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CONTROLLABILITY OF DISCRETE SEMILINAR IMPULSIVE SYSTEMS AND APPLICATIONS

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Abstract. For many control systems in real life impulses are intrinsic properties that do not modify their controllability. So we conjecture that under certain conditions the abrupt changes as perturbations of a system do not modify certain properties such as controllability. In other words, the controllability is robust by looking the impulses as perturbations. In this regard, here we prove the exact controllability and the approximate controllability of a semilinear difference equation with impulses. We prove that, under some conditions on the nonlinear term and the impulses, the exact controllability and the approximate controllability of the linear equation are preserved. Finally, we apply this result to a discrete version of the semilinear heat and wave equations.

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1 Introduction

For many control systems in real life impulses are intrinsic properties that do not modify their controllability. So we conjecture that under certain conditions the abrupt changes as perturbations of a system do not modify certain properties such as controllability. In other words, the controllability is robust by looking the impulses as perturbations. In this regard, here we prove the exact controllability and the approximate controllability of semilinear difference equations with impulses.

One of the main sources for applications of discrete control systems methods are continuous control systems; that is to say, those models described by differential equations instead of difference equations. The reason for this is that while physical systems are modelled by differential equations, control laws are implemented often in a digital computer, whose inputs and outputs are sequences. A common approach to design controls in this case is to obtain a difference equation model that approximates the continuous system that will be controlled.

Considering this observation and using some ideas presented in [1, 2, 3, 4, 5, 6] we will give sufficient conditions for the exact controllability and the approximate controllability of the following semilinear impulsive difference equation

$$\begin{cases} z(n+1) = A(n)z(n) + B(n)u(n) + f(n, z(n), u(n)), & n > m \in \mathbb{N}^*, \\ z(m) = z_0, \\ z(m_k) = z(m_k - 0) + I_k(m_k, z(m_k), u(m_k)), & k = 1, 2, \dots, p, \end{cases} \tag{1.1}$$

where $\mathbb{N}^* = \mathbb{N} \cup \{0\}$, $m < m_1 < m_2 < \cdots < m_p < n$, $m_i \in \mathbb{N} = \{1, 2, 3, \cdots, \}$, $k = 1, \dots, p$, $z(n) \in Z$, $u(n) \in U$, Z and U are Hilbert spaces, $A \in l^\infty(\mathbb{N}, L(Z))$, $B \in l^\infty(\mathbb{N}, L(U, Z))$, $u \in l^2(\mathbb{N}, U)$, $L(U, Z)$ denotes the space of all bounded linear operators from U to Z, $L(Z, Z) = L(Z)$ and $f, I_k \colon \mathbb{N} \times Z \times U \to Z$ are nonlinear perturbations; $l^2(\mathbb{N}, U) = \{s \colon \mathbb{N} \to U : \sum_{n=1}^{\infty} \|s(n)\|_U^2 < \infty\}$ and $l^\infty(\mathbb{N}, L(U, Z)) = \{D \colon \mathbb{N} \to L(U, Z) : \sup_{n \in \mathbb{N}} \|D(n)\|_{L(U,Z)} < \infty\}$.

In (1.1), $z(m_k - 0)$ represents the value of $z(m_k)$ determined by the first equation of (1.1), which is used to find the new value $z(m_k)$ in the third equation of (1.1) and with this value we calculate $z(m_k + 1)$ in the first equation of (1.1).

Consider the set $\Delta = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n \geq m\}$ and let $\Phi = \{\Phi(n, m)\}_{(n,m) \in \Delta}$ be the evolution operator associated to A, *i.e.*,

$$\Phi(n, m) = \begin{cases} A(n-1) \cdots A(m), & n > m, \\ I_d, & n = m, \end{cases}$$

where I_d is the identity operator in the space of bounded and linear operators $L(Z)$. Then, for $z_0 \in Z$ the equation (1.1) has a unique solution given by

$$\begin{aligned} z_u(n) = \Phi(n, m)z_0 &+ \sum_{k=m+1}^{n} \Phi(n, k)\big[B(k-1)u(k-1) + f(k-1, z(k-1), u(k-1))\big] \\ &+ \sum_{i=1}^{p} \Phi(n, m_i) I_i(m_i, z(m_i), u(m_i)), \quad n > m \in \mathbb{N}^*. \end{aligned} \tag{1.2}$$

Corresponding to the nonlinear system (1.1) we consider also the linear system:

$$\begin{cases} z(n+1) = A(n)z(n) + B(n)u(n), & n > m \in \mathbb{N}^*, \\ z(m) = z_0. \end{cases} \tag{1.3}$$

We will use the following notation $[m, n_0]_{\mathbb{N}} = [m, n_0] \cap \mathbb{N}^*$ with $0 \leq m < n_0$.

Definition 1 (Exact controllability of system (1.1)**)** *The system* (1.1) *is said to be exactly controllable on* $[m, n_0]_{\mathbb{N}}$ *if for every* z_0, $z_1 \in Z$ *there exists* $u \in l^2(\mathbb{N}, U)$ *such that the corresponding solution of* (1.1) *satisfies* $z(m) = z_0$ *and* $z(n_0) = z_1$.

Definition 2 (Approximate controllability of system (1.1)**)** *The system* (1.1) *is said to be approximately controllable on* $[m, n_0]_{\mathbb{N}}$ *if for every* z_0, $z_1 \in Z$ *and* $\epsilon > 0$ *there exists* $u \in l^2(\mathbb{N}, U)$ *such that the corresponding solution of* (1.1) *satisfies* $\|z(n_0) - z_1\| < \epsilon$.

Definition 3 (Approximate controllability on free time of (1.1)**)** *The system* (1.1) *is said to be approximately controllable on free time if for every* z_0, $z_1 \in Z$ *and* $\epsilon > 0$ *there exists* $u \in l^2(\mathbb{N}, U)$ *and* $n_0 \in \mathbb{N}$ *such that the corresponding solution of* (1.1) *satisfies* $\|z(n_0) - z_1\| < \epsilon$.

Remark 1 *To avoid trivialities and contradictions, in order to analyse the controllability on* $[m, n_0]_{\mathbb{N}}$ *of the semilinear system with impulses* (1.1)*, we shall assume the following condition*:

$$m < m_1 < m_2 < \cdots < m_p < n_0. \tag{1.4}$$

Let us assume the following conditions for $k \in \mathbb{N}, u_1, u_2 \in l^2(\mathbb{N}, U), z_1, z_2 \in Z$,

$$\|f(k, z_2, u_2) - f(k, z_1, u_1)\| \leq L_1\{\|z_2 - z_1\| + \|u_2 - u_1\|\}, \tag{1.5}$$

$$\|I_i(k, z_2, u_2) - I_i(k, z_1, u_1)\| \leq L_2\{\|z_2 - z_1\| + \|u_2 - u_1\|\}, \quad i = 1, \dots, p, \tag{1.6}$$

and for $n \in \mathbb{N}, u \in l^2(\mathbb{N}, U), z \in Z$,

$$\|\Phi(n, k) f(k-1, z(k-1), u(k-1))\| \leq M_k, \quad 1 \leq k \leq n, \tag{1.7}$$

$$\sum_{k=1}^{\infty} M_k < \infty. \tag{1.8}$$

We will prove the following statements:

(A) If conditions (1.5)–(1.6) hold and the linear system (1.3) is exactly controllable on $[0, n_0]_{\mathbb{N}}$ for some $n_0 > m_p$, then the semilinear system (1.1) is exactly controllable on $[0, n_0]_{\mathbb{N}}$.

(B) If the conditions (1.7)–(1.8) hold and the linear system (1.3) is approximately controllable on $[m, n_0]_{\mathbb{N}}$ for all $0 \leq m < n_0$ with $n_0 \geq 1$, then the semilinear system (1.1) is approximately controllable on free time.

We have already obtained some results on exact and approximate controllability for linear and semilinear difference equations without impulses [13, 14, 15, 16].

2 Controllability of the linear equation without impulses

In this section we will present some characterization of the exact controllability and the approximate controllability for the linear difference equation (1.3), which appears in the foregoing references. To this end, we note that the solution of (1.3) is given by the discrete variation of constants formula

$$z(n) = \Phi(n,m)z_0 + \sum_{k=m+1}^{n} \Phi(n,k)B(k-1)u(k-1), \quad n > m. \tag{2.1}$$

Definition 4 *For the linear system* (1.3) *we define the following concepts*:

(a) *the controllability map* $\mathcal{B}^{mn_0} \colon l^2(\mathbb{N}, U) \to Z$ (*for* $0 < m < n_0 \in \mathbb{N}$) *is defined by*

$$\mathcal{B}^{mn_0}u = \sum_{k=m+1}^{n_0} \Phi(n_0,k)B(k-1)u(k-1); \tag{2.2}$$

(b) *the grammian map* (*for* $0 < m < n_0 \in \mathbb{N}$) *is defined by* $L_{\mathcal{B}^{mn_0}} = \mathcal{B}^{mn_0}\mathcal{B}^{mn_0*}$.

The proof of the following Proposition can be seen in [13].

Proposition 1 *The adjoint* $\mathcal{B}^{mn_0*} \colon Z \to l^2(\mathbb{N}, U)$ *of the operator* $\mathcal{B}^{mn_0}$ *is given by*

$$(\mathcal{B}^{mn_0*}z)(k-1) = \begin{cases} B^*(k-1)\Phi^*(n_0,k)z, & k \le n_0, \\ 0, & k > n_0, \end{cases} \tag{2.3}$$

and

$$L_{\mathcal{B}^{mn_0}}z = \sum_{k=m+1}^{n_0} \Phi(n_0,k)B(k-1)B^*(k-1)\Phi^*(n_0,k)z, \quad z \in Z. \tag{2.4}$$

When $(m = 0)$ in (1.3) instead of $\mathcal{B}^{mn_0}$ we write $\mathcal{B}^{n_0}$ and instead of $L_{\mathcal{B}^{mn_0}}$ we write $L_{\mathcal{B}^{n_0}}$.

The following Theorem holds in general for a linear bounded operator $G \colon W \to Z$ between Hilbert spaces W and Z (see [1, 2, 7, 8, 14, 15]).

Theorem 1 (1) *The equation* (1.3) *is exactly controllable on* $[0, n_0]_{\mathbb{N}}$ *for some* $n_0 \in \mathbb{N}$ *if and only if one of the following statements holds*:

(a) $\mathrm{Range}(\mathcal{B}^{n_0}) = Z$,

(b) *there exists* $\gamma > 0$ *such that*

$$\langle L_{\mathcal{B}^{n_0}}z, z\rangle \ge \gamma\|z\|_Z^2,$$

(c) *there exists* $\gamma > 0$ *such that*

$$\|\mathcal{B}^{n_0*}z\|_{l^2(\mathbb{N},U)} \ge \gamma\|z\|_Z, \;\; z \in Z.$$

(2) *The linear system* (1.3) *is approximately controllable on* $[m, n_0]_{\mathbb{N}}$ *if and only if one of the following statements holds*:

(a) $\overline{\text{Range}}(\mathcal{B}^{mn_0}) = Z$,

(b) $\text{Ker}(\mathcal{B}^{mn_0*}) = \{0\}$,

(c) $\langle L_{\mathcal{B}^{mn_0}} z, z\rangle > 0$, $z \neq 0$ *in* Z,

(d) *if* $B^*(k-1)\Phi^*(n_0, k)z = 0$ *for* $k \leq n_0$, *then* $z = 0$,

(e) $\lim_{\alpha\to 0^+} \alpha(\alpha I + L_{\mathcal{B}^{mn_0}})^{-1}z = 0$,

(f) *for all* $z \in Z$ *we have* $\mathcal{B}^{mn_0}u_\alpha = z - \alpha(\alpha I + L_{\mathcal{B}^{mn_0}})^{-1}z$, *where*

$$u_\alpha = \mathcal{B}^{mn_0*}(\alpha I + L_{\mathcal{B}^{mn_0}})^{-1}z, \quad \alpha \in (0, 1];$$

so, $\lim_{\alpha\to 0} \mathcal{B}^{mn_0}u_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha(\alpha I + L_{\mathcal{B}^{mn_0}})^{-1}z, \quad \alpha \in (0, 1].$$

Remark 2 *The foregoing theorem implies that the family of operators* $\Gamma_{\alpha mn_0} : Z \to l^2(\mathbb{N}, U)$ *defined by*

$$\Gamma_{\alpha mn_0}z = \mathcal{B}^{mn_0*}(\alpha I + L_{\mathcal{B}^{mn_0}})^{-1}z, \quad \alpha \in (0, 1],$$

is an approximate right inverse of the operator $\mathcal{B}^{mn_0}$, i.e.,

$$\lim_{\alpha\to 0^+} \mathcal{B}^{mn_0}\Gamma_{\alpha mn_0} = I$$

in the strong topology.

The following Lemma can be found in [13].

Lemma 1 *The equation* (1.3) *is exactly controllable on* $[0, n_0]_\mathbb{N}$ *for* $n_0 \in \mathbb{N}$ *if and only if* $L_{\mathcal{B}^{n_0}}$ *is invertible. Moreover, in this case* $S = \mathcal{B}^{n_0*}L^{-1}_{\mathcal{B}_{n_0}}$ *is a right inverse of* $\mathcal{B}^{n_0}$ *and the control* $u \in l^2(\mathbb{N}, U)$ *steering an initial state* z_0 *to a final state* z_1 *is given by*:

$$u = \mathcal{B}^{n_0*}L^{-1}_{\mathcal{B}^{n_0}}(z_1 - \Phi(n_0, 0)z_0). \tag{2.5}$$

Lemma 2 *If the linear system* (1.3) *is approximately controllable on* $[m, n_0]_\mathbb{N}$*, a sequence of controls steering the initial state* z_0 *to a* ε*-neighbourhood of a final state* z_1 *in time* n_0 *is given by*

$$u_\alpha = \mathcal{B}^{mn_0*}(\alpha I + L_{\mathcal{B}^{mn_0}})^{-1}(z_1 - \Phi(n_0, m)z_0),$$

with corresponding solutions $y(n) = y(n, m, y_0, u_\alpha)$ *of the initial value problem*

$$\begin{cases} y(n+1) = A(n)y(n) + B(n)u_\alpha(n), & n > m \in \mathbb{N}^*, \\ y(m) = y_0, \end{cases} \tag{2.6}$$

satisfying

$$\lim_{\alpha\to 0^+} y(n, m, y_0, u_\alpha) = z_1, \tag{2.7}$$

i.e.,

$$\lim_{\alpha\to 0^+} y(n) = \lim_{\alpha\to 0^+} \left\{ \Phi(n, m)z(m) + \sum_{k=m+1}^{n} \Phi(n, k)Bu^m_\alpha(k-1) \right\} = z_1. \tag{2.8}$$

3 Exact controllability of the nonlinear difference equation with impulse

In this section we shall study the exact controllability of the nonlinear difference equation with impulses (1.1) on $[0, n_0]_{\mathbb{N}}$ with $n_0 > m_p$.

The technique applied in the main result of this section can be used in a more general problem since it is based on the following Theorem used to characterize center manifolds in dynamical system theory.

Theorem 2 *Let Z be a Banach space and $K\colon Z \to Z$ a Lipschitz function with a Lipschitz constant $k < 1$ and consider $G(z) = z + Kz$. Then G is a homeomorphism whose inverse is a Lipschitz function with a Lipschitz constant $(1-k)^{-1}$.*

Corresponding to the nonlinear system (1.1) we consider also the linear system (1.3) with $m = 0$:

$$\begin{cases} z(n+1) = A(n)z(n) + B(n)u(n), \quad n \in \mathbb{N}^*, \\ z(0) = z_0. \end{cases} \tag{3.1}$$

Then, the solution of (3.1) is given by the discrete variation of constants formula:

$$z(n) = \Phi(n,0)z(0) + \sum_{k=1}^{n} \Phi(n,k)B(k-1)u(k-1), \ n \in \mathbb{N}. \tag{3.2}$$

Consider the following nonlinear operator $\mathcal{B}^{n_0}_{f,I}\colon l^2(\mathbb{N}, U) \to Z$ defined by

$$\begin{aligned} \mathcal{B}^{n_0}_{f,I}u = \sum_{k=1}^{n_0} \Phi(n_0,k)B(k-1)u(k-1) + \sum_{k=1}^{n_0} \Phi(n_0,k)f(k-1, z(k-1), u(k-1)) \\ + \sum_{i=1}^{p} \Phi(n_0, m_i)I_i(m_i, z(m_i), u(m_i)). \end{aligned} \tag{3.3}$$

Then, the following proposition is a characterization of the exact controllability of the nonlinear system (1.1) on $[0, n_0]_{\mathbb{N}}$.

Proposition 2 *The system* (1.1) *is exactly controllable on $[0, n_0]_{\mathbb{N}}$ for some n_0 if and only if $\mathcal{R}(\mathcal{B}^{n_0}_{f,I}) = Z$.*

Proof. Assume that (1.1) is exactly controllable on $[0, n_0]_{\mathbb{N}}$ for some $n_0 \in \mathbb{N}$. Given $z \in Z$, we can find $z_0, z_1 \in Z$ such that

$$z_1 = \Phi(n_0, 0)z_0 + z. \tag{3.4}$$

Then, there exists a control $u \in l^2(\mathbb{N}, U)$ such that $z_u(0) = z_0$ and $z_u(n_0) = z_1$. So,

$$\begin{aligned} z_1 = z_u(n_0) = \Phi(n_0,0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0,k)B(k-1)u(k-1) \\ + \sum_{k=1}^{n_0} \Phi(n_0,k)f(k-1, z_u(k-1), u(k-1)) + \sum_{i=1}^{p} \Phi(n_0, m_i)I_i(m_i, z(m_i), u(m_i)). \end{aligned} \tag{3.5}$$

By substituting (3.4) in (3.5), we obtain

$$\Phi(n_0,0)z_0 + z = \Phi(n_0,0)z_0 + \sum_{k=1}^{n_0}\Phi(n_0,k)B(k-1)u(k-1) + \sum_{k=1}^{n_0}\Phi(n_0,k)f(k-1,z_u(k-1),u(k-1)) + \sum_{i=1}^{p}\Phi(n_0,m_i)I_i(m_i,z(m_i),u(m_i)).$$

Then,

$$z = \sum_{k=1}^{n_0}\Phi(n_0,k)B(k-1)u(k-1) + \sum_{k=1}^{n_0}\Phi(n_0,k)f(k-1,z_u(k-1),u(k-1)) + \sum_{i=1}^{p}\Phi(n_0,m_i)I_i(m_i,z(m_i),u(m_i)) = \mathcal{B}_{f,I}^{n_0}u.$$

So, $\mathrm{Range}(\mathcal{B}_{f,I}^{n_0}) = Z$.

Assume now that $\mathrm{Range}(\mathcal{B}_{f,I}^{n_0}) = Z$. Consider $z \in Z$ such that

$$z = z_1 - \Phi(n_0,0)z_0, \tag{3.6}$$

with $z_0, z_1 \in Z$. Then there exists a control $u \in l^2(\mathbb{N},U)$ such that

$$\mathcal{B}_{f,I}^{n_0}u = z. \tag{3.7}$$

Then, by substituting (3.6) in (3.7), we obtain

$$z = z_1 - \Phi(n_0,0)z_0 = \sum_{k=1}^{n_0}\Phi(n_0,k)B(k-1)u(k-1) + \sum_{k=1}^{n_0}\Phi(n_0,k)f(k-1,z_u(k-1),u(k-1)) + \sum_{i=1}^{p}\Phi(n_0,m_i)I_i(m_i,z(m_i),u(m_i)).$$

Therefore,

$$z_u(n_0) = \Phi(n_0,0)z_0 + \sum_{k=1}^{n_0}\Phi(n_0,k)B(k-1)u(k-1) + \sum_{k=1}^{n_0}\Phi(n_0,k)f(k-1,z_u(k-1),u(k-1)) + \sum_{i=1}^{p}\Phi(n_0,m_i)I_i(m_i,z(m_i),u(m_i)).$$

So, we have found a solution $z_u(\cdot)$ of (1.1) such that $z_u(n_0) = z_1$ and $z_u(0) = z_0$, *i.e.*, (1.1) is exactly controllable on $[0,n_0]_{\mathbb{N}}$ for some $n_0 \in \mathbb{N}$. □

Lemma 3 *Let u_1, $u_2 \in l^2(\mathbb{N},U)$ and let z_1, z_2 be the corresponding solutions of* (1.1). *Then, the following inequality holds*:

$$\|z_1(j) - z_2(j)\|_Z \leq [M(\|B\| + L_1) + L_2]\sqrt{n_0}e^{(ML_1+L_2)n_0}\|u_1 - u_2\|_{l^2(\mathbb{N},U)}, \tag{3.8}$$

where $j \leq n_0$ and $M = \sup_{1\leq j,k\leq n_0}\{\|\Phi(j,k)\|\}$.

Proof. Let z_1, z_2 be solutions of (1.1) corresponding to u_1, u_2, respectively. Then

$$\begin{aligned}
\|z_1(j)-z_2(j)\| &\leq \sum_{k=1}^{j}\|\Phi(j,k)\|\|B\|\|u_1(k-1)-u_2(k-1)\| \\
&\quad+\sum_{k=1}^{j}\|\Phi(j,k)\|\|f(k-1,z_1(k-1),u_1(k-1))-f(k-1,z_2(k-1),u_2(k-1))\| \\
&\quad+\sum_{i=1}^{p}\|\Phi(j,m_i)\|\|I_i(k-1,z_1(m_i),u_1(m_i))-I_i(m_i,z_2(m_i),u_2(m_i))\| \\
&\leq M[\|B\|+L_1]\sum_{k=1}^{j}\|u_1(k-1)-u_2(k-1)\|+L_2\sum_{i=1}^{p}\|u_1(m_i)-u_2(m_i)\| \\
&\quad+ML_1\sum_{k=1}^{j}\|z_1(k-1)-z_2(k-1)\|+L_2\sum_{i=1}^{p}\|z_1(m_i)-z_2(m_i)\| \\
&\leq [M(\|B\|+L_1)+L_2]\sqrt{n_0}\|u_1-u_2\|+[ML_1+L_2]\sum_{k=1}^{j}\|z_1(k-1)-z_2(k-1)\|.
\end{aligned}$$

Using the Discrete Gronwall Inequality (see [11, Corollary 1.6.2]), we obtain

$$\|z_1(j)-z_2(j)\|_Z \leq [M(\|B\|+L_1)+L_2]\sqrt{n_0}e^{(ML_1+L_2)n_0}\|u_1-u_2\|_{l^2(\mathbb{N},U)},$$

for $j \leq n_0$. □

Now, we are ready to formulate and prove the main result of this section.

Theorem 3 *Under conditions* (1.5) *and* (1.6)*, if the system* (3.1) *is exactly controllable and the following estimate holds*

$$L_K = M(L_1+L_2)(\Gamma+1)\|\mathcal{B}^{n_0*}\|\|L^{-1}_{\mathcal{B}^{n_0}}\|\sqrt{n_0} < 1, \tag{3.9}$$

where $\Gamma = [M(\|B\|+L_1)+L_2]\sqrt{n_0}e^{(ML_1+L_2)n_0}$*, then the system* (1.1) *is exactly controllable on* $[0,n_0]_{\mathbb{N}}$.

Proof. We want to prove that

$$\mathcal{B}^{n_0}_{f,I}(l^2(\mathbb{N};U)) = \text{Range}(\mathcal{B}^{n_0}_{f,I}) = Z.$$

But, from the exact controllability of the linear system (1.3) we know due to Lemma 1 that the operator $S = \mathcal{B}^{n_0*}L^{-1}_{B_{n_0}}$ is a right inverse of $\mathcal{B}^{n_0}$. Then, it is enough to prove that the operator $\widetilde{\mathcal{B}}^{n_0}_{f,I} = \mathcal{B}^{n_0}_{f,I}\circ S$ is surjective. From the equation (2.2) we obtain the following expression for this operator:

$$\begin{aligned}
\widetilde{\mathcal{B}}^{n_0}_{f,I}\xi = \xi &+ \sum_{k=1}^{n_0}\Phi(n_0,k)f(k-1,z(k-1),S(\xi)(k-1)) \\
&+\sum_{i=1}^{p}\Phi(n_0,m_i)I_i(m_i,z(m_i),u(m_i)).
\end{aligned}\tag{3.10}$$

Now, if we define the operator $K\colon Z \to Z$ by

$$\begin{aligned} K\xi = \sum_{k=1}^{n_0} \Phi(n_0,k) f(k-1, z(k-1), S(\xi)(k-1)) \\ + \sum_{i=1}^{p} \Phi(n_0, m_i) I_i(m_i, z(m_i), u(m_i)), \end{aligned} \tag{3.11}$$

where $z = z_\xi$ is the solution of (1.1) corresponding to the control $u = S(\xi)$, then the equation (3.10) takes the form

$$\widetilde{\mathcal{B}}^{n_0}_{f,I} = I_d + K. \tag{3.12}$$

The function K is globally Lipschitz. In fact, let z_1, z_2 be solutions of (1.1) corresponding to the controls $S\xi_1$, $S\xi_2$, respectively. Then

$$\begin{aligned} &\|K\xi_1 - K\xi_2\| \\ &\quad\le \sum_{k=1}^{n_0} \|\Phi(n_0,k)\| \|f(k-1, z_1(k-1), S(\xi_1)(k-1)) - f(k-1, z_2(k-1), S(\xi_2)(k-1))\| \\ &\qquad + \sum_{i=1}^{p} \|\Phi(n_0, m_i)\| \|I_i(m_i, z_1(m_i), S(\xi_1)(m_i)) - I_i(m_i, z_2(m_i), S(\xi_2)(m_i))\| \\ &\quad\le \sum_{k=1}^{n_0} M L_1 \{\|z_1(k-1) - z_2(k-1)\| + \|(S\xi_1)(k-1) - (S\xi_2)(k-1)\|\} \\ &\qquad + \sum_{i=1}^{p} M L_2 \{\|z_1(m_i) - z_2(m_i)\| + \|(S\xi_1)(m_i) - (S\xi_2)(m_i)\|\} \\ &\quad\le \sum_{k=1}^{n_0} M(L_1+L_2)(\Gamma+1)\|(S\xi_1)(k-1) - (S\xi_2)(k-1)\| \\ &\quad\le M(L_1+L_2)(\Gamma+1)\sqrt{n_0}\|S\xi_1 - S\xi_2\|_{l^2(\mathbb{N},U)} \\ &\quad\le M(L_1+L_2)(\Gamma+1)\|\mathcal{B}^{n_0*}\|\|L^{-1}_{\mathcal{B}^{n_0}}\|\sqrt{n_0}\|\xi_1 - \xi_2\|. \end{aligned}$$

Therefore, the operator K satisfies Lipschitz' condition with the Lipschitz constant $L_K = M(L_1+L_2)(\Gamma+1)\|\mathcal{B}^{n_0*}\|\|L^{-1}_{\mathcal{B}^{n_0}}\|\sqrt{n_0}$, and the assumption (3.9) implies that $L_K < 1$. Hence, from Theorem 2 we get that $\widetilde{\mathcal{B}}^{n_0}_{f,I} = I_d + K$ is a homeomorphism and consequently the operator $\mathcal{B}^{n_0}_{f,I}$ is surjective; that is to say,

$$\mathcal{B}^{n_0}_{f,I}(l^2(\mathbb{N};U)) = \mathrm{Range}(\mathcal{B}^{n_0}_{f,I}) = Z. \qquad \square$$

4 Approximate controllability of the nonlinear impulsive equations

In this section we shall study the approximate controllability on free time of the nonlinear difference equation with impulse (1.1).

Theorem 4 *Under conditions* (1.7) *and* (1.8)*, if the system* (1.3) *is approximately controllable on* $[m, n_0]_\mathbb{N}$ *for all* $0 \le m < n_0$ *and* $n_0 \ge 1$*, then the system* (1.1) *is approximately controllable on free time.*

Proof. Given an initial state z_0, a final state z_1 and $\varepsilon > 0$, we want to find a control $u_\alpha^l \in l^2(\mathbb{N}, U)$ steering the system from z_0 to an ε-neighbourhood of z_1 on time n_0. Specifically, the corresponding solution $z_\alpha^l(n) = z(n, 0, z_0, u_\alpha^l)$ of initial value problem (1.1) at time n_0 satisfies

$$\|z_\alpha^l(n_0) - z_1\| \le \varepsilon. \tag{4.1}$$

Consider any $u \in l^2(\mathbb{N}, U)$ and the corresponding solution $z(n) = z(n, 0, z_0, u)$ of initial value problem (1.1). For $\alpha \in (0, 1]$, we define the control $u_\alpha^l \in l^2(\mathbb{N}, U)$ as follows:

$$u_\alpha^l(n) = \begin{cases} u(n), & \text{if } 0 < n \le l,\ n \in \mathbb{N}, \\ u_\alpha(n), & \text{if } l < n \le n_0,\ n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where

$$u_\alpha = \mathcal{B}^{ln_0 *}(\alpha I_d + L_{\mathcal{B}^{ln_0}})^{-1}(z_1 - \Phi(n_0, l) z_0). \tag{4.3}$$

Now, assume that $m_p < l < n_0$. Then the corresponding solution $z_\alpha^l(n) = z(n, 0, z_0, u_\alpha^l)$ of initial value problem (1.1) at time n_0 can be written as follows:

$$\begin{aligned} z_\alpha^l(n_0) &= \Phi(n_0, 0) z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k) B u_\alpha^l(k-1) \\ &\quad + \sum_{k=1}^{n_0} \Phi(n_0, k) f(k-1, z_\alpha^l(k-1), u_\alpha^l(k-1)) \\ &\quad + \sum_{i=1}^{p} \Phi(n_0, m_i) I_i(m_i, z_\alpha^l(m_i), u_\alpha^l(m_i)) \\ &= \Phi(n_0, l) \Bigg\{ \Phi(l, 0) z_0 + \sum_{k=1}^{l} \Phi(l, k) B u_\alpha^l(k-1) \\ &\quad + \sum_{k=1}^{l} \Phi(l, k) f(k-1, z_\alpha^l(k-1), u_\alpha^l(k-1)) \\ &\quad + \sum_{i=1}^{p} \Phi(l, m_i) I_i(m_i, z_\alpha^l(m_i), u_\alpha^l(m_i)) \Bigg\} \\ &\quad + \sum_{k=l+1}^{n_0} \Phi(n_0, k) B u_\alpha^l(k-1) \\ &\quad + \sum_{k=l+1}^{n_0} \Phi(n_0, k) f(k-1, z_\alpha^l(k-1), u_\alpha^l(k-1)) \\ &= \Phi(n_0, l) z(l) + \sum_{k=l+1}^{n_0} \Phi(n_0, k) B u_\alpha^l(k-1) \\ &\quad + \sum_{k=l+1}^{n_0} \Phi(n_0, k) f(k-1, z_\alpha^l(k-1), u_\alpha^l(k-1)). \end{aligned}$$

Therefore, the solution $z^l_\alpha(n) = z(n, 0, z_0, u^l_\alpha)$ of initial value problem (1.1) at time n_0 can be written as follows:

$$z^l_\alpha(n_0) = \Phi(n_0, l)z(l) + \sum_{k=l+1}^{n_0} \Phi(n_0, k)Bu_\alpha(k-1) + \sum_{k=l+1}^{n_0} \Phi(n_0, k)f(k-1, z^l_\alpha(k-1), u_\alpha(k-1)).$$

The corresponding solution $y^l_\alpha(n) = y(n, l, z(l), u_\alpha)$ of initial value problem (2.6) at time n_0 is given by

$$y(n_0) = \Phi(n_0, m)z(l) + \sum_{k=l+1}^{n_0} \Phi(n_0, k)Bu_\alpha(k-1). \tag{4.4}$$

Hence, for $l, n_0 \in \mathbb{N}$ big enough, with $m_p < l < n_0$, we obtain that

$$\begin{aligned}\|z^l_\alpha(n_0) - y^l_\alpha(n_0)\| &\leq \sum_{k=l+1}^{n_0} \|\Phi(n_0, k)\| \|f(k-1, z^l_\alpha(k-1), u_\alpha(k-1))\| \\ &\leq \sum_{k=l+1}^{n_0} M_k < \frac{\varepsilon}{2}.\end{aligned}$$

On the other hand, from Lemma 2, there exists $\alpha > 0$ such that

$$\|y^l_\alpha(n_0) - z_1\| \leq \frac{\varepsilon}{2}.$$

Therefore, from the above two inequalities we get the following estimate

$$\|z^l_\alpha(n_0) - z_1\| \leq \|z^l_\alpha(n_0) - y^l_\alpha(n_0)\| + \|y^l_\alpha(n_0) - z_1\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which completes the proof of the theorem. □

5 Applications

Now, as an application of the main results of this paper we shall consider two important examples, a flow-discretization of the controlled nonlinear 1D heat equation and the controlled nD wave equation with impulses.

In general, given a controlled evolution equation

$$z' = Az + Bu, \;\; z \in Z, u \in U, t > 0,$$

where $z \in Z$, $u \in U$, Z and U are Hilbert spaces, A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, one can consider a discretization on its flow, the same that is used in [9] and [10] to study the exponential dichotomy of evolution equations. That is to say,

$$z(n+1) = T(n)z(n) + Bu(n), \;\; n \in \mathbb{N}^*,$$

where the control $u = \{u(n)\}_{n\geq 1}$ belongs to $l^2(\mathbb{N}, U)$ and $z(n) \in Z$.

Example 1 (Heat equation) We shall consider a discrete version on flow of the controlled nonlinear heat equation in 1 dimension

$$\begin{cases} y_t = y_{xx} + u(t,x), \\ y_x(t,0) = y_x(t,1) = 0, \\ y(0,x) = y_0(x). \end{cases} \tag{5.1}$$

The system (5.1) can be written as an abstract system in the space $Z = L^2[0,1]$ as follows:

$$\begin{cases} z' = -Az + Bu(t), \quad z \in Z, \\ z(0) = z_0, \end{cases} \tag{5.2}$$

where $B = I_d$, the control function u belongs to $L^2(0,r;Z)$ and the operator A is given by $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$. A has the following spectral decomposition.

(a) For all $z \in D(A)$ we have

$$Az = \sum_{n=1}^{\infty} \lambda_n E_n(z),$$

where $\lambda_n = n^2\pi^2$, $E_n(z) = \pi^2\langle z, \phi_n\rangle\phi_n$ and $\phi_n(x) = \sin(n\pi x)$.

(b) $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n z, \quad z \in Z, \quad t \geq 0. \tag{5.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{n=1}^{\infty} E_n z, \quad z \in Z.$$

Now, the discretization of (5.3) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n), \quad z \in Z, \\ z(0) = z_0. \end{cases} \tag{5.4}$$

In this case, $T^*(t) = T(t)$ and $B = I_d$. The system (5.4) is exactly controllable (see [13]). Therefore, if we consider a perturbation with impulses of the equation (5.4), say

$$\begin{cases} z(n+1) = T(n)z(n) + u(n) + f(n, z(n), u(n)), \quad z \in Z, \\ z(0) = z_0, \\ z(m_k) = z(m_k - 0) + I_k(m_k, z(m_k), u(m_k)), \quad k = 1, 2, \ldots, p. \end{cases} \tag{5.5}$$

where the nonlinear term f and the impulses satisfy the conditions of Theorem 3, then we have immediately that (5.5) is exactly controllable.

Example 2 (Wave equation) Now, we shall consider a discretization on flow of the controlled nonlinear wave equation

$$\begin{cases} y_{tt} - \Delta y = u(t,x), \ x \in \Omega, \\ y = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ y(0,x) = y_0(x), \ y_t(0,x) = y_1(x), \ x \in \Omega, \end{cases} \tag{5.6}$$

where Ω is a bounded domain in $\mathbb{R}^n$, the distributed control $u \in L^2(0,\tau; L^2(\Omega))$, $y_0 \in H^2(\Omega) \cap H_0^1$, $y_1 \in L_2(\Omega)$. The system (5.6) can be written as an abstract second order equation in the Hilbert space $X = L^2[0,1]$ as follows:

$$\begin{cases} y'' = -Ay + u(t), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \tag{5.7}$$

where the operator A is given by $A\phi = -\Delta\phi$ with the domain $D(A) = H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R})$.

The operator A has the following properties: the spectrum of A consists only of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$, each one with multiplicity γ_n equal to the dimension of the corresponding eigenspace.

(a) There exists a orthonormal and complete set $\{\phi_n\}$ of eigenvectors of A.

(b) For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} \langle \xi, \phi_{n,k} \rangle \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n \xi, \tag{5.8}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in X and

$$E_n x = \sum_{k=1}^{\gamma_n} \langle \xi, \phi_{n,k} \rangle \phi_{n,k}. \tag{5.9}$$

So, $\{E_n\}$ is an orthonormal and complete family of projections in X and $x = \sum_{n=1}^{\infty} E_n x$, $x \in X$.

(c) $-A$ generates an analytical semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{5.10}$$

(d) The spaces of fractional powers X^r are given by:

$$X^r = D(A^r) = \left\{ x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty \right\}, \ \ r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \ \ x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{5.11}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert space with the norm

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2.$$

Now, using the change of variables $y' = v$, the second order equation (5.7) can be written as a first order system of ordinary differential equations in the Hilbert space $Z = X^{1/2} \times X$ as

$$\begin{cases} z' = \mathcal{A}z + Bu(t), z \in Z, \\ z(0) = z_0, \end{cases} \tag{5.12}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \tag{5.13}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times X$, $u \in L^2(0, \tau, U)$ with $U = X$.

The proof of the following Theorem follows directly from Lemma 2.1 in [12].

Theorem 5 *The operator* $\mathcal{A}$ *given by* (5.13) *is the infinitesimal generator of a strongly continuous group* $\{T(t)\}_{t \in \mathbb{R}}$ *given by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, t \geq 0, \tag{5.14}$$

where $\{P_j\}_{j \geq 1}$ *is a complete family of orthogonal projections in the Hilbert space* Z *given by*

$$P_j = \operatorname{diag}[E_j, E_j], \quad n \geq 1 \tag{5.15}$$

and

$$A_j = R_j P_j, \qquad R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}, \quad j \geq 1. \tag{5.16}$$

Note that

$$R_j^* = \begin{bmatrix} 0 & -1 \\ \lambda_j & 0 \end{bmatrix}, \quad A_j = R_j P_j, \quad A_j^* = R_j^* P_j, \quad j \geq 1,$$

and there exist $M > 1$ such that $\|T(t)\| \leq M$.

Now, the discretization of (5.12) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n), \quad z \in Z, \\ z(m) = z_0, \end{cases} \tag{5.17}$$

where

$$u \in l^2(\mathbb{N}, U), \quad B \colon U \to Z, \quad Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

In this case, the evolution operator associated to $T(\cdot)$, is given by

$$\Phi(m, n) = T(m-1)T(m-2)\dots T(n), \quad n < m,$$

and $\Phi(m,m) = I$. Note that $\Phi(m,n) = T(\Theta(m,n))$, where $\Theta(m,n) = \frac{m^2-n^2+n-m}{2} \in \mathbb{N}$, $m > n$.

It has already been shown that (5.17) is approximately controllable on $[m, n_0]_{\mathbb{N}}$ (see [16]). So, if we consider a perturbation with impulses of the equation (5.17), say

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n) + f(n, z(n), u(n)), & z \in Z, \\ z(0) = z_0, \\ z(m_k) = z(m_k - 0) + I_k(m_k, z(m_k), u(m_k)), & k = 1, 2, \dots, p. \end{cases} \tag{5.18}$$

where the nonlinear term f and the impulses I_k, $k = 1, \dots, p$, are suitable functions, then from the results obtained in Section 4, we see that (5.18) is approximately controllable on free time.

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