
\mu = \left(|L_g|_C + \frac{(2 + L.L_h)|L_f|_C}{\alpha\Gamma(\alpha)} T_0^\alpha + \frac{|L_A|_C}{\alpha\Gamma(\alpha)} T_0^\alpha \right) \left(1 + \|\phi_A\|_{L^1([0,T_0],\mathbb{R}^+)} \right) < 1.
$$

From the part (iii) of Lemma 1, we have

$$
u(t) = \int_0^t \mathcal{S}'(t-s)[\widehat{A}_u(s) + F_u(s) + G_u(s)]\, \mathrm{d}s + \widehat{A}_u(t) + F_u(t) + G_u(t).
$$

Now, we introduce the map $\mathcal{F}\colon C_L([0,T_0],X) \to C_L([0,T_0],X)$ by the formula

$$\mathcal{F}u(t) = \int_0^t \mathcal{S}'(t-s)[\widehat{A}_u(s) + F_u(s) + G_u(s)]\,\mathrm{d}s + \widehat{A}_u(t) + F_u(t) + G_u(t).$$

We observe that

$$\begin{aligned}
&\int_0^t \|\mathcal{S}'(t-s)[\widehat{A}_u(s) + F_u(s) + G_u(s)]\|\,\mathrm{d}s \\
&\qquad \le \int_0^t \phi_A(t-s)\|[\widehat{A}_u(s) + F_u(s) + G_u(s)]\|_{D(A)}\,\mathrm{d}s \\
&\qquad \le \int_0^t \phi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|A^*(\tau)\|_{D(A)}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad\quad + \int_0^t \phi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|f(\tau,u(\tau),u[h(\tau,u(\tau))])\|_{D(A)}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad\quad + \|G_u\|_C\int_0^t \phi_A(t-s)\,\mathrm{d}s \\
&\qquad \le \frac{1}{\Gamma(\alpha)}\|A^*\|_C\int_0^t \phi_A(t-s)\frac{s^\alpha}{\alpha}\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\|f\|_C\int_0^t \phi_A(t-s)\frac{s^\alpha}{\alpha}\,\mathrm{d}s \\
&\qquad\quad + \|G_u\|_C\|\phi_A\|_{L^1} \\
&\qquad \le \left(\|A^*\|_C\frac{T_0^\alpha}{\alpha\Gamma(\alpha)} + \|f\|_C\frac{T_0^\alpha}{\alpha\Gamma(\alpha)} + \|G_u\|_C\right)\|\phi_A\|_{L^1}.
\end{aligned}$$

This shows that the function $s \mapsto \mathcal{S}'(t-s)[\widehat{A}_u(s) + F_u(s) + G_u(s)]$ is integrable and $\mathcal{F}u \in C_L([0,T_0],X)$. Thus, $\mathcal{F}$ is well-defined. For any $u,v \in C_L([0,T_0],X)$ we have

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| &\le \|\widehat{A}_u(t) - \widehat{A}_v(t)\| + \|F_u(t) - F_v(t)\| + \|G_u(t) - G_v(t)\| \\
&\quad + \int_0^t \left\|\mathcal{S}'(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}(A_u^*(\tau) - A_v^*(\tau))\,\mathrm{d}\tau\right\|\mathrm{d}s \\
&\quad + \int_0^t \left\|\mathcal{S}'(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(f(\tau,u(\tau),u[h(\tau,u(\tau))])\right. \\
&\qquad \left. - f(\tau,v(\tau),v[h(\tau,v(\tau))])\Big)\mathrm{d}\tau\right\|\mathrm{d}s \\
&\quad + \int_0^t \|\mathcal{S}'(t-s)(G_u(s) - G_v(s))\|\,\mathrm{d}s \\
&\le \frac{|L_A|_C}{\alpha\Gamma(\alpha)}T_0^\alpha\|u-v\|_C + \frac{(2+LL_h)|L_f|_C}{\alpha\Gamma(\alpha)}T_0^\alpha\|u-v\|_C \\
&\quad + |L_g|_C\|u-v\|_C + \int_0^t \phi_A(t-s)\,\mathrm{d}s\frac{|L_A|_C}{\alpha\Gamma(\alpha)}T_0^\alpha\|u-v\|_C \\
&\quad + \int_0^t \phi_A(t-s)\,\mathrm{d}s\frac{(2+LL_h)|L_f|_C}{\alpha\Gamma(\alpha)}T_0^\alpha\|u-v\|_C \\
&\quad + \int_0^t \phi_A(t-s)\,\mathrm{d}s|L_g|_C\|u-v\|_C \\
&\le \mu\|u-v\|_C.
\end{aligned}$$

By the Banach fixed point theorem, there exists a unique fixed point $u(\cdot)$ of the map $\mathcal{F}$. Hence, $u(\cdot)$ is the mild solution of (1.1). $\square$

Now, we discuss our results assuming Hölder type properties for the functions f, g.

Lemma 2 *Assume that* $\alpha \in (\frac{1}{2}, 1)$ *and* $u \in C(J,X)$. *Then,* $F_u \in C^{1-\alpha}(J,X)$ *and* $[|F_u|]_{C^{1-\alpha}([0,T_0],X)} \leq \frac{T_0^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)\|f(\cdot,u(\cdot),u(h(\cdot,u(\cdot))))\|_{C([0,T_0],X)}$.

Proof. For $t \in [0,T_0)$ and $h > 0$ such that $t+h \in [0,T_0]$, we have

$$\begin{aligned}
\|F_u(t+h)-F_u(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left[\frac{1}{(t+h-s)^{1-\alpha}}+\frac{1}{|(s-t)|^{1-\alpha}}\right]\|f\|_X \,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+h}\frac{\|f\|_X}{(t+h-s)^{1-\alpha}}\,\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{2}{|(s-t)|^{1-\alpha}}\frac{h^{1-\alpha}}{(s-t)^{1-\alpha}}\right)\|f\|_X\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+h}\frac{\|f\|_X}{(t+h-s)^{1-\alpha}}\,\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)}\|f(\cdot,u(\cdot),u(h(\cdot,u(\cdot))))\|_C h^{1-\alpha}\int_0^t \frac{2}{(t-s)^{2(1-\alpha)}}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\|f(\cdot,u(\cdot),u(h(\cdot,u(\cdot))))\|_C\int_t^{t+h}\frac{1}{(t+h-s)^{1-\alpha}}\,\mathrm{d}s.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|F_u(t+h)-F_u(t)\| &\leq \frac{1}{\Gamma(\alpha)}\|f(\cdot,u(\cdot),u(h(\cdot,u(\cdot))))\|_C\left(h^{1-\alpha}\frac{2T_0^{2\alpha-1}}{2\alpha-1}+\frac{h^\alpha}{\alpha}\right) \\
&\leq \frac{1}{\Gamma(\alpha)}\|f(\cdot,u(\cdot),u(h(\cdot,u(\cdot))))\|_C\left(\frac{2T_0^{2\alpha-1}}{2\alpha-1}+\frac{T_0^{2\alpha-1}}{\alpha}\right)h^{1-\alpha}.
\end{aligned}$$

We get $[|F_u|]_{C^{1-\alpha}([0,T_0],X)} \leq \frac{T_0^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)\|f(\cdot,u(\cdot),u(h(\cdot,u(\cdot))))\|_C$. □

Lemma 3 *Assume that* $\alpha \in (\frac{1}{2}, 1)$ *and* $u \in C(J,X)$. *Then,* $\widehat{A}_u \in C^{1-\alpha}(J,X)$ *and* $[|\widehat{A}_u|]_{C^{1-\alpha}([0,T_0],X)} \leq \frac{T_0^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)\|A_u^*\|_{C([0,T_0],X)}$.

Proof. For $t \in [0,T_0)$ and $h > 0$ such that $t+h \in [0,T_0]$, we have

$$\begin{aligned}
\|\widehat{A}_u(t+h)-\widehat{A}_u(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left[\frac{1}{(t+h-s)^{1-\alpha}}+\frac{1}{|(s-t)|^{1-\alpha}}\right]\|A_u^*(s)\|_X \,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+h}\frac{\|A_u^*(s)\|_X}{(t+h-s)^{1-\alpha}}\,\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{2}{|(s-t)|^{1-\alpha}}\frac{h^{1-\alpha}}{(s-t)^{1-\alpha}}\right)\|A_u^*(s)\|_X\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+h}\frac{\|A_u^*(s)|\|_X}{(t+h-s)^{1-\alpha}}\,\mathrm{d}s.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|\widehat{A}_u(t+h)-\widehat{A}_u(t)\| &\leq \frac{1}{\Gamma(\alpha)}\|A_u^*(\cdot)\|_C h^{1-\alpha}\int_0^t \frac{2}{(t-s)^{2(1-\alpha)}}\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\|A_u^*(\cdot)\|_C \int_t^{t+h} \frac{1}{(t+h-s)^{1-\alpha}}\,\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha)}\|A_u^*(\cdot)\|_C\left(h^{1-\alpha}\frac{2T_0^{2\alpha-1}}{2\alpha-1}+\frac{h^\alpha}{\alpha}\right) \\ &\leq \frac{1}{\Gamma(\alpha)}\|A_u^*(\cdot)\|_C\left(\frac{2T_0^{2\alpha-1}}{2\alpha-1}+\frac{T_0^{2\alpha-1}}{\alpha}\right)h^{1-\alpha}.\end{aligned}$$

Hence, we get $[|\widehat{A}_u|]_{C^{1-\alpha}([0,T_0],X)} \leq \frac{T_0^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)\|A_u^*(\cdot)\|_C$. □

Theorem 2 *Assume that $\alpha \in (\frac{1}{2},1)$, $f \in C(J_1, L(X))$, $g \in C^{1-\alpha}(J_1, L(X))$ and $\mathcal{S}(\cdot)u_0 \in C^{1-\alpha}([0,T],X)$, where $J_1=[0,T]$. If there exists $0<T_0\leq T$ such that*

$$\left(M_0+\frac{2M_1}{1-\alpha}+\frac{M_2}{(1-\alpha)\alpha}\right)\|g\|_{C^{1-\alpha}([0,T_0],L(X))}<1,$$

then there exists a unique mild solution of (1.1) *on* $[0,T_1]$ *for* $0<T_1<T_0$.

Proof. We choose a suitable T_0 such that $0<T_1<T_0$ and

$$\begin{aligned}&\left(M_0+\frac{2M_1}{1-\alpha}+\frac{M_2}{(1-\alpha)\alpha}+\lambda(g,T_1)\right)\|g\|_{C^{1-\alpha}([0,T_1],L(X))} \\ &\quad+\left(\frac{M_0T_1^\alpha}{\alpha\Gamma(\alpha)}+\frac{M_1T_1^\alpha}{(1-\alpha)\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)+\lambda(f,T_1)\right)\|f\|_{C(L(X))} \\ &\quad+\left(\frac{M_0T_1^\alpha}{\alpha\Gamma(\alpha)}+\frac{M_1T_1^\alpha}{(1-\alpha)\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)+\lambda(A_u^*,T_1)\right)\|A_u^*\|_{C(L(X))}<1,\end{aligned}$$

where $\lambda(g,T_1)$, $\lambda(f,T_1)$ and $\lambda(A_u^*,T_1)$ are given by

$$\begin{aligned}\lambda(g,T_1)&=\left(M_0T_1^{1-\alpha}+M_1\left(\frac{T_1^{1-\alpha}}{1-\alpha}+\frac{T_1^\alpha}{\alpha}\right)\right), \\ \lambda(f,T_1)&=\frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)\left(M_1\left(\frac{2}{1-\alpha}+\frac{T_1^\alpha}{\alpha}\right)+M_0+\frac{M_2}{(1-\alpha)\alpha}\right), \\ \lambda(A_u^*,T_1)&=\frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\right)\left(M_1\left(\frac{2}{1-\alpha}+\frac{T_1^\alpha}{\alpha}\right)+M_0+\frac{M_2}{(1-\alpha)\alpha}\right).\end{aligned}$$

We define the map $\mathcal{F}\colon C^{1-\alpha}([0,T_1],X)\to C^{1-\alpha}([0,T_1],X)$ by $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3$, where

$$\begin{aligned}(\mathcal{F}_1u)(t)&=\mathcal{S}(t)(G_u(t)-u_0)+\int_0^t \mathcal{S}'(t-s)[G_u(s)-G_u(t)]\,\mathrm{d}s+\mathcal{S}(t)u_0, \\ (\mathcal{F}_2u)(t)&=\mathcal{S}(t)F_u(t)+\int_0^t \mathcal{S}'(t-s)[F_u(s)-F_u(t)]\,\mathrm{d}s, \\ (\mathcal{F}_3u)(t)&=\mathcal{S}(t)\widehat{A}_u(t)+\int_0^t \mathcal{S}'(t-s)[\widehat{A}_u(s)-\widehat{A}_u(t)]\,\mathrm{d}s.\end{aligned}$$

Now, our aim is to prove that $\mathcal{F}$ is actually well-defined. For this we take $t \in [0, T_1]$. We have

$$\begin{aligned}
&\int_0^t \mathcal{S}'(t-s)[G_u(s) - G_u(t)]\,\mathrm{d}s \\
&\quad \le M_1 \int_0^t \frac{\|g(s, u(a(s))) - g(t, u(a(t)))\|}{t-s}\,\mathrm{d}s \\
&\quad \le M_1 \int_0^t \frac{\|(g(s) - g(t))u(a(t))\| + \|g(t)(u(a(t)) - u(a(s)))\|}{t-s}\,\mathrm{d}s \\
&\quad \le M_1\big([|g|]_{C^{1-\alpha}}\|u\|_C + \|g\|_C[|u|]_{C^{1-\alpha}}\big) \int_0^t \frac{(t-s)^{1-\alpha}}{t-s}\,\mathrm{d}s \\
&\quad \le M_1\|g\|_{C^\alpha L(X)}\|u\|_{C^\alpha(X)}\frac{(T_1)^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

This shows that the function $s \mapsto \mathcal{S}'(t-s)[G_u(s) - G_u(t)]$ is integrable on the interval $[0, T_1]$ and $t \mapsto \int_0^t \mathcal{S}'(t-s)[G_u(s) - G_u(t)]\,\mathrm{d}s \in C([0, T_1], X)$. Similarly,

$$\begin{aligned}
\int_0^t \mathcal{S}'(t-s)[F_u(s) - F_u(t)]\,\mathrm{d}s &\le M_1\|f\|_{C(X)}\frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1} + \frac{1}{\alpha}\right)\int_0^t \frac{(t-s)^{1-\alpha}}{t-s}\,\mathrm{d}s \\
&\le M_1\|f\|_{C(L(X))}\|u\|_{C(X)}\frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1} + \frac{1}{\alpha}\right)\frac{T_1^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

This shows that the function $s \mapsto \mathcal{S}'(t-s)[F_u(s) - F_u(t)]$ is integrable on the interval $[0, T_1]$ and $t \mapsto \int_0^t \mathcal{S}'(t-s)[F_u(s) - F_u(t)]\,\mathrm{d}s \in C([0, T_1], X)$. Similarly,

$$\begin{aligned}
\int_0^t \mathcal{S}'(t-s)[\widehat{A}_u(s) - \widehat{A}_u(t)]\,\mathrm{d}s &\le M_1\|A_u^*\|_{C(X)}\frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1} + \frac{1}{\alpha}\right)\int_0^t \frac{(t-s)^{1-\alpha}}{t-s}\,\mathrm{d}s \\
&\le M_1\|A_u^*\|_{C(L(X))}\|u\|_{C(X)}\frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\left(\frac{2}{2\alpha-1} + \frac{1}{\alpha}\right)\frac{T_1^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

This allows us to say that $t \mapsto \int_0^t \mathcal{S}'(t-s)[\widehat{A}_u(s) - \widehat{A}_u(t)]\,\mathrm{d}s \in C([0, T_1], X)$. Hence, we conclude that $\mathcal{F} \in C([0, T_1], X)$. It can be easily seen that

$$\begin{aligned}
\|\mathcal{F}u\|_{C(X)} \le{}& \left(M_0T_1^{1-\alpha} + M_1\frac{T_1^{1-\alpha}}{1-\alpha}\right)\|g\|_{C^\alpha L(X)}\|u\|_{C^\alpha(X)} + \|\mathcal{S}(\cdot)u_0\|_{C(X)} \\
&+ \left(M_0\frac{T_1^\alpha}{\alpha\Gamma(\alpha)} + M_1\frac{T_1^\alpha}{\Gamma(\alpha)(1-\alpha)}\left(\frac{2}{2\alpha-1} + \frac{1}{\alpha}\right)\right)\|f\|_{C(L(X))}\|u\|_{C(X)} \\
&+ \left(M_0\frac{T_1^\alpha}{\alpha\Gamma(\alpha)} + M_1\frac{T_1^\alpha}{\Gamma(\alpha)(1-\alpha)}\left(\frac{2}{2\alpha-1} + \frac{1}{\alpha}\right)\right)\|A_u^*\|_{C(L(X))}\|u\|_{C(X)}.
\end{aligned}$$

Now, we prove that $\mathcal{F}_i \in C^{1-\alpha}([0, T_1], X)$, $i = 1, 2, 3$. By taking $\ln(\frac{t+h}{h}) \le \frac{h^\alpha}{\alpha}$ for all $t \in [0, T_1)$ and $t + h \in [0, T_1]$, where $h > 0$, we have

$$\begin{aligned}
&\|(\mathcal{F}_1u)(t+h) - (\mathcal{F}_1u)(t)\| \\
&\quad \le \|(\mathcal{S}(t+h) - \mathcal{S}(t))(G_u(t) - u_0)\| \\
&\qquad + \|\mathcal{S}(t+h)\|_{L(X)}\|(G_u(t+h) - G_u(t))\| \\
&\qquad + \int_t^{t+h} \|\mathcal{S}'(t+h-s)\|_{L(X)}\|(G_u(s) - G_u(t+h))\|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \|\mathcal{S}'(h+s) - \mathcal{S}'(s)\|_{L(X)} \|(G_u(t+s) - G_u(t))\| \,\mathrm{d}s \\
&+ \int_0^t \|\mathcal{S}'(h+s)\|_{L(X)} \|G_u(t) - G_u(t+h)\| \,\mathrm{d}s + \|\mathcal{S}'u_0\|_{C^{1-\alpha}(X)} h^{1-\alpha}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\|(\mathcal{F}_1 u)(t+h) - (\mathcal{F}_1 u)(t)\| \\
&\quad \le \int_t^{t+h} \|\mathcal{S}'(s)(G_u(t) - u_0)\| \,\mathrm{d}s + M_0 \|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} h^{1-\alpha} \\
&\qquad + \int_t^{t+h} \|\mathcal{S}'(t+h-s)\| \|g||_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} (t+h-s)^{1-\alpha} \,\mathrm{d}s \\
&\qquad + \int_0^t \int_t^{t+h} \|\mathcal{S}''(\eta)\|_{L(X)} \|g||_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} (s)^{1-\alpha} \,\mathrm{d}\eta \,\mathrm{d}s \\
&\qquad + \|g||_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} (h)^{1-\alpha} \int_0^t \|\mathcal{S}'(h+s)\| \,\mathrm{d}s + \|\mathcal{S}'u_0\| h^{1-\alpha} \\
&\quad \le M_1 \|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}} \int_t^{t+h} \frac{s^{1-\alpha}}{s} \,\mathrm{d}s \\
&\qquad + M_0 \|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} h^{1-\alpha} \\
&\qquad + M_1 \|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} \int_t^{t+h} \frac{\mathrm{d}s}{(t+h-s)^{\alpha}} \\
&\qquad + M_2 \|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} \int_0^t \int_s^{s+h} \frac{1}{\eta^{1+\alpha}} \,\mathrm{d}\eta \,\mathrm{d}s \\
&\qquad + \|g||_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} (h)^{1-\alpha} M_1 \ln\left(\frac{t+h}{h}\right) \\
&\qquad + \|\mathcal{S}'u_0\|_{C^{1-\alpha}(X)} h^{1-\alpha}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\|(\mathcal{F}_1 u)(t+h) - (\mathcal{F}_1 u)(t)\| \\
&\quad \le \Bigg[\|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} \left(M_1 \left(\frac{2}{1-\alpha} + \frac{T_1^{\alpha}}{\alpha} \right) + M_0 + \frac{M_2}{\alpha(1-\alpha)} \right) \\
&\qquad + \|\mathcal{S}'u_0\|_{C^{1-\alpha}(X)} \Bigg] h^{1-\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
&[|\mathcal{F}_1 u|]_{C^{1-\alpha}(X)} \\
&\quad \le \|g\|_{C^{1-\alpha}(L(X))} \|u\|_{C^{1-\alpha}(X)} \left(M_1 \left(\frac{2}{1-\alpha} + \frac{T_1^{\alpha}}{\alpha} \right) + M_0 + \frac{M_2}{\alpha(1-\alpha)} \right) \\
&\qquad + \|\mathcal{S}'u_0\|_{C^{1-\alpha}(X)}.
\end{aligned}
$$

Similarly, by Lemma 2, we have

$$[|\mathcal{F}_2 u|]_{C^{1-\alpha}(X)} \\ \leq \frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\Big(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\Big)\|f\|_{C(L(X))}\|u\|_{C(X)}\Big(M_1\Big(\frac{2}{1-\alpha}+\frac{T_1^\alpha}{\alpha}\Big) \\ + M_0 + \frac{M_2}{\alpha(1-\alpha)}\Big)$$

and by Lemma 3, we obtain

$$[|\mathcal{F}_3 u|]_{C^{1-\alpha}(X)} \\ \leq \frac{T_1^{2\alpha-1}}{\Gamma(\alpha)}\Big(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\Big)\|A_u^*\|_{C(L(X))}\|u\|_{C(X)}\Big(M_1\Big(\frac{2}{1-\alpha}+\frac{T_1^\alpha}{\alpha}\Big) \\ + M_0 + \frac{M_2}{\alpha(1-\alpha)}\Big).$$

This implies that $\mathcal{F}u \in C^{1-\alpha}([0,T_1],X)$. By using the above inequalities, we arrive at the following expression

$$\|\mathcal{F}u-\mathcal{F}v\| \\ \leq \Big[M_0+\frac{2M_1}{1-\alpha}+\frac{M_2}{(1-\alpha)\alpha}+\lambda(g,T_1)\Big]\|g\|_{C^{1-\alpha}(L(X))}\|u-v\|_{C^{1-\alpha}(X)} \\ + \Big[\frac{M_0T_1^\alpha}{\alpha\Gamma(\alpha)}+\frac{M_1T_1^\alpha}{(1-\alpha)\Gamma(\alpha)}\Big(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\Big)+\lambda(f,T_1)\Big]\|f\|_{C(L(X))}\|u-v\|_{C^{1-\alpha}(X)} \\ + \Big[\frac{M_0T_1^\alpha}{\alpha\Gamma(\alpha)}+\frac{M_1T_1^\alpha}{(1-\alpha)\Gamma(\alpha)}\Big(\frac{2}{2\alpha-1}+\frac{1}{\alpha}\Big)+\lambda(A_u^*,T_1)\Big]\|A_u^*\|_{C(L(X))}\|u-v\|_{C^{1-\alpha}(X)}.$$

This shows that $\mathcal{F}$ is a contraction mapping. Hence, by Banach's fixed point theorem, $\mathcal{F}$ has a unique fixed point. Thus, the proof is complete. □

4 Example

In this section, we consider an example of a partial differential equation of neutral type with retarded argument. The main aim is to show the applicability of our result. Let $X=L^2(0,1)$. We consider the following partial differential equations with deviated argument:

$$\begin{aligned} &\partial_t^\alpha[w(t,x)+f_1(t,w(a(t),x))]-m(t)\partial_x^2 w(t,x)=f_2(x,w(t,x))+f_3(t,x,w(t,x)), \\ &\qquad\qquad x\in(0,1),\ t>0, \\ &w(t,0)=w(t,1)=0,\ t\in[0,T],\ 0<T<\infty, \\ &w(0,x)=u_0,\ x\in(0,1), \end{aligned} \tag{4.1}$$

where

$$f_2(x,w(t,x))=\int_0^x K(x,s)w(s,h(t)(a_1|w(s,t)|+b_1|w_s(s,t)|))\,\mathrm{d}s.$$

Further, we assume that $a_1,b_1\geq 0$, $(a_1,b_1)\neq(0,0)$, $h\colon \mathbb{R}^+\to\mathbb{R}^+$ is locally continuous in t with $h(0)=0$ and $K\colon[0,1]\times[0,1]\to\mathbb{R}$. We define the operator A as follows

$$A(t)u=m(t)u'' \quad \text{with} \quad u\in D(A)=\{u\in H_0^1(0,1)\cap H^2(0,1):\ u''\in X\}. \tag{4.2}$$

We may choose $m(t)$ such that $m(0) = 1$. Hence, $A(0) = A$, which is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$ such that $S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u, \phi_n)\phi_n$, where $\phi_n = \sqrt{\frac{2}{\pi}} \sin nx$, $n = 1, 2, 3, \cdots$.

The equation (4.1) can be reformulated as the following abstract equation in $X = L^2(0, 1)$:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}[u(t) + g(t, u(a(t)))] = A(t)u(t) + f(t, u(t), u[h(u(t), t)]), \quad t > 0,$$
$$u(0) = u_0,$$

where $u(t) = w(t, \cdot)$, that is, $u(t)(x) = w(t, x)$, $x \in (0, 1)$. The function $g\colon \mathbb{R}^+ \times X \to X$ is such that $g(t, u(a(t)))(x) = f_1(t, w(a(t), x))$ and the operator A is same as in formula (4.2). The function $f\colon \mathbb{R}^+ \times X \times X \to X$ is given by

$$f(t, \psi, \xi)(x) = f_2(x, \xi) + f_3(t, x, \psi),$$

where $f_2\colon [0, 1] \times X \to H_0^1(0, 1)$ is given by

$$f_2(t, \xi) = \int_0^x K(x, y)\xi(y)\,\mathrm{d}y.$$

And

$$\|f_3(t, x, \psi)\| \le Q(x, t)(1 + \|\psi\|_{H^2(0,1)})$$

with $Q(\cdot, t) \in X$ and Q is continuous in its second argument. We can easily verify that the function f satisfies the assumption (**A1**). For more details see [7]. For the function a we can take:

(i) $a(t) = kt$, where $t \in [0, T]$ and $0 < k \le 1$;

(ii) $a(t) = kt^n$ for $t \in I = [0, 1]$, $k \in (0, 1]$ and $n \in \mathbb{N}$;

(iii) $a(t) = k \sin t$ for $t \in I = [0, \frac{\pi}{2}]$ and $k \in (0, 1]$.

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