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NONLINEAR PARABOLIC EQUATIONS WITH DIFFUSE MEASURE DATA

AMINE MARAH*
Faculté des Sciences et Thechniqes, Université Hassan 1, B.P. 764, Settat, Morocco

HICHAM REDWANE†
Faculté des Sciences Juridiques, Économiques et Sociales, Université Hassan 1, B.P. 764, Settat, Morocco

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Abstract. We prove the existence of a renormalized solution for a class of nonlinear parabolic equations

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}\big(a(x,t,\nabla u)\big) = \mu,$$

where the right-hand side is a diffuse measure, $b(x,u)$ is an unbounded function of u, and where $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u.

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$ and let $Q := \Omega \times (0,T)$. We prove the existence of a renormalized solution for a class of nonlinear parabolic equations of the type:

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}\big(a(x,t,\nabla u)\big) = \mu \quad \text{in } Q, \tag{1.1}$$

$$b(x,u)(t=0) = b(x,u_0) \quad \text{in } \Omega, \tag{1.2}$$

*e-mail address: aminemarah90@gmail.com
†e-mail address: redwane_hicham@yahoo.fr

$$u = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{1.3}$$

In the problem (1.1)–(1.3) the framework is as follows: the data μ is a measure and $b(x, u_0)$ belongs to $L^1(\Omega)$. The operator $-\operatorname{div}(a(x, t, \nabla u))$ is a Leray–Lions operator which is coercive and which grows like $|\nabla u|^{p-1}$ with respect to ∇u (see the assumptions (3.4), (3.5) and (3.6) in Section 3).

In this paper we use the framework of renormalized solutions. This notion was introduced by P.-L. Lions and Di Perna [22] for the study of the Boltzmann equation (see also P.-L. Lions [16] for a few applications to fluid mechanics models; see also [5, 17] for a nonlinear parabolic equation with natural growth). A large number of papers was then devoted to the study of the existence of renormalized solutions of parabolic problems with rough data under various assumptions and in different contexts (for a review on classical results see [1, 2, 3, 5, 6, 7, 10, 11, 17, 18]).

Concerning the datum μ, we restrict ourselves to the space of measures with bounded total variation over Q that do not charge the sets of zero p-capacity (see Section 2 for the definition), the so-called diffuse measures or soft measures, and we will use the symbol $\mu \in \mathcal{M}_0(Q)$ to denote them. The existence and uniqueness of a renormalized solution of (1.1)–(1.3) was proved in [23] in the case where $b(x, u) = u$, $u_0 \in L^1(\Omega)$ and for every measure μ which does not charge the sets of zero p-capacity. In the case where μ is of bounded total variation over Q, $b(x, u) = u$ and $u_0 \in L^1(\Omega)$, the existence of a renormalized solution was proved in [19], and in the case where $b(x, u) = b(u)$, $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_0(Q)$ the existence and uniqueness of a renormalized solution was proved in [8].

We organize the paper as follows. In Section 2 we give some preliminaries; in particular, we provide the definition of a parabolic capacity and some its basic properties. Section 3 is devoted to specifying the assumptions on b, a, u_0 and μ and to giving the definition of a renormalized solution of (1.1)–(1.3). In Section 4 we establish the existence of such a solution (Theorem 2). In Section 5 (Appendix), we prove Proposition 2 which states that the formulation of a renormalized solution does not depend on the decomposition of μ.

2 Preliminaries on parabolic capacity

We recall the notion of a p-capacity associated to our problem (for further details see [20, 21, 23]). For any fixed $T > 0$ let $Q = \Omega \times (0, T)$, and let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ is endowed with its natural norm $\|.\|_{W_0^{1,p}(\Omega)} + \|.\|_{L^2(\Omega)}$ and

$$W = \Big\{ u \in L^p(0, T; V) : u_t \in L^{p'}(0, T; V') \Big\}$$

is endowed with its natural norm $\|.\|_{L^p(0,T;V)} + \|.\|_{L^{p'}(0,T;V')}$. Let us also remark that W is continuously embedded in $C([0, T], L^2(\Omega))$, and if $1 < p < \infty$, then $\mathcal{C}_0^\infty(Q)$ is dense in W. Let $U \subseteq Q$ be an open set. We define the parabolic p-capacity of U as

$$\operatorname{cap}_p(U) = \inf \Big\{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \Big\},$$

where as usual we set $\inf\{\emptyset\} = +\infty$. Then for any Borel set $B \subseteq Q$ we set

$$\operatorname{cap}_p(B) = \inf \Big\{ \operatorname{cap}_p(U) : U \text{ open subset of } Q, B \subseteq U \Big\}.$$

By $\mathcal{M}_b(Q)$ we will denote the set of all Radon measures with bounded variation on Q. Moreover, as we already mentioned, by $\mathcal{M}_0(Q)$ we will denote the set of all measures with bounded total variation over Q that do not charge the sets of zero p-capacity, that is, if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$ for all $E \subseteq Q$ such that $\operatorname{cap}_p(E) = 0$.

In [23] the authors proved the following decomposition theorem.

Theorem 1 *Let μ be a bounded measure on Q. If $\mu \in \mathcal{M}_0(Q)$, then there exists (f, g_1, g_2) such that $f \in L^1(Q)$, $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and*

$$\int_Q \phi \,\mathrm{d}\mu = \int_Q f\phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g_1, \phi\rangle \,\mathrm{d}t + \int_0^T \langle \phi_t, g_2\rangle \,\mathrm{d}t, \quad \phi \in \mathcal{C}_c^\infty(Q).$$

Such a triplet (f, g_1, g_2) will be called a decomposition of μ.

Note that the decomposition of μ is not uniquely determined.

In the proof of the existence result we will use the density argument, and so we need the following preliminary result whose prove can be found, for instance, in [23].

Proposition 1 *Let $\mu \in \mathcal{M}_0(Q)$. Then there exists a decomposition $(f, \operatorname{div}(G), g)$ of μ in the sense of Theorem 1 and an approximation μ^ε of μ satisfying the following conditions*:

$$\mu^\varepsilon \in \mathcal{C}_c^\infty(Q) \ : \|\mu^\varepsilon\|_{L^1(Q)} \le C,$$

$$\int_Q \phi\mu^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_Q f^\varepsilon\phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \operatorname{div}(G^\varepsilon), \phi\rangle \,\mathrm{d}t + \int_0^T \langle \phi_t, g_2^\varepsilon\rangle \,\mathrm{d}t \quad \textit{for all } \phi \in \mathcal{C}_c^\infty(Q),$$

and

$$f^\varepsilon \in \mathcal{C}_c^\infty(Q) : f^\varepsilon \to f \textit{ in } L^1(Q) \textit{ as } \varepsilon \to 0,$$
$$G^\varepsilon \in (\mathcal{C}_c^\infty(Q))^N : G^\varepsilon \to G \textit{ in } (L^{p'}(Q))^N \textit{ as } \varepsilon \to 0,$$
$$g^\varepsilon \in \mathcal{C}_c^\infty(Q) : g^\varepsilon \to g \textit{ in } L^p(0,T;W_0^{1,p}(\Omega) \cap L^2(\Omega)) \textit{ as } \varepsilon \to 0.$$

Here are some notations we will use throughout this paper. For any non-negative real number k by $T_k(r) = \min(k, \max(r, -k))$ we denote the truncation function at level k. By $\langle ., .\rangle$ we mean the duality between suitable spaces in which functions are involved. In particular, we will consider both the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and the duality between $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $W^{-1,p'}(\Omega) + L^1(\Omega)$.

3 Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true.

Suppose that Ω is a bounded open set on $\mathbb{R}^N$, $N \ge 1$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$. Furthermore,

$$b,\ \frac{\partial b}{\partial s} \colon \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \nabla_x b \colon \Omega \times \mathbb{R} \to \mathbb{R}^N \tag{3.1}$$

are Carathéodory functions such that for almost every $x \in \Omega$, $b(x,s)$ is a strictly increasing $\mathcal{C}^1$-function with $b(x,0)=0$. For every $s \in \mathbb{R}$, the function $b(x,s)$ is in $W^{1,p}(\Omega)$.

There exist λ, $\Lambda > 0$ such that

$$\lambda \leq \frac{\partial b(x,s)}{\partial s} \leq \Lambda \tag{3.2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$. There exists a function B in $L^p(\Omega)$ such that

$$\Big|\nabla_x b(x,s)\Big| \leq B(x), \tag{3.3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$. We also assume that

$$a\colon Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function} \tag{3.4}$$

and

$$a(x,t,\xi).\xi \geq \alpha |\xi|^p \tag{3.5}$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ is a given real number. Moreover,

$$|a(x,t,\xi)| \leq \beta\big(L(x,t) + |\xi|^{p-1}\big) \tag{3.6}$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\beta > 0$ is a given real number, L is a non-negative function in $L^{p'}(Q)$. We also assume that

$$[a(x,t,\xi) - a(x,t,\xi')][\xi - \xi'] > 0 \tag{3.7}$$

for any $(\xi,\xi') \in \mathbb{R}^{2N}$ and for almost every $(x,t) \in Q$. Finally, we assume that

$$\mu \in \mathcal{M}_0(Q) \tag{3.8}$$

and that

$$u_0 \text{ is a measurable function defined on } \Omega \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{3.9}$$

The definition of a renormalized solution for the problem (1.1)–(1.3) is given below.

Definition 1 *A measurable function u defined on Q (let $v := b(x,u) - g$) is a renormalized solution of the problem* (1.1)–(1.3) *if*

$$T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for all } k \geq 0 \text{ and } v \in L^\infty(0,T;L^1(\Omega)), \tag{3.10}$$

$$\int_{\{(t,x)\in Q\,:\, n\leq |v| \leq n+1\}} a(x,t,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0 \text{ as } n \to +\infty, \tag{3.11}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise $\mathcal{C}^1(\mathbb{R})$ and such that S' has a compact support we have

$$\begin{aligned}\frac{\partial S(v)}{\partial t} - \operatorname{div}\big(S'(v)a(x,t,\nabla u)\big) + {}& S''(v)a(x,t,\nabla u)\nabla v \\ &= fS'(v) - \operatorname{div}\big(GS'(v)\big) + S''(v)G\nabla v \text{ in } \mathcal{D}'(Q),\end{aligned} \tag{3.12}$$

$$S(v)(t=0) = S(b(x,u_0)) \text{ in } L^1(\Omega). \tag{3.13}$$

Remark 1 *Note that all terms in* (3.12) *are well-defined. Indeed, let $k>0$ be such that* $\operatorname{supp}(S') \subset [-k,k]$. *We have*

$$\nabla S(v) = S'(T_k(v))\nabla T_k(v) \in (L^p(Q))^N.$$

Then $S(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ and $\frac{\partial S(v)}{\partial t} \in \mathcal{D}'(Q)$.

The term $S'(v)a(x,t,\nabla u)$ can be identified with:

$$S'(T_k(v))a\Big(x,t,\Big(\frac{\partial b(x,s)}{\partial s}\Big)^{-1}(\nabla T_k(v) + (\nabla g - \nabla_x b(x,u))\chi_{\{|v|\le k\}})\Big) \text{ a.e. in } Q.$$

Using the assumption (3.6), *we obtain*

$$\begin{aligned}|S'(v)a(x,t,\nabla u)| \le \beta\, \|S'\|_{L^\infty(\mathbb{R})}\Big[L(x,t) + \\ + \Big(\frac{\partial b(x,u)}{\partial s}\Big)^{-(p-1)}\Big|\nabla T_k(v) + \nabla g - \nabla_x b(x,u)\Big|^{p-1}\Big] \text{ a.e. in } Q.\end{aligned} \tag{3.14}$$

Further, using (3.2), (3.3) *and* (3.10), *we deduce that*: $S'(v)a(x,t,\nabla u) \in (L^{p'}(Q))^N$. *The term $S''(v)a(x,t,\nabla u)\nabla v$ can be identified with*

$$S''(T_k(v))a\Big(x,t,\Big(\frac{\partial b(x,u)}{\partial s}\Big)^{-1}(\nabla T_k(v) + (\nabla g - \nabla_x b(x,u))\chi_{\{|v|\le k\}})\Big)\nabla T_k(v) \text{ a.e. in } Q.$$

In view of (3.2), (3.3), (3.10), (3.14) *and the Hölder inequality we obtain*

$$S''(v)a(x,t,\nabla u)\nabla v \in L^1(Q).$$

Finally, $fS''(v)$ and $S''(v)G\nabla v \in L^1(Q)$ and $GS'(v) \in (L^{p'}(Q))^N$.

We also have $\frac{\partial S(v)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and $S(v) \in L^p(0,T;W_0^{1,p}(\Omega))$, which implies that $S(v) \in \mathcal{C}^0([0,T],L^1(\Omega))$ (see [17]) *and* (3.13) *makes a weak sense.*

It should be observed (*see* [23]) *that since $\mu \in \mathcal{M}_0(Q)$, then μ does not charge set at $t=0$ and in the weak sense we can suppose that $g(x,0)=0$ for a.e. $x\in\Omega$.*

Note that the formulation of a renormalized solution does not depend on the decomposition of μ. The proof of this fact relies on the following result.

Lemma 1 *Let $\mu \in \mathcal{M}_0(Q)$, and let $(f,\operatorname{div}(G),g)$ and $(\overline{f},\operatorname{div}(\overline{G}),\overline{g})$ to be two different decompositions of μ in the sense of Theorem* 1. *Then we have $(g-\overline{g})_t = \overline{f} - f - \operatorname{div}\big(\overline{G}-G\big)$ in distributional sense, $g-\overline{g} \in C([0,T];L^1(\Omega))$ and $(g-\overline{g})(0)=0$.*

Proof. See [23]. □

Proposition 2 *Let u be a renormalized solution of* (1.1)–(1.3). *Then u satisfies* (3.10)–(3.13) *for every decomposition $(\overline{f},\operatorname{div}(\overline{G}),\overline{g})$ of μ.*

Proof. See Appendix. □

4 Existence result

This section is devoted to establishing the following existence theorem.

Theorem 2 *Under the assumptions* (3.1)–(3.9) *there exists at least a renormalized solution* u *of the problem* (1.1)–(1.3).

Proof. The proof is divided into 6 steps. In Step 1 we introduce an approximate problem. Step 2 is devoted to establishing a few a priori estimates. In Step 3 the limit u of the approximate solutions u^ε is introduced and $v := b(x,u) - g$ is shown to belong to $L^\infty(0,T;L^1(\Omega))$ and to satisfy (3.10). In Step 4 we define a time regularization of the field $T_k(u)$ and we establish Lemma 2, which allows us to control the parabolic contribution that arises in the monotonicity method when passing to the limit. Step 5 is devoted to proving an energy estimate (Lemma 3). At last, Step 6 is devoted to proving that u satisfies (3.11), (3.12) and (3.13) of the Definition 1.

Step 1. For a fixed $\varepsilon > 0$ let us introduce the following regularizations of the data:

$$b_\varepsilon(x,s) = b(x, T_{\frac{1}{\varepsilon}}(s)) + \varepsilon s \quad \text{a.e. in } \Omega, \text{ for every } s \in \mathbb{R}, \tag{4.1}$$

$$u_0^\varepsilon \in \mathcal{C}_c^\infty(\Omega) : b_\varepsilon(x,u_0^\varepsilon) \to b(x,u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.2}$$

In view of Proposition 1 we can find

$$\mu^\varepsilon \in \mathcal{C}_c^\infty(Q) \ : \|\mu^\varepsilon\|_{L^1(Q)} \le C \quad \text{and} \quad \mu^\varepsilon = f^\varepsilon - \operatorname{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} \tag{4.3}$$

such that

$$f^\varepsilon \in \mathcal{C}_c^\infty(Q) : f^\varepsilon \to f \text{ in } L^1(Q) \text{ as } \varepsilon \to 0, \tag{4.4}$$

$$G^\varepsilon \in (\mathcal{C}_c^\infty(Q))^N : G^\varepsilon \to G \text{ in } (L^{p'}(Q))^N \text{ as } \varepsilon \to 0, \tag{4.5}$$

$$g^\varepsilon \in \mathcal{C}_c^\infty(Q) : g^\varepsilon \to g \text{ in } L^p(0,T;W_0^{1,p}(\Omega) \cap L^2(\Omega)) \text{ as } \varepsilon \to 0. \tag{4.6}$$

Let us now consider the following regularized problem:

$$u^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega)), \tag{4.7}$$

$$\int_0^T \left\langle \frac{\partial v^\varepsilon}{\partial t}, \varphi \right\rangle \mathrm{d}t + \int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q f^\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q G^\varepsilon \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \tag{4.8}$$
$$\text{for all } \varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q),$$

$$b(x,u^\varepsilon)(t=0) \ = \ b(x,u_0^\varepsilon) \text{ in } \Omega, \tag{4.9}$$

where $v^\varepsilon := b_\varepsilon(x,u^\varepsilon)$ and $\langle .,. \rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. In view of (4.1), b_ε satisfies (3.1), and due to (3.2) for $\varepsilon > 0$ we have

$$\lambda \le \frac{\partial b_\varepsilon(x,s)}{\partial s} \le \Lambda + 1 \text{ and } |\nabla_x b_\varepsilon(x,s)| \le B(x) \text{ a.e. in } \Omega, \text{ for all } s \in \mathbb{R}. \tag{4.10}$$

As a consequence, proving the existence of a weak solution $u^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ of (4.7)–(4.9) is an easy task (see e.g. [15]).

Step 2. We use $T_k(v^\varepsilon) \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ as a test function in (4.7). Employing the integration by parts formula for the evolution term (see [12]), we get

$$\begin{aligned}\int_\Omega \overline{T_k}(v^\varepsilon)\,\mathrm{d}x &+ \int_0^t \int_\Omega a(x,s,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t \int_\Omega f^\varepsilon T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega G^\varepsilon \nabla T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \overline{T_k}(b_\varepsilon(x,u_0^\varepsilon))\,\mathrm{d}x\end{aligned} \tag{4.11}$$

for almost every $t \in (0,T)$; here $\overline{T_k}(r) = \int_0^r T_k(s)\,\mathrm{d}s$.

Using the assumptions (3.5)–(3.6) and the definition of $\overline{T_k}$ in (4.11), we obtain

$$\begin{aligned}&\int_\Omega \overline{T_k}(b_\varepsilon(x,u^\varepsilon) - g^\varepsilon)\,\mathrm{d}x + \alpha \int_{E_k} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s} |\nabla u^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s \\ &\le k\|f^\varepsilon\|_{L^1(Q)} + \int_{E_k} G^\varepsilon \nabla T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \beta \int_{E_k} L(x,s)|\nabla g^\varepsilon|\,\mathrm{d}x\,\mathrm{d}s + \beta\int_{E_k} |\nabla u^\varepsilon|^{p-1}|\nabla g^\varepsilon|\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_{E_k} |a(x,t,\nabla u^\varepsilon)\nabla_x b_\varepsilon(x,u^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}s + k\|b_\varepsilon(x,u_0^\varepsilon)\|_{L^1(Q)},\end{aligned} \tag{4.12}$$

where $E_k = \{(x,s) : |v^\varepsilon| \le K\}$. Using (4.10), by means of the Young inequality, we obtain

$$\begin{aligned}&\beta \int_{E_k} |\nabla u^\varepsilon|^{p-1}|\nabla g^\varepsilon|\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{\beta}{\lambda}\int_{E_k} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}|\nabla u^\varepsilon|^{p-1}|\nabla g^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{\alpha}{4p'}\int_{E_k} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}|\nabla u^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{p}\Big(\Lambda+1\Big)\Big(\frac{4\beta}{\alpha\lambda}\Big)^{p-1}\int_{E_k}|\nabla g^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\int_{E_k} |a(x,t,\nabla u^\varepsilon)\nabla_x b_\varepsilon(x,u^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \le \frac{\alpha}{4p'}\int_{E_k}\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}|\nabla u^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s + \frac{T}{p}\Big(\Lambda+1\Big)\Big(\frac{4\beta}{\alpha\lambda}\Big)^{p-1}\|B\|^p_{L^p(\Omega)};\end{aligned}$$

we also obtain

$$\begin{aligned}&\int_{E_k}\Big|G^\varepsilon\nabla T_k(v^\varepsilon)\Big|\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \le \frac{\alpha}{2p}\int_{E_k}\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}|\nabla u^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s + C\Big(\|B\|^p_{L^p(\Omega)} + \|G^\varepsilon\|^{p'}_{L^{p'}(Q)} + \|\nabla g^\varepsilon\|^p_{L^p(Q)}\Big),\end{aligned}$$

where C is a constant independent of ε.

Hence

$$\begin{aligned}&\int_\Omega \overline{T_k}(v^\varepsilon)\,\mathrm{d}x + \frac{\alpha}{2}\int_{E_k}\frac{\partial b_\varepsilon(x,s)}{\partial s}|\nabla u^\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \le C\Big(\|f^\varepsilon\|_{L^1(Q)} + \|L\|^{p'}_{L^{p'}(Q)} + \|\nabla g^\varepsilon\|^p_{L^p(Q)} + \|b_\varepsilon(x,u_0^\varepsilon)\|_{L^1(\Omega)} + \\ &\qquad\qquad + \|B\|^p_{L^p(\Omega)} + \|G^\varepsilon\|^{p'}_{L^{p'}(Q)}\Big).\end{aligned} \tag{4.13}$$

In view of the properties of $\overline{T_k}$ ($\overline{T_k} \geq 0$, $\overline{T_k}(s) \geq |s| - 1$ for all $s \in \mathbb{R}$), b_ε, g^ε, G^ε, f^ε, and since $\|b_\varepsilon(x, u_0^\varepsilon)\|_{L^1(\Omega)}$ is bounded, from (4.13) we deduce that

$$v^\varepsilon \text{ is bounded in } L^\infty(0, T; L^1(\Omega)). \tag{4.14}$$

By using (4.10) and (4.13) we deduce that

$$T_k(v^\varepsilon) \text{ is bounded in } L^p(0, T; W_0^{1,p}(\Omega)), \tag{4.15}$$

independently of ε and for any $k \geq 0$.

Proceeding as in [2, 3, 7], for any $S \in W^{1,\infty}(\mathbb{R})$ such that S' has a compact support ($\operatorname{supp}(S') \subset [-k, k]$), we have

$$S(v^\varepsilon) \text{ is bounded in } L^p(0, T; W_0^{1,p}(\Omega)), \tag{4.16}$$

$$\frac{\partial S(v^\varepsilon)}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega)), \tag{4.17}$$

independently of ε.

In fact, as a consequence of (4.15), by Stampacchia's theorem, we obtain (4.16). To show that (4.17) holds true, we multiply the equation (4.7) by $S'(v^\varepsilon)$ and obtain

$$\begin{aligned} &\frac{\partial S(v^\varepsilon)}{\partial t} - \operatorname{div}\big(S'(v^\varepsilon) a(x, t, \nabla u^\varepsilon)\big) + S''(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \\ &\qquad = f^\varepsilon S'(v^\varepsilon) - \operatorname{div}\big(G^\varepsilon S'(v^\varepsilon)\big) + G^\varepsilon S''(v^\varepsilon) \nabla v^\varepsilon \text{ in } \mathcal{D}'(Q). \end{aligned} \tag{4.18}$$

We have

$$\Big|S'(v^\varepsilon) a(x, t, \nabla u^\varepsilon)\Big| \leq \beta \|S'\|_{L^\infty(\mathbb{R})} \Big[L(x,t) + \frac{1}{\lambda^{p-1}} \Big|\nabla T_k(v^\varepsilon) + \nabla g^\varepsilon - \nabla_x b_\varepsilon(x, u^\varepsilon)\Big|^{p-1}\Big]. \tag{4.19}$$

As a consequence, each terms on the right-hand side of (4.18) is bounded either in $L^{p'}(0, T; W^{-1,p'}(\Omega))$ or in $L^1(Q)$, and we then obtain (4.17).

Now we look of an energy estimate of the approximating solutions. For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined by $\theta_n(r) = T_{n+1}(r) - T_n(r)$. Note that $\|\theta_n\|_{L^\infty(\mathbb{R})} \leq 1$ for any $n \geq 1$ and that $\theta_n(r) \to 0$ as $n \to \infty$ for every $r \in \mathbb{R}$. Using $\theta_n(v^\varepsilon)$ as a test function in (4.7), we get

$$\begin{aligned} &\int_\Omega \overline{\theta_n}(v^\varepsilon)\, \mathrm{d}x + \int_Q a(x, t, \nabla u^\varepsilon) \nabla \theta_n(v^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t \\ &\qquad = \int_Q f^\varepsilon \theta_n(v^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t + \int_Q G^\varepsilon \nabla \theta_n(v^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega \overline{\theta_n}(b_\varepsilon(x, u_0^\varepsilon))\, \mathrm{d}x, \end{aligned} \tag{4.20}$$

where $\overline{\theta_n}(r) = \int_0^r \theta_n(s)\, \mathrm{d}s \geq 0$. Hence

$$\begin{aligned} &\int_{\{n \leq |v^\varepsilon| \leq n+1\}} \frac{\partial b_\varepsilon(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon\, \mathrm{d}x\, \mathrm{d}t \\ &\quad \leq \int_{\{n \leq |v^\varepsilon| \leq n+1\}} a(x, t, \nabla u^\varepsilon) \nabla g^\varepsilon\, \mathrm{d}x\, \mathrm{d}t \\ &\qquad - \int_{\{n \leq |v^\varepsilon| \leq n+1\}} a(x, t, \nabla u^\varepsilon) \nabla_x b_\varepsilon(x, u^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t \\ &\qquad + \int_Q f^\varepsilon \theta_n(v^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t + \int_Q G^\varepsilon \nabla \theta_n(v^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega \overline{\theta_n}(b_\varepsilon(x, u_0^\varepsilon))\, \mathrm{d}x. \end{aligned} \tag{4.21}$$

By (3.6), (4.10) and the Young inequality, we obtain

$$\begin{aligned}\int_{\{n\le|v^\varepsilon|\le n+1\}} &\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s} a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\\ &\le C\int_{\{|v^\varepsilon)|\ge n\}}\left(|L(x,t)|^{p'}+|\nabla g^\varepsilon|^p+|G^\varepsilon|^{p'}+|B|^p\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{\{|v^\varepsilon|\ge n\}}|f^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t+\int_{\{|b_\varepsilon(u_0^\varepsilon)|\ge n\}}|b_\varepsilon(x,u_0^\varepsilon)|\,\mathrm{d}x.\end{aligned}\tag{4.22}$$

Step 3. Again, we argue as in [2, 3, 4, 5, 7, 17]. Estimates (4.16) and (4.17) imply that for a subsequence still indexed by ε we have

$$v^\varepsilon \;\to\; v := b(x,u)-g \text{ a.e. in } Q,\tag{4.23}$$

$$u^\varepsilon \;\to\; u \text{ a.e. in } Q,\tag{4.24}$$

$$T_k(v^\varepsilon)\rightharpoonup T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\tag{4.25}$$

$$a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\rightharpoonup \sigma_k \text{ weakly in } (L^{p'}(Q))^N.\tag{4.26}$$

By (3.1), (4.10), (4.24) and the Lebesgue convergence theorem we obtain

$$\nabla_x b_\varepsilon(x,u^\epsilon)\to\nabla_x b(x,u) \text{ strongly in } (L^p(Q))^N,\tag{4.27}$$

as ε tends to zero for any $k>0$ and where for any $k>0$, σ_k belongs to $(L^{p'}(Q))^N$.

Now, we establish that $b(x,u)-g$ belongs to $L^\infty(0,T;L^1(\Omega))$. Indeed, using (4.13) and the fact that $\overline{T_k}(s)\ge|s|-1$, we obtain

$$\begin{aligned}\int_\Omega|v^\varepsilon(t)|\,\mathrm{d}x\le C\Big(&\|f^\varepsilon\|_{L^1(Q)}+\|L\|^{p'}_{L^{p'}(Q)}+\|\nabla g^\varepsilon\|^p_{L^p(Q)}+\|b_\varepsilon(x,u_0^\varepsilon)\|_{L^1(\Omega)}\\ &+\|B\|^p_{L^p(\Omega)}+\|G^\varepsilon\|^{p'}_{L^{p'}(Q)}+\mathrm{meas}(\Omega)\Big)\text{ a.e. in }(0,T),\end{aligned}\tag{4.28}$$

where C is a constant independent of ε. Using (4.1)–(4.6) and (4.23), we deduce that $v:=b(x,u)-g$ belongs to $L^\infty(0,T;L^1(\Omega))$.

We are now in a position to exploit (4.22). Since v^ε is bounded in $L^\infty(0,T;L^1(\Omega))$, we have

$$\lim_{n\to\infty}\Big(\sup_\varepsilon \mathrm{meas}\{|v^\varepsilon|\ge n\}\Big)=0.\tag{4.29}$$

Using the equi-integrability of the sequences $|f^\varepsilon|$, $|b_\varepsilon(x,u_0^\varepsilon)|$, $|\nabla g^\varepsilon|^p$ and $|G^\varepsilon|^{p'}$ in $L^1(Q)$ we deduce that

$$\lim_{n\to\infty}\left(\sup_\varepsilon\int_{\{n\le|v^\varepsilon|\le n+1\}}\frac{\partial b_\varepsilon(x,u)}{\partial s}a(x,s,\nabla u^\varepsilon)\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right)=0.\tag{4.30}$$

Step 4. In this step (in order to perform the monotonicity method which will be developed in Step 5 and Step 6) for a fixed $k\ge0$ we introduce a time regularization of the function $T_k(u)$. This kind of a regularization was first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [14]). More recently, it has been exploited in [9, 13] to solve a few nonlinear evolution problems with L^1 or measure data. This specific time regularization of $T_k(u)$ (for fixed

$k \geq 0$) is defined as follows. Let $(v_0^\zeta)_\mu$ in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ be such that $\|v_0^\zeta\|_{L^\infty(\Omega)} \leq k$ for all $\zeta > 0$, and $v_0^\zeta \to T_k(u_0)$ a.e. in Ω with $\frac{1}{\zeta}\|v_0^\zeta\|_{L^p(\Omega)} \to 0$ as $\zeta \to +\infty$. For fixed $k \geq 0$ and $\zeta > 0$ let us consider the unique solution $T_k(u)_\zeta \in L^\infty(Q) \cap L^p(0, T, W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial T_k(u)_\zeta}{\partial t} + \zeta(T_k(u)_\zeta - T_k(u)) = 0 \text{ in } \mathcal{D}'(Q), \tag{4.31}$$

$$T_k(u)_\zeta(t=0) = v_0^\zeta \text{ in } \Omega. \tag{4.32}$$

The behaviour of $T_k(u)_\zeta$ as $\zeta \to +\infty$ was investigated in [14] (see also [13]) and we just recall here that (4.31) and (4.32) imply that:

$$T_k(u)_\zeta \to T_k(u) \text{ strongly in } L^p(0, T, W_0^{1,p}(\Omega)) \text{ a.e. in } Q \text{ as } \zeta \to +\infty \tag{4.33}$$

with $\|T_k(u)_\zeta\|_{L^\infty(\Omega)} \leq k$ for any ζ, and $\frac{\partial T_k(u)_\zeta}{\partial t} \in L^p(0, T, W_0^{1,p}(\Omega))$.

Let $h \in W^{1,\infty}(\mathbb{R})$ be a non-negative function with a compact support. The main estimate is the following

Lemma 2 *Let $v^\varepsilon = b_\varepsilon(x, u^\varepsilon) - g^\varepsilon$. Then we have*

$$\varliminf_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial v^\varepsilon}{\partial t}, h(v^\varepsilon)\big(T_k(v^\varepsilon) - T_k(v)_\zeta\big) \right\rangle \mathrm{d}t \geq 0.$$

Proof. See Lemma 1 in [4]. □

Step 5. In this step we identify the weak limit σ_k and we prove the weak L^1 convergence of the truncated energy $a(x, t, \nabla u^\varepsilon)\nabla T_k(v^\varepsilon)$ as ε tends to zero.

Lemma 3 *The subsequence of u^ε in Step 3 for any $k \geq 0$ satisfies*:

$$\varlimsup_{\varepsilon\to 0} \int_Q a(x, t, \nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \leq \int_Q \int_\Omega \sigma_k \nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.34}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_Q \frac{\partial b_\varepsilon(x, u^\varepsilon)}{\partial s} \Big[a(x, t, \nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq k\}}) - a(x, t, \nabla u \chi_{\{|v| \leq k\}})\Big] \times \\ \times \Big[\nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq k\}} - \nabla u \chi_{\{|v| \leq k\}}\Big]\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned} \tag{4.35}$$

Moreover, for a fixed $k \geq 0$, we have

$$\sigma_k = a(x, t, \nabla u)\chi_{\{|v| \leq k\}} \textit{ a.e. in } Q, \tag{4.36}$$

$$a(x, t, \nabla u^\varepsilon)\nabla T_k(v^\varepsilon) \rightharpoonup a(x, t, \nabla u)\nabla T_k(v) \textit{ weakly in } L^1(Q) \tag{4.37}$$

as ε tends to 0.

Proof. First, we prove that (4.34) holds true. For a fixed $k \geq 0$ let $W_\zeta^\varepsilon = (T_k(v^\varepsilon) - T_k(v)_\zeta)$. Let us introduce a sequence of increasing $\mathcal{C}^\infty(\mathbb{R})$-functions S_n such that

$$S_n(r) = r \text{ for } |r| \leq n,\ \operatorname{supp}(S_n') \subset [-(n+1), n+1],\ \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1, \text{ for any } n \geq 1.$$

We choose $S'_n(v^\varepsilon)W^\varepsilon_\zeta$ as test function in (4.7) and obtain

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial v^\varepsilon}{\partial t}, S'_n(v^\varepsilon)W^\varepsilon_\zeta \Big\rangle \,\mathrm{d}t + \int_Q S'_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon)\nabla W^\varepsilon_\zeta \,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''_n(v^\varepsilon) W^\varepsilon_\zeta a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_Q f^\varepsilon S'_n(v^\varepsilon) W^\varepsilon_\zeta \,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q G^\varepsilon S'_n(v^\varepsilon)\nabla W^\varepsilon_\zeta \,\mathrm{d}x\,\mathrm{d}t + \int_Q S''_n(v^\varepsilon) W^\varepsilon_\zeta G^\varepsilon \nabla v^\varepsilon \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.38}$$

In the following we pass to the limit in (4.38) as ε tends to 0, then ζ tends to ∞ and then n tends to ∞, the real number $k \geq 0$ being kept fixed. In order to perform this task we prove bellow the following results for fixed $k \geq 0$:

$$\varliminf_{\zeta\to\infty} \lim_{\varepsilon\to 0} \int_0^T \Big\langle \frac{\partial v^\varepsilon}{\partial t}, S'_n(v^\varepsilon)W^\varepsilon_\zeta \Big\rangle \,\mathrm{d}t \geq 0, \tag{4.39}$$

for any $n \geq k$

$$\lim_{n\to\infty}\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0} \int_Q S''_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon) W^\varepsilon_\zeta \nabla v^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.40}$$

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0} \int_Q f^\varepsilon S'_n(v^\varepsilon) W^\varepsilon_\zeta \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.41}$$

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0} \int_Q G^\varepsilon S'_n(v^\varepsilon)\nabla W^\varepsilon_\zeta \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.42}$$

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0} \int_Q S''_n(v^\varepsilon) W^\varepsilon_\zeta G^\varepsilon \nabla v^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.43}$$

Let us prove (4.39). In view of the definition of W^ε_ζ, Lemma 3 applies with $h = S'_n$ for fixed $n \geq k$. As a consequence, (4.39) holds true.

Now, we pass to the proof of (4.40). For any $n \geq 1$ and any $\zeta > 0$, we have $\operatorname{supp}(S''_n) \subset [-(n+1), -n] \cup [n, n+1]$, $\|W^\varepsilon_\zeta\|_{L^\infty(Q)} \leq 2k$ and $\|S''_n\|_{L^\infty(\mathbb{R})} \leq 1$. As a consequence,

$$\begin{aligned}&\left|\int_Q S''_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon) W^\varepsilon_\zeta \nabla v^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| \\ &\quad\leq 2k \int_{\{n\leq |v^\varepsilon|\leq n+1\}} |a(x,t,\nabla u^\varepsilon)\nabla_x b_\varepsilon(x,u^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + 2k \int_{\{n\leq |v^\varepsilon|\leq n+1\}} |a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon| \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + 2k \int_{\{n\leq |v^\varepsilon|\leq n+1\}} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}\, a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By the assumptions (3.5), (3.6), (4.10) and Young's inequality we obtain

$$\begin{aligned}&\int_{\{n\leq |v^\varepsilon|\leq n+1\}} |a(x,t,\nabla u^\varepsilon)\nabla_x b_\varepsilon(x,u^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leq \beta \int_{\{n\leq |v^\varepsilon|\leq n+1\}} \Big(|\nabla u^\varepsilon|^{p-1} |\nabla_x b_\varepsilon(x,u^\varepsilon)| + L(x,t)|\nabla_x b_\varepsilon(x,u^\varepsilon)| \Big) \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}&\leq \frac{\beta}{\lambda\alpha p'}\int_{\{n\leq|v^\varepsilon|\leq n+1\}}\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}\,a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\&\quad+C\int_{\{|v^\varepsilon|\geq n\}}\left(|B|^p+|L|^{p'}\right)\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}&\int_{\{n\leq|v^\varepsilon|\leq n+1\}}|a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leq C\int_{\{n\leq|v^\varepsilon|\leq n+1\}}\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}\,a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\&\qquad+C\int_{\{|v^\varepsilon|\geq n\}}\left(|\nabla g^\varepsilon|^p+|L|^{p'}\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\int_Q S_n''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)W_\zeta^\varepsilon\nabla v^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right|\\&\quad\leq C\int_{\{n\leq|v^\varepsilon|\leq n+1\}}\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\&\qquad+C\int_{\{|v^\varepsilon|\geq n\}}\left(|B|^p+|L|^{p'}+|\nabla g^\varepsilon|^p\right)\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{4.44}$$

for any $n\geq 1$, where C is a constant independent of n. Using the assumptions (4.29)–(4.30) and the equi-integrability of the sequence $|\nabla g^\varepsilon|^p$ in $L^1(Q)$, we are able to pass to the limits in (4.44) as n tends to ∞ and to establish (4.40).

Now, we prove (4.41). For a fixed $n\geq 1$, in view (4.4) and (4.23), the Lebesgue convergence theorem implies that for any $\zeta>0$ and $n\geq 1$ we have

$$\lim_{\varepsilon\to 0}\int_Q f^\varepsilon S_n'(v^\varepsilon)W_\zeta^\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_Q fS_n'(v)W_\zeta\,\mathrm{d}x\,\mathrm{d}t.$$

Using (4.33) we are able to pass to the limit in the above equality as ζ tends to ∞ to obtain (4.41).

Let us now prove (4.42). Using (4.5) and (4.23) we see that to $S_n'(v^\varepsilon)G^\varepsilon$ tends to $S_n'(v)G$ strongly in $(L^{p'}(Q))^N$ as ε tends to 0. For a fixed $\zeta>0$, we have that W_ζ^ε tends to $T_k(v)-T_k(v)_\zeta$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, and a.e. in Q as ε tends to 0. So we deduce that

$$\lim_{\varepsilon\to 0}\int_Q G^\varepsilon S_n'(v^\varepsilon)\nabla W_\zeta^\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_Q GS_n'(v)\nabla(T_k(v)-T_k(v)_\zeta)\,\mathrm{d}x\,\mathrm{d}t\tag{4.45}$$

for any $\zeta>0$. Appealing now to (4.33) and passing to the limit in (4.45) as $\zeta\to\infty$ allows to conclude that (4.42) holds true.

Finally, we prove (4.43). From (4.5) and (4.25) it follows that

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0}\int_Q\nabla S_n'(v^\varepsilon)W_\zeta^\varepsilon G^\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\lim_{\zeta\to\infty}\int_Q\nabla S_n'(v)W_\zeta G\,\mathrm{d}x\,\mathrm{d}t=0$$

for any $n\geq 1$.

Now, we turn back to the proof of Lemma 3. Due to (4.39)–(4.43), we are in a position to pass in (4.38) to the limit superior when ε tends to zero, then to the limit superior when ζ tends to ∞ and then to the limit as n tends to ∞. Thus, for any $k \geq 0$, we obtain

$$\lim_{n\to\infty} \overline{\lim_{\zeta\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_Q S_n'(v^\varepsilon) a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon) - T_k(v)_\zeta)\,\mathrm{d}x\,\mathrm{d}t \leq 0.$$

Since $S_n'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon) = a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)$ for $k \leq n$, the above inequality implies that for $k \leq n$ we get

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}} \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ \leq \lim_{n\to\infty} \overline{\lim_{\zeta\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_Q S_n'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla T_k(v)_\zeta\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.46}$$

Due to (4.23) and (4.26), we see that $S_n'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)$ converges to $S_n'(v)\sigma_{n+1}$ weakly in $(L^{p'}(Q))^N$ as ε tends to zero. The strong convergence of $T_k(v)_\zeta$ to $T_k(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ as ζ tends to ∞ allows then to conclude that for all $k \leq n$ we have

$$\begin{aligned}\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0}\int_Q S_n'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla T_k(v)_\zeta\,\mathrm{d}x\,\mathrm{d}t \\ = \int_Q S_n'(v)\sigma_{n+1}\nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \sigma_{n+1}\nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.47}$$

Now for $k \leq n$, we have

$$S_n'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq k\}} = a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq k\}} \text{ a.e. in } Q.$$

Letting ε tend to 0, we obtain $\sigma_{n+1}\chi_{\{|v|\leq k\}} = \sigma_k\chi_{\{|v|\leq k\}}$ a.e. in $Q\backslash\{|v| = k\}$ for $k \leq n$. Then for $k \leq n$ we have $\sigma_{n+1}\nabla T_k(v) = \sigma_k\nabla T_k(v)$ a.e. in Q. Recalling (4.46) and (4.47) allows to conclude that (4.34) holds true.

Now, we are going to prove (4.35). Let $k \geq 0$ be fixed. We use (4.11) and the monotone character (3.7) of $a(x,t,\xi)$ with respect to ξ to obtain

$$\begin{aligned}A^\varepsilon = \int_Q \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}\Big(a(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}) - a(x,t,\nabla u\chi_{\{|v|\leq k\}})\Big)\times \\ \times\Big(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} - \nabla u\chi_{\{|v|\leq k\}}\Big)\,\mathrm{d}x\,\mathrm{d}t \geq 0.\end{aligned} \tag{4.48}$$

The left-hand side of the inequality (4.48) can be split as follows: $A^\varepsilon = A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon$, where:

$$A_1^\varepsilon = \int_Q \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}\,\mathrm{d}x\,\mathrm{d}t,$$

$$A_2^\varepsilon = -\int_Q \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u^\varepsilon)\nabla u\chi_{\{|v^\varepsilon|\leq k\}}\chi_{\{|v|\leq k\}}\,\mathrm{d}x\,\mathrm{d}t,$$

$$A_3^\varepsilon = -\int_Q \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}a(x,t,\nabla u)(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}) - \nabla u\chi_{\{|v|\leq k\}}))\,\mathrm{d}x\,\mathrm{d}t.$$

We pass in A_1^ε, A_2^ε and A_3^ε to the limit superior as ε tends to 0. Let us remark that we have $v^\varepsilon = b_\varepsilon(x,u^\varepsilon) - g^\varepsilon$ and $\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} = \nabla T_k(v^\varepsilon) - (\nabla_x b_\varepsilon(x,u^\varepsilon) + \nabla g^\varepsilon)\chi_{\{|v^\varepsilon|\leq k\}}$ a.e.

in Q, and we also have that $\chi_{\{|v^\varepsilon|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ for almost every k (see [9]). Using (4.34), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} A_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad + \lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad - \lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\nabla_x b_\varepsilon(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_Q \sigma_k\nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t - \int_Q \sigma_k\nabla_x b(x,u)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \sigma_k\nabla g\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{4.49}$$

As a consequence of (4.6) and (4.25)–(4.27) we obtain

$$\lim_{\varepsilon\to 0} A_2^\varepsilon = -\int_Q \sigma_k(\nabla T_k(v) - \nabla_x b(x,u) + \nabla g)\,\mathrm{d}x\,\mathrm{d}t.\tag{4.50}$$

In view of (4.6), (4.25) and (4.27) we have

$$\begin{aligned}\lim_{\varepsilon\to 0} A_3^\varepsilon = -\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u)\Big(&\nabla T_k(v^\varepsilon) - (\nabla_x b_\varepsilon(x,u^\varepsilon) + \nabla g^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\\ &- \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s}\Big(\frac{\partial b(x,u)}{\partial s}\Big)^{-1}\Big(\nabla T_k(v) - (\nabla_x b(x,u) + \nabla g)\chi_{\{|v|\le k\}}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}\tag{4.51}$$

Taking the limit superior in (4.48) as ε tends to 0 and using (4.49), (4.50) and (4.51), we conclude that (4.35) holds true.

Finally, we sketch the idea of the proof of (4.36)–(4.37). Using (4.35) and the usual Minty argument we see that (4.36)–(4.37) holds true. □

Step 6. In this step we prove that u satisfies (3.10) and (3.11). To this end, for any fixed $n\ge 1$, we have

$$\begin{aligned}&\int_{\{n\le|v^\varepsilon|\le n+1\}} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s} a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_n(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n+1\}}\nabla g^\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n\}}\nabla g^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n+1\}}\nabla_x b_\varepsilon(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n\}}\nabla_x b_\varepsilon(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

According to (4.26), (4.27), (4.36) and (4.37) for a fixed $n\ge 0$ we can pass to the limit in the above equality as ε tends to 0 and obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\{n\le |v^\varepsilon|\le n+1\}} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s} a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q a(x,t,\nabla u)\nabla T_{n+1}(v)\,\mathrm{d}x\,\mathrm{d}t - \int_Q a(x,t,\nabla u)\nabla T_n(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q a(x,t,\nabla u)\chi_{\{|v|\le n+1\}}\nabla g\,\mathrm{d}x\,\mathrm{d}t - \int_Q a(x,t,\nabla u)\chi_{\{|v|\le n\}}\nabla g\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_Q a(x,t,\nabla u)\chi_{\{|v|\le n+1\}}\nabla_x b(x,u)\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(x,t,\nabla u)\chi_{\{|v|\le n\}}\nabla_x b(x,u)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\{n\le |v|\le n+1\}} \frac{\partial b(x,u)}{\partial s} a(x,t,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.52}
$$

Taking the limit in (4.52) as n tends to ∞ and using the estimate (4.30), in view of (3.2), we deduce that u satisfies (3.11).

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support and let k be a positive real number such that $\operatorname{supp}(S') \subset [-k,k]$. By multiplying the approximate equation (4.7) by $S'(v^\varepsilon)$, we easily see that

$$
\begin{aligned}
\frac{\partial S(v^\varepsilon)}{\partial t} - \operatorname{div}\big(S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\big) + S''(v^\varepsilon)a(x,t,\nabla u)\nabla v^\varepsilon& \\
= f^\varepsilon S'(v^\varepsilon) - \operatorname{div}\big(G^\varepsilon S'(v^\varepsilon)\big) + G^\varepsilon S''(v^\varepsilon)\nabla v^\varepsilon \ &\text{ in } \mathcal{D}'(Q).
\end{aligned}
\tag{4.53}
$$

In what follows, we pass with ε tending to 0 in each term of (4.53). Since S is bounded and $S(v^\varepsilon)$ converges to $S(v)$ a.e. in Q and in $L^\infty(Q)$ *-weak, $\frac{\partial S(v^\varepsilon)}{\partial t}$ converges to $\frac{\partial S(v)}{\partial t}$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\operatorname{supp}(S') \subset [-k,k]$, we have $S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon) = S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}$ a.e. in Q. The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (4.36) of Lemma 3 imply that $S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)$ converges to $S'(v)a(x,t,\nabla u)$ weakly in $(L^{p'}(Q))^N$ as ε tends to 0. The pointwise convergence of v^ε to v, the bounded character of S'' and (4.37) of Lemma 3 allow us to conclude that $S''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)$ converges to $S''(v)a(x,t,\nabla u)\nabla T_k(v)$ weakly in $L^1(Q)$ as ε tends to 0. We use (4.4), (4.5), (4.6), (4.23) and (4.25) to deduce that $f^\varepsilon S'(v^\varepsilon)$ converges to $fS'(v)$ strongly in $L^1(Q)$, the term $G^\varepsilon S'(v^\varepsilon)$ converges to $GS'(v)$ strongly in $(L^{p'}(Q))^N$ and $G^\varepsilon S''(v^\varepsilon)\nabla v^\varepsilon$ converges to $GS''(v)\nabla v$ weakly in $L^1(Q)$.

As a consequence of the above convergence result, we are in a position to pass to the limit in the equation (4.53) as ε tends to 0 and to conclude that u satisfies (3.12).

It remains to show that $S(v)$ satisfies the initial condition (3.13). To this end, firstly note that $S(v^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (4.53) and the above considerations on the behaviour of the terms of this equation show that $\frac{\partial S(v^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type lemma (see e.g. Corollary 4 in [25]) implies that $S(v^\varepsilon)$ lies in a compact set of $\mathcal{C}([0,T];W^{-1,s}(\Omega))$ for any $s < \inf(p', \frac{N}{N-1})$. It follows that, on one hand, $S(v^\varepsilon)(t=0)$ converges to $S(v)(t=0)$ strongly in $W^{-1,s}(\Omega)$, and on the other hand, the smoothness of S implies that $S(v^\varepsilon)(t=0)$ converges to $S(b(x,u))(t=0)$ strongly in $L^q(\Omega)$ for all $q<\infty$. Due to (4.2), we conclude that $S(v^\varepsilon)(t=0) = S(b_\varepsilon(x,u_0^\varepsilon))$ converges to $S(b(x,u)(t=0)$ strongly in $L^q(\Omega)$. Then we conclude that $S(v)(t=0) = S(b(x,u_0))$ in Ω.

As a conclusion of Step 3 and Step 6, the proof of Theorem 2 is complete. □

5 Appendix

We prove Proposition 2.

Proof. (Sketch) Assume that u satisfies Definition 1 for $(f, \mathrm{div}(G), g)$ and let $(\overline{f}, \mathrm{div}(\overline{G}), \overline{g})$ be a decomposition of μ. Note that by Lemma 1, since $g - \overline{g} \in C([0,T]; L^1(\Omega))$, we infer that $\overline{v} := b(x,u) - \overline{g} \in L^\infty(0,T; L^1(\Omega))$ and $\overline{v}$ is also almost everywhere finite. We prove that $T_K(\overline{v}) \in L^p(0,T; W_0^{1,p}(\Omega))$ for every $K > 0$. We can reason as in the proof of Proposition 3.10 in [23]. We use the definition of S_n' (see the proof of Lemma 3) and we choose as a test function $T_K(S_n(v) + g - \overline{g})$ in (3.11). Using Lemma 1 we obtain

$$I_1 + I_2 = I_3 + I_4 + I_5 + I_6, \tag{5.1}$$

where

$$\begin{aligned}
I_1 &= \int_0^T \left\langle \frac{\partial(S_n(v) + g - \overline{g})}{\partial t}, T_K(S_n(v) + g - \overline{g}) \right\rangle \mathrm{d}t,\\
I_2 &= \int_Q S_n'(v) a(x,t,\nabla u) \nabla T_K(S_n(v) + g - \overline{g}) \,\mathrm{d}x\,\mathrm{d}t,\\
I_3 &= -\int_Q S_n''(v) a(x,t,\nabla u) \nabla v T_K(S_n(v) + g - \overline{g}) \,\mathrm{d}x\,\mathrm{d}t,\\
I_4 &= \int_Q ((S_n'(v) - 1) f + \overline{f}) T_K(S_n(v) + g - \overline{g}) \,\mathrm{d}x\,\mathrm{d}t,\\
I_5 &= \int_Q ((S_n'(v) - 1) G + \overline{G}) \nabla T_K(S_n(v) + g - \overline{g}) \,\mathrm{d}x\,\mathrm{d}t,\\
I_6 &= \int_Q (S_n''(v) G \nabla(v) T_K(S_n(v) + g - \overline{g}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We use the integration by parts formula (see for example [12]), the initial condition (3.12) and Lemma 1 to obtain

$$I_1 = \int_\Omega \overline{T_K}(S_n(v) + g - \overline{g})(T) \,\mathrm{d}x - \int_\Omega \overline{T_K}(S_n(b(x,u_0))) \,\mathrm{d}x, \tag{5.2}$$

where $\overline{T_K}(r) = \int_0^r T_K(s)\,\mathrm{d}s$ is a positive Lipschitz continuous function. Using (5.2) and the definition of S_n, we obtain

$$I_1 \geq -K \int_\Omega |b(x,u_0)| \,\mathrm{d}x \quad \text{for all } n \geq 1. \tag{5.3}$$

Let $E_K = \big\{(x,t) : |S_n(v) + g - \overline{g}| \leq K\big\}$. Then we have

$$\begin{aligned}
I_2 &= \int_{E_K} |S_n'(v)|^2 a(x,t,\nabla u) \left[\nabla_x b(x,u) + \frac{\partial b(x,u)}{\partial s} \nabla u\right] \mathrm{d}x\,\mathrm{d}t\\
&\qquad - \int_{E_K} |S_n'(v)|^2 a(x,t,\nabla u) \nabla g \,\mathrm{d}x\,\mathrm{d}t + \int_{E_K} S_n'(v) a(x,t,\nabla u) \nabla(g - \overline{g}) \,\mathrm{d}x\,\mathrm{d}t\\
&\equiv I_{21} + I_{22} + I_{23}.
\end{aligned} \tag{5.4}$$

Using (3.1) and the properties of S_n: $(S'_n(s))^p \le S'_n(s)$, $(S'_n(s))^{p'} \le S'_n(s)$, $S'_n(s) \le S'_n(s)^2 + \chi_{\{n\le|s|\le n+1\}}$, we obtain

$$\begin{aligned} I_{21} \ge \frac{\alpha}{\lambda^{p-1}} &\int_{E_K} |S'_n(v)|^p |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_{\{n\le|v|\le n+1\}} \frac{\partial b(x,u)}{\partial s} a(x,t,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.5}$$

Using (3.1), (3.4), (3.5) and Young's inequality, we deduce that

$$\begin{aligned} |I_{22}| + |I_{23}| \le C&\Big(\|L\|_{L^{p'}}^{p'} + \|\nabla g\|_{L^p}^p + \|\nabla(g-\overline{g})\|_{L^p}^p\Big) \\ &+ C\int_{\{n\le|v|\le n+1\}} b'(u) a(x,t,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t \\ &+ \frac{\alpha}{4\,\lambda^{p-1}} \int_{E_K} |S'_n(v)|^p |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} |I_3| + |I_6| \le C&\Big(\|L\|_{L^{p'}}^{p'} + \|\nabla g\|_{L^p}^p + \|G\|_{L^{p'}}^{p'}\Big) \\ &+ C\int_{\{n\le|v|\le n+1\}} a(x,t,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.7}$$

Using (3.4) and Young's inequality, we obtain

$$\begin{aligned} |I_4| + |I_5| \le C&\Big(\|f\|_{L^1} + \|\overline{f}\|_{L^1} + \|G\|_{L^{p'}}^{p'} + \|\overline{G}\|_{L^{p'}}^{p'} + \|\nabla g\|_{L^p}^p + \|\nabla \overline{g}\|_{L^p}^p\Big) \\ &+ C\int_{\{n\le|v|\le n+1\}} a(x,t,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t \\ &+ \frac{\alpha}{4\,\lambda^{p-1}} \int_{E_K} |S'_n(v)|^p |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.8}$$

Using (5.1) to (5.8), we deduce that

$$\begin{aligned} &\frac{\alpha}{4\,\lambda^{p-1}} \int_{E_K} |S'_n(v)|^p |\nabla b(x,u)|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le \frac{C\alpha}{4\lambda^{p-1}} \int_{E_K} |S'_n(v)|^p |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t + \frac{C\alpha}{4\lambda^{p-1}} \int_{E_K} |S'_n(v)|^p |B|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le C\int_{\{n\le|v|\le n+1\}} a(x,t,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + C\big(\|f\|_{L^1(Q)} + \|\overline{f}\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)} + \|G\|_{L^{p'}}^{p'} + \|\overline{G}\|_{L^{p'}}^{p'} + \|\nabla g\|_{L^p}^p + \|\nabla \overline{g}\|_{L^p}^p\big). \end{aligned} \tag{5.9}$$

Using the properties of S_n and the fact that g belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce that $\int_Q \chi_{E_n} |\nabla S_n(v)|^p \,\mathrm{d}x\,\mathrm{d}t \le C$ for all $n \ge 1$. And then, since $g, \overline{g}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we have $\int_Q |\nabla T_K(S_n(v) + g - \overline{g}|^p \,\mathrm{d}x\,\mathrm{d}t \le C$ for all $n \ge 1$. It follows that $T_K(S_n(v) + g - \overline{g})$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and converges to $T_K(\overline{v})$ a.e. in Q and weakly in $L^p(0,T;W_0^{1,p}(\Omega))$. Then $T_K(\overline{v}) \in L^p(0,T;W_0^{1,p}(\Omega))$.

Now, we prove that (3.11) holds true for $\overline{g}$. Using the admissible test function $\theta_h(S_n(v)+g-\overline{g})$ in (3.12) with $S=S_n$, $\theta_h(s)=T_{h+1}(s)-T_h(s)$, the coercive character (3.5), the properties (3.2) of b and the Young inequality, we are able to deduce that

$$\begin{aligned}
&\lambda\int_{F_n}|S_n'(v)|^2 a(x,u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le C\int_Q(|f|+|\overline{f}|)|\theta_h(S_n(v)+g-\overline{g})|\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\overline{\theta_h}(S_n(b(x,u_0)))\,\mathrm{d}x\\
&\quad\quad+C\int_{F_n}(|G|^{p'}+|\overline{G}|^{p'}+|\nabla g|^p+|\nabla\overline{g}|^p+|L(x,t)|^{p'})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad+C\Lambda\int_{\{n\le|v|\le n+1\}}a(x,t,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t+w(n),
\end{aligned}\tag{5.10}$$

where $F_n=\big\{h\le|S_n(v)+g-\overline{g}|\le h+1\big\}$. Taking the limit in (5.10) as n tends to $+\infty$, using (3.11) and the convergence of χ_{F_n} to $\chi_{\{h\le|v|\le h+1\}}$, we can show that for any $h>0$.

$$\begin{aligned}
&\int_{\{h\le|\overline{v}|\le h+1\}}a(x,t,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le C\int_{\{|v|>h\}}(|f|+|\overline{f}|)\,\mathrm{d}t\,\mathrm{d}x+C\int_{\{|b(x,u_0)|>h\}}|b(x,u_0)|\,\mathrm{d}x\\
&\quad\quad+C\int_{\{h\le|\overline{v}|\le h+1\}}\Big[|f|+|\overline{f}|+|G|^{p'}+|\overline{G}|^{p'}+|\nabla g|^p+|\nabla\overline{g}|^p+|L|^{p'}\Big]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{5.11}$$

Note that v is almost everywhere finite, and so passing to the limit in (5.11) as h tends to $+\infty$ yields

$$\lim_{h\to+\infty}\int_{\{h\le|\overline{v}|\le h+1\}}a(x,t,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t=0.\tag{5.12}$$

In the following we prove that the renormalized equation (3.12) and the initial condition (3.13) hold with $\overline{g}$ as well.

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let φ be a function in $C_c^\infty(Q)$. We choose $S'(S_n(v)+g+\overline{g})\varphi$ as a test function in (3.12) and we have

$$\begin{aligned}
&\int_0^T\Big\langle\frac{\partial(S_n(v)+g-\overline{g})}{\partial t},S'(S_n(v)+g+\overline{g})\varphi\Big\rangle\,\mathrm{d}t\\
&\quad+\int_Q S_n'(v)a(x,t,\nabla u)\nabla S'(S_n(v)+g+\overline{g})\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q S_n'(v)a(x,t,\nabla u)\nabla\varphi S'(S_n(v)+g+\overline{g})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q S_n''(v)a(x,t,\nabla u)\nabla(v)S'(S_n(v)+g+\overline{g})\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_Q((S_n'(v)-1)f+\overline{f})S'(S_n(v)+g+\overline{g})\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q((S_n'(v)-1)G+\overline{G})\nabla\varphi S'(S_n(v)+g+\overline{g})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q((S_n'(v)-1)G+\overline{G})\nabla S'(S_n(v)+g+\overline{g})\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q S_n''(v)G\nabla(\overline{v})S'(S_n(v)+g+\overline{g})\varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{5.13}$$

In what follows we pass to the limit in each term of (5.13) as n tends to 0. For the parabolic contribution in (5.13), we write

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial(S_n(v)+g-\overline{g})}{\partial t}, S'(S_n(v)+g+\overline{g})\varphi \right\rangle \mathrm{d}t \\
&\quad= \int_0^T \left\langle \frac{\partial S((S_n(v)+g-\overline{g}))}{\partial t}, \varphi \right\rangle \mathrm{d}t = -\int_Q S((S_n(v)+g-\overline{g}))\varphi_t \\
&\quad= \int_0^T \left\langle \frac{\partial S(\overline{v})}{\partial t}, \varphi \right\rangle \mathrm{d}t + w(n).
\end{aligned} \tag{5.14}$$

Recall that, since $\operatorname{supp}(S') \subset [-K, K]$ and

$$\operatorname{supp}\big(S_n'(v)S'(S_n(v)+g-\overline{g})\big) \subset \big\{|v| \le n+1 \ : \ |\overline{v}| \le K+1\big\},$$

∇u may be replaced by $w \equiv \left(\frac{\partial b(x,s)}{\partial s}\right)^{-1}\Big[\nabla(T_{K+1}(\overline{v}) + \overline{g} - \nabla_x b(x,u)\Big]$ in all the terms of (5.13). Using the definition of S_n, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} \int_Q S_n'(v) a(x,t,\nabla u)\nabla S'(S_n(v)+g-\overline{g})\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \lim_{n\to+\infty} \int_Q S_n'(v) a(x,t,w)\nabla S'(S_n(v)+g-\overline{g})\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_Q a(x,t,w)\nabla S'(\overline{v})\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q a(x,t,\nabla u)\nabla S'(\overline{v})\varphi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
&\lim_{n\to+\infty} \int_Q S_n'(v) a(x,t,\nabla u)\nabla\varphi S'(S_n(v)+g-\overline{g}) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \lim_{n\to+\infty} \int_Q S_n'(v) a(x,t,w)\nabla\varphi S'(S_n(v)+g-\overline{g}) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_Q a(x,t,w)\nabla\varphi S'(\overline{v}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q a(x,t,\nabla u)\nabla\varphi S'(\overline{v}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.16}$$

The definition of S_n', ($S_n'' \to 0$) allows us to deduce that

$$\begin{aligned}
&\lim_{n\to+\infty} \int_Q S_n''(v) a(x,t,\nabla u)\nabla v S'(S_n(v)+g-\overline{g})\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \lim_{n\to+\infty} \int_Q S_n''(v) a(x,t,w)\nabla(T_{K+1}(\overline{v})+\overline{g}-g) S'(S_n(v)+g-\overline{g})\varphi \,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned} \tag{5.17}$$

Repeating the arguments that lead to (5.15), (5.16) and (5.17), we obtain

$$\lim_{n\to+\infty} \int_Q ((S_n'(v)-1)f + \overline{f}) S'(S_n(v)+g+\overline{g})\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q \overline{f} S'(\overline{v})\varphi \,\mathrm{d}x\,\mathrm{d}t, \tag{5.18}$$

$$\lim_{n\to+\infty} \int_Q ((S_n'(v)-1)G + \overline{G})\nabla\varphi S'(S_n(v)+g+\overline{g}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \overline{G}\nabla\varphi S'(\overline{v}) \,\mathrm{d}x\,\mathrm{d}t, \tag{5.19}$$

$$\lim_{n\to+\infty} \int_Q ((S_n'(v)-1)G + \overline{G})\nabla S'(S_n(v)+g+\overline{g})\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q \overline{G}\nabla S'(\overline{v})\varphi \,\mathrm{d}x\,\mathrm{d}t, \tag{5.20}$$

$$\lim_{n\to+\infty}\int_Q S_n''(v)G\nabla\overline{v}S'(S_n(v)+g+\overline{g})\varphi\,\mathrm{d}x\,\mathrm{d}t=0. \tag{5.21}$$

As a consequence of the above convergence results, we are in a position to pass to the limit in (5.13) as n tends to $+\infty$ and to conclude that u satisfies (3.12) (with $\overline{g}$ instead of g).

It remains to show that $S(\overline{v})$ satisfies the initial condition (3.13). To this end, for $\psi\in C_0^\infty(\Omega)$ we take $\varphi=(T-t)\psi$ in (5.13); it possible to obtain

$$\begin{aligned}
\lim_{n\to+\infty}\int_0^T\Big\langle\frac{\partial(S_n(v)+g-\overline{g})}{\partial t},S'(S_n(v)+g+\overline{g})\varphi\Big\rangle\,\mathrm{d}t&\\
+\int_Q S'(\overline{v})a(x,t,\nabla u)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q a(x,t,\nabla u)\nabla S'(\overline{v})\varphi\,\mathrm{d}x\,\mathrm{d}t&\\
=\int_Q\overline{f}S'(\overline{v})\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\overline{G}\nabla\varphi S'(\overline{v})\,\mathrm{d}x\,\mathrm{d}t+\int_Q G\nabla S'(\overline{v})\varphi\,\mathrm{d}x\,\mathrm{d}t.&
\end{aligned} \tag{5.22}$$

Employing the integration-by-parts formula for the evolution term, using $S_n(v)(t=0)=S_n(b(x,u_0))$ and $(g-\overline{g})(0)=0$, we get

$$\begin{aligned}
&\int_0^T\Big\langle\frac{\partial(S_n(v)+g-\overline{g})}{\partial t},S'(S_n(v)+g+\overline{g})\varphi\Big\rangle\,\mathrm{d}t\\
&\quad=\int_0^T\Big\langle\frac{\partial S(S_n(v)+g-\overline{g})}{\partial t},\varphi\Big\rangle\,\mathrm{d}t\\
&\quad=-\int_\Omega S(S_n(b(u_0)))\varphi(0)\,\mathrm{d}x-\int_Q S(S_n(v)+g-\overline{g})\varphi_t\,\mathrm{d}x\\
&\quad=-\int_\Omega S(b(x,u_0))\varphi(0)\,\mathrm{d}x-\int_Q S(\overline{v})\varphi_t\,\mathrm{d}x+w(n).
\end{aligned} \tag{5.23}$$

Secondly, we use φ as a test function in (3.12) (with $\overline{g}$). This leads to

$$\begin{aligned}
&-\int_\Omega S(\overline{v})(0)\,\mathrm{d}x-\int_Q S(\overline{v})\varphi_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q S'(\overline{v})a(x,t,\nabla u)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q a(x,t,\nabla u)\nabla S'(\overline{v})\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\int_Q\overline{f}S'(\overline{v})\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\overline{G}\nabla\varphi S'(\overline{v})\,\mathrm{d}x\,\mathrm{d}t+\int_Q G\nabla S'(\overline{v})\varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.24}$$

From (5.22), (5.23) and (5.24) we conclude that $\int_\Omega S(\overline{v})(0)\psi\,\mathrm{d}x=\int_\Omega S(b(x,u_0))\psi\,\mathrm{d}x$ for all $\psi\in C_0^\infty(\Omega)$, and so $S(\overline{v})(t=0)=S(b(x,u_0))$ in Ω. The proof of Proposition 2 is complete. □

Remark 2 *Let us mention that the question of the uniqueness of a renormalized solution for* (1.1)–(1.3) *still remains open. Note that some recent results in this directions which may be useful to show the uniqueness of a renormalized solution for* (1.1)–(1.3) *are contained in* [8] *and* [24].

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