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QUADRATIC MEAN ALMOST PERIODIC MILD SOLUTIONS TO A FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATION IN HILBERT SPACES

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Abstract. In this paper, we prove the existence and uniqueness of quadratic mean almost periodic mild solutions for a class of fractional stochastic differential equations of Sobolev type in a real separable Hilbert space. To establish our main results, we use the Banach contraction mapping principle, fractional calculus, stochastic analysis and an analytic semigroup of linear operators. An example is given to illustrate the theory.

Keywords: Analytic semigroups of linear operators, stochastic fractional differential equations, quadratic mean almost periodic.

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1 Introduction

Stochastic differential equations have attracted great interest due to their applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations (see for instance [6, 16, 18, 20, 23, 24, 30, 31, 32, 33, 34, 35] and the references therein).

The existence of periodic and almost periodic solutions for stochastic differential equations was obtained. We refer the reader to [2, 5, 7, 15] and references therein.

On the other hand, recently fractional differential equations have found numerous applications in various fields of science and engineering [1, 4, 9, 10, 14, 17, 25, 26, 27].

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The qualitative properties of stochastic fractional differential equations have been considered only in few publications. El-Borai *et al.* [13] studied the existence, uniqueness and continuity of the solution of a fractional stochastic integral equation; M. Kerboua *et al.* [21] derived a set of sufficient conditions for approximate controllability of Sobolev type nonlocal fractional stochastic dynamic systems in Hilbert spaces by using a stochastic version of the well-known fixed point theorem and semigroup theory. Moreover, theory of neutral differential equations is of both theoretical and practical interests. For a large class of electrical net-works containing lossless transmission lines, the describing equations can be reduced to neutral differential equations; N. Ding [12] has derived the exponential stability in mean square of mild solutions for neutral stochastic partial functional differential equations with impulses by applying the impulsive Gronwall–Bellman inequality, the stochastic analytic techniques, the fractional power of operator, and semigroup theory; Sakthivel *et al.* [36] have studied the existence and asymptotic stability in pth moment of a mild solution to a class of nonlinear fractional neutral stochastic differential equations with infinite delays in Hilbert spaces with the help of semigroup theory and fixed point technique.

It should be mentioned that there is no work yet reported on the quadratic mean almost periodic mild solutions to a fractional Sobolev type stochastic differential equations in Hilbert spaces. Motivated by the above facts, the main purpose of this paper is to investigate the existence and uniqueness of quadratic mean almost periodic solutions to a class of neutral fractional Sobolev type stochastic differential equation in the abstract form

$$ {}^{C}D_t^q\big[Ex(t) - h\left(t, x(t)\right)\big] = Ax(t) + \sigma\left(t, x(t)\right)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \quad t \in \mathbb{R}, \tag{1.1} $$

where ${}^{C}D_t^q$ denotes the Caputo fractional derivative operator of order q, $0 < q < 1$, $A\colon D(A) \subset L^2(\Omega, X) \to L^2(\Omega, X)$ and $E\colon D(E) \subset L^2(\Omega, X) \to L^2(\Omega, X)$ are linear operators from a Hilbert space $L^2(\Omega, X)$ to $L^2(\Omega, X)$. The functions $h\colon \mathbb{R} \times L^2(\Omega, X) \to L^2(\Omega, X_\alpha)$ and $\sigma\colon \mathbb{R} \times L^2(\Omega, X) \to L^2(\Omega, L_2^0)$ are jointly continuous functions, $w(t)$ is a Brownian motion.

The paper is organized as follows. In Section 2, we present some essential facts in fractional calculus, semigroup theory and stochastic analysis that will be used to obtain our main results. In Sections 3, the existence and uniqueness of mean square almost periodic mild solutions is proved. An example is given to illustrate our results in Section 4.

2 Preliminaries

This section is mainly concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows. For more details on this section, we refer the reader to [2, 3, 10, 8].

Let X, E and U be separable Hilbert spaces. By $L(E, X)$ we denote the set of all linear bounded operators from E into X which is equipped with the usual operator norm $||\cdot||$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We let $L^2(E, X)$ denote the space of all Hilbert–Schmidt operators $\Phi\colon E \to X$, equipped with the Hilbert–Schmidt norm $||\cdot||_2$.

For a symmetric nonnegative operator $Q \in L^2(E, X)$ with finite trace we suppose that $\{w(t) : t \in \mathbb{R}\}$ is a Q-Wiener process defined on $(\Omega, \mathcal{F}, P)$ and with values in E. So, actually, w can be obtained as follows: let $w_i(t), t \in \mathbb{R}, i = 1, 2$, be independent E-valued Q-Wiener processes; then

$$ w(t) = \begin{cases} w_1(t) & \text{if } t \geq 0, \\ w_2(-t) & \text{if } t \leq 0 \end{cases} $$

is Q-Wiener process with $\mathbb{R}$ as time parameter. We then let $\mathcal{F}_t = \sigma\{w(s) : s \le t\}$ be the σ-algebra generated by w.

Let $E(\cdot)$ denote the expectation with respect to the measure P. Let $C(\mathbb{R}, L^2(\Omega, X))$ be the Banach space of continuous maps from $\mathbb{R}$ into $L^2(\Omega, X)$ satisfying $E\,||x(t)||^2 < \infty$. For details, we refer the reader to [8, 32, 35] and references therein.

Let $E_0 = Q^{\frac{1}{2}}E$ and $L_0^2(E_0, X)$ with respect to the norm

$$\|\Phi\|_{L_0^2}^2 = \left\|\Phi Q^{\frac{1}{2}}\right\|_2^2 = \operatorname{Tr}\left(\Phi Q \Phi^*\right).$$

We introduce the following assumptions on the operators L and M:

(i) E and A are linear operators, and A is closed;

(ii) $D(E) \subset D(A)$ and E is bijective;

(iii) $E^{-1}\colon L^2(\Omega, X) \to D(E) \subset L^2(\Omega, X)$ is a linear compact operator.

From (iii) we deduce that E^{-1} is a bounded operator; for short, we denote its norm by $C = \left|\left|E^{-1}\right|\right|$. Note that (iii) also implies that E is closed since the fact: E^{-1} is closed and injective, then its inverse is also closed. From (i)–(iii) and the closed graph theorem we obtain the boundedness of the linear operator $AE^{-1}\colon L^2(\Omega, X) \to L^2(\Omega, X)$. Consequently, AE^{-1} generates a semigroup $\left\{S(t) := e^{AE^{-1}t} : t \ge 0\right\}$. We suppose that $K_0 = \sup_{t\ge 0} ||S(t)|| < \infty$.

Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent of A. Then for $0 < \alpha \le 1$ it is possible to define the fractional power $(-A)^\alpha$ as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in $L^2(\Omega, X)$, and we denote by $L^2(\Omega, X_\alpha)$ the Banach space $D((-A)^\alpha)$ endowed with the norm $||x||_\alpha = ||(-A)^\alpha x||_{L^2(\Omega,X)}$, which is equivalent to the graph norm of $(-A)^\alpha$.

The following properties hold by [28].

Lemma 1 (see [28]) *Suppose that $0 \in \rho(A)$. Then we know that there exist constants $K_0 \ge 1$, $\delta > 0$ such that $||S(t)|| \le K_0 e^{-\delta t}$ for $t \ge 0$, and for every $0 < \alpha \le 1$*

(i) *we have for each $x \in D((-A)^\alpha)$,*

$$S(t)(-A)^\alpha x = (-A)^\alpha S(t)x;$$

(ii) *there exists $K_\alpha > 0$ such that*

$$||(-A)^\alpha S(t)|| \le \frac{K_\alpha}{t^\alpha} e^{-\delta t}.$$

Definition 1 *The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a, b], \mathbb{R}^+)$ is given by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

where Γ is the gamma function.

If $a = 0$, we can write $I^{\alpha} f(t) = (g_{\alpha} * f)(t)$, where

$$g_{\alpha}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and as usual "$*$" denotes the convolution of functions. Moreover, $\lim_{\alpha \to 0} g_{\alpha}(t) = \delta(t)$, where δ is the Dirac delta function.

Definition 2 *The Riemann–Liouville derivative of order* $n - 1 < \alpha < n$, $n \in \mathbb{N}$, *for a function* $f \in C([0, +\infty))$ *is given by*

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} \,\mathrm{d}s, \quad t > 0.$$

Definition 3 *The Caputo derivative of order* $n-1 < \alpha < n$, $n \in \mathbb{N}$, *for a function* $f \in C([0, +\infty))$ *is given by*

$$^{C}D^{\alpha} f(t) = {}^{L}D^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0.$$

Remark 1 *The following properties hold* (*see,* e.g., [37]):

(i) *if* $f(t) \in C^n([0, \infty))$, *then*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} \,\mathrm{d}s = I^{n-\alpha} f^{(n)}(s), \ t > 0, \ n-1 < \alpha < n, \ n \in \mathbb{N};$$

(ii) *the Caputo derivative of a constant is equal to zero*;

(iii) *if* f *is an abstract function with values in* X, *then integrals which appear in Definitions* 1, 2 *and* 3 *are taken in Bochner's sense.*

According to previous definitions, it is suitable to rewrite problem (1.1) as the equivalent integral equation

$$\begin{aligned} x(t) = {} & [x(a) - h(a, x(a))] + E^{-1} h(t, x(t)) + \frac{1}{\Gamma(q)} \int_a^t E^{-1} (t-s)^{q-1} Ax(s) \,\mathrm{d}s \\ & + \frac{1}{\Gamma(q)} \int_a^t E^{-1} (t-s)^{q-1} \sigma(s, x(s)) \,\mathrm{d}w(s) \end{aligned} \tag{2.1}$$

for all $t \geq a$ and for each $a \in \mathbb{R}$.

In the following results and definitions, we let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ and $(Z, ||\cdot||_Z)$ be Banach spaces and let $L^2(\Omega; X)$, $L^2(\Omega; Y)$ and $L^2(\Omega; Z)$ be their corresponding L^2-spaces, respectively.

Definition 4 (see [2]) *A stochastic process* $x \colon \mathbb{R} \to L^2(\Omega; X)$ *is said to be continuous, whenever*

$$\lim_{t \to s} E\, ||x(t) - x(s)||_X^2 = 0.$$

Definition 5 (see [2]) *A continuous stochastic process* $x\colon \mathbb{R} \to L^2(\Omega; X)$ *is said to be quadratic mean almost periodic if for each* $\epsilon > 0$ *there exists* $l(\epsilon) > 0$ *such that any interval of length* $l(\epsilon)$ *contains at least a number* τ *for which*

$$\sup_{t\in\mathbb{R}} E\,||x(t+\tau) - x(t)||_X^2 < \epsilon.$$

The collection of all stochastic processes $x : \mathbb{R} \to L^2(\Omega; X)$ *which are quadratic mean almost periodic is then denoted by* $AP(\mathbb{R}; L^2(\Omega, X))$.

Lemma 2 (see [2]) *If* x *belongs to* $AP(\mathbb{R}; L^2(\Omega, X))$, *then the following hold true*:

(i) *the mapping* $t \to E\,||x(t)||_X^2$ *is uniformly continuous*;

(ii) *there exists a constant* $N > 0$ *such that* $E\,||x(t)||_X^2 \leq N$ *for each* $t \in \mathbb{R}$;

(iii) x *is stochastically bounded.*

Let $C(\mathbb{R}, L^2(\Omega; X))$ denote the space of all continuous stochastic processes $x\colon \mathbb{R} \to L^2(\Omega; X)$. The notation $CUB(\mathbb{R}; L^2(\Omega; X))$ stands for the collection of all stochastic processes $x\colon \mathbb{R} \to L^2(\Omega; X)$, which are continuous and uniformly bounded. It is known from [2] that $CUB(\mathbb{R}; L^2(\Omega; X))$ is a Banach space endowed with the norm:

$$||x||_\infty = \sup_{t\in\mathbb{R}} \left(E\,||x(t)||_X^2\right)^{\frac{1}{2}}.$$

Lemma 3 (see [2]) $AP\left(\mathbb{R}; L^2(\Omega, X)\right) \subset CUB(\mathbb{R}; L^2(\Omega; X))$ *is a closed subspace.*

Lemma 4 (see [2]) $AP\left(\mathbb{R}; L^2(\Omega, X), ||\cdot||_{AP(\mathbb{R};L^2(\Omega,X))}\right)$ *is a Banach space endowed with the norm*:

$$||x||_{AP(\mathbb{R};L^2(\Omega,X))} = \sup_{t\in\mathbb{R}} \left(E\,||x(t)||_X^2\right)^{\frac{1}{2}}.$$

Definition 6 (see [2]) *A function* $F\colon \mathbb{R} \times L^2(\Omega; Y) \to L^2(\Omega; Z),\ (t, y) \to F(t, y)$, *which is jointly continuous, is said to be quadratic mean almost periodic in* $t \in \mathbb{R}$ *uniformly in* $y \in \mathbb{B}$, *where* $\mathbb{B} \subset L^2(\Omega; Y)$ *is compact, if for any* $\epsilon > 0$ *there exists* $l(\epsilon, \mathbb{B}) > 0$ *such that any interval of length* $l(\epsilon, \mathbb{B})$ *contains at least a number* τ *for which*

$$\sup_{t\in\mathbb{R}} E\,||F\,(t+\tau, y) - F\,(t, y)||_Z^2 < \epsilon$$

for each stochastic process $y\colon \mathbb{R} \to \mathbb{B}$.

Lemma 5 (see [2]) *Let* $F\colon \mathbb{R} \times L^2(\Omega; Y) \to L^2(\Omega; Z),\ (t, y) \to F(t, y)$ *be a quadratic mean almost periodic process in* $t \in \mathbb{R}$ *uniformly in* $y \in \mathbb{B}$, *where* $\mathbb{B} \subset L^2(\Omega; Y)$ *is compact. Suppose that* F *is Lipschitz in the following sense*:

$$E\,||F\,(t, x) - F\,(t, y)||_Z^2 \leq \tilde{M} E\,||x - y||_Y^2$$

for all $x, y \in L^2(\Omega; Y)$ *and for each* $t \in \mathbb{R}$, *where* $\tilde{M} > 0$. *Then for any quadratic mean almost periodic process* $\Psi\colon \mathbb{R} \to L^2(\Omega; Y)$ *the stochastic process* $t \to F(t, \Psi(t))$ *is quadratic mean almost periodic.*

Definition 7 *A $\mathcal{F}_t$-progressively process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of the problem* (1.1) *on $\mathbb{R}$ if the function $s \to (t-s)^{q-1} A\mathcal{T}_q(t-s)h(s,x(s))$ is integrable on $(-\infty,t)$ for each $t\in\mathbb{R}$, and $x(t)$ satisfies*

$$\begin{aligned} x(t) &= \mathcal{S}_q(t-a)[x(a)-h(a,x(a))] + E^{-1}h(t,x(t)) \\ &\quad + \int_a^t (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s,x(s))\,\mathrm{d}s \\ &\quad + \int_a^t (t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\sigma(s,x(s))\,\mathrm{d}w(s) \end{aligned}$$

for all $t\geq a$ and for each $a\in\mathbb{R}$, where

$$\mathcal{S}_q(t) = \int_0^{+\infty} \xi_q(\theta) S(t^q\theta)\,\mathrm{d}\theta \qquad \textit{and} \qquad \mathcal{T}_q(t) = q\int_0^{+\infty} \theta\xi_q(\theta) S(t^q\theta)\,\mathrm{d}\theta;$$

here $S(t)$ is a C_0-semigroup generated by the linear operator $AE^{-1}\colon X\to X$ and ξ_q is a probability density function defined on $(0,\infty)$, that is, $\xi_q(\theta)\geq 0$, $\theta\in(0,\infty)$ and $\int_0^\infty \xi_q(\theta)\,\mathrm{d}\theta = 1$.

Lemma 6 (see [37]) *The operators $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ have the following properties*:

(i) *for any fixed $t\geq 0$, $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ are linear and bounded operators,* i.e., *for any $x\in L^2(\Omega,X)$,*

$$||\mathcal{S}_q(t)x|| \leq K_0\,||x||\,, \qquad ||\mathcal{T}_q(t)x|| \leq \frac{qK_0}{\Gamma(1+q)}\,||x||\,;$$

(ii) *the operators $\{\mathcal{S}_q(t)\}_{t\geq 0}$ and $\{\mathcal{T}_q(t)\}_{t\geq 0}$ are strongly continuous and compact*;

(iii) *for any $x\in L^2(\Omega,X)$, $\beta\in(0,1)$ and $\alpha\in(0,1]$, we have*

$$A\mathcal{T}_q(t)x = A^{1-\beta}\mathcal{T}_q(t)A^\beta x, \quad t\geq 0,$$

$$||(-A)^\alpha\mathcal{T}_q(t)|| \leq \frac{qK_\alpha\Gamma(2-\alpha)}{t^{q\alpha}\Gamma(1+q(1-\alpha))}e^{-\delta t}, \quad t\geq 0.$$

For the problem (1.1), we impose the following assumptions.

(A1) The linear operator $AE^{-1}\colon D\left(AE^{-1}\right)\subset L^2(\Omega,X)\to L^2(\Omega,X)$ generates a semigroup $\left\{S(t) := e^{AE^{-1}t} : t\geq 0\right\}$ on $L^2(\Omega,X)$ such that $||S(t)||\leq K_0 e^{-\delta t}$ for $K_0\geq 1$ and $\delta>0$.

(A2) There exists a positive number $\alpha\in(0,1)$ such that $h\colon\mathbb{R}\times L^2(\Omega,X)\to L^2(\Omega,X_\alpha)$ is quadratic mean almost periodic in $t\in\mathbb{R}$ uniformly in $x\in\mathbb{B}_1$, where $\mathbb{B}_1\subset L^2(\Omega,X)$ is a compact subspace. Moreover, h is Lipschitz in the sense that there exists $K_h>0$ such that

$$E\,||(-A)^\alpha h(t,x) - (-A)^\alpha h(t,y)||^2 \leq K_h E\,||x-y||^2$$

for all $t\in\mathbb{R}$ and for all stochastic processes $x,y\in L^2(\Omega,X)$.

(A3) The function $\sigma\colon\mathbb{R}\times L^2(\Omega,X)\to L^2(\Omega,L_2^0)$ is quadratic mean almost periodic in $t\in\mathbb{R}$ uniformly in $x\in\mathbb{B}_3$, where $\mathbb{B}_3\subset L^2(\Omega,X)$ is a compact subspace. Moreover, σ is Lipschitz in the sense that there exists $K_\sigma>0$ such that

$$E\,||\sigma(t,x)-\sigma(t,y)||^2_{L_2^0} \leq K_\sigma E\,||x-y||^2$$

for all $t\in\mathbb{R}$ and for all stochastic processes $x,y\in L^2(\Omega,X)$.

3 Main results

This section is devoted to proving the existence and uniqueness of a quadratic-mean almost periodic solution of neutral fractional Sobolev type stochastic differential equation (1.1)

Theorem 1 *Assume the conditions* **(A1)–(A3)** *are satisfied. Then the problem* (1.1) *admits a unique quadratic mean almost periodic mild solution on* $\mathbb{R}$ *provided that*

$$\begin{aligned} L_0 = \Big\{ & 3C^2 \left|\left|(-A)^{-\alpha}\right|\right|^2 K_h + 3C^2 K_h \delta^{-2q\alpha} \left(K_{1-\alpha}\Gamma\left(1+\alpha\right)\right)^2 \\ & +3C^2 \left(\frac{qK_\alpha \Gamma\left(2-\alpha\right)}{\Gamma\left(1+q\left(1-\alpha\right)\right)}\right)^2 \operatorname{Tr} Q K_\sigma \left(2\delta\right)^{-2q(1-\alpha)+3} \Gamma\left(2q\left(1-\alpha\right)\right) \Big\} < 1, \end{aligned} \tag{3.1}$$

where $\Gamma(\cdot)$ *is the gamma function.*

Proof. Let $\Psi\colon AP(\mathbb{R}; L^2(\Omega, X)) \to C(\mathbb{R}, L^2(\Omega, X))$ be the operator defined by

$$\begin{aligned} \Psi x\left(t\right) = {} & \mathcal{S}_q(t-a)[x(a) - h(a, x(a))] + E^{-1}h(t, x(t)) \\ & + \int_{-\infty}^{t} (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s, x(s))\,\mathrm{d}s \\ & + \int_{-\infty}^{t} (t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\sigma\left(s, x\left(s\right)\right)\,\mathrm{d}w\left(s\right), \quad t \in \mathbb{R}. \end{aligned}$$

First we prove that Ψx is well-defined. From Lemma 5, we infer that $s \to h(s, x(s))$ is in $AP(\mathbb{R}, L^2(\Omega, X_\alpha))$. Thus, using Lemma 2 (ii) it follows that there exists a constant $\tilde{K}_h > 0$ such that $E\left|\left|(-A)^\alpha h\left(t, x\right)\right|\right|^2 \leq \tilde{K}_h$ for all $t \in \mathbb{R}$. Moreover, from the continuity of $s \to (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)$ and $s \to \mathcal{T}_q(t-s)$ in the uniform operator topology on $(-\infty, t)$ for each $t \in \mathbb{R}$ and the estimate

$$\begin{aligned} & E\left|\left|\int_{-\infty}^{t} (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s, x(s))\,\mathrm{d}s\right|\right|^2 \\ & = E\left|\left|\int_{-\infty}^{t} (t-s)^{q-1} E^{-1}\left(-A\right)^{1-\alpha} \mathcal{T}_q(t-s)\left(-A\right)^{\alpha} h(s, x(s))\,\mathrm{d}s\right|\right|^2 \\ & \leq E\left(\int_{-\infty}^{t} (t-s)^{q-1} \left|\left|E^{-1}\left(-A\right)^{1-\alpha} \mathcal{T}_q(t-s)\right|\right| \times \left|\left|(-A)^{\alpha} h(s, x(s))\right|\right|\,\mathrm{d}s\right)^2 \\ & \leq \left(\frac{qK_{1-\alpha}\Gamma\left(1+\alpha\right)}{\Gamma\left(1+q\alpha\right)}\right)^2 \times C^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\left(t-s\right)^{q\alpha-1}\left|\left|(-A)^{\alpha} h(s, x(s))\right|\right|\,\mathrm{d}s\right)^2 \\ & \leq \left(\frac{qK_{1-\alpha}\Gamma\left(1+\alpha\right)}{\Gamma\left(1+q\alpha\right)}\right)^2 \times C^2 \left(\int_{-\infty}^{t} e^{-\delta(t-s)}\left(t-s\right)^{q\alpha-1}\,\mathrm{d}s\right) \\ & \quad \times \left(\int_{-\infty}^{t} e^{-\delta(t-s)}\left(t-s\right)^{q\alpha-1} E\left|\left|(-A)^{\alpha} h(s, x(s))\right|\right|^2\,\mathrm{d}s\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2\tilde{K}_h^2 \left(\int_{-\infty}^t e^{-\delta(t-s)}(t-s)^{q\alpha-1}\,\mathrm{d}s\right)^2 \\
&\leq \left(\frac{qC_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2\tilde{K}_h^2 \times \delta^{-2q\alpha}\left[\Gamma(q\alpha)\right]^2 \\
&\leq \left(C_{1-\alpha}\Gamma(\alpha)\right)^2 \times C^2 \times \tilde{K}_h^2 \times \delta^{-2q\alpha},
\end{aligned}
$$

it follows that $s \to (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s,x(s))$ and $(t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\sigma(s,x(s))$ are integrable on $(-\infty,t)$ for every $t \in \mathbb{R}$. Therefore, Ψx is well-defined and continuous.

Next, we show that $\Psi x(t) \in AP(\mathbb{R}, L^2(\Omega, X))$. We define

$$
\begin{aligned}
\Psi_1 x(t) &= \int_{-\infty}^t (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s,x(s))\,\mathrm{d}s, \\
\Psi_2 x(t) &= \int_{-\infty}^t (t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\sigma(s,x(s))\,\mathrm{d}w(s).
\end{aligned}
$$

Let us show that $\Psi_1 x(t)$ is quadratic mean almost periodic. Now, since $h(\cdot,x(\cdot)) \in AP(\mathbb{R}, L^2(\Omega, X_\alpha))$, by Definition 5, it follows that for any $\epsilon > 0$, there exists $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains at least a number τ with the property that

$$
E\left|\left|(-A)^\alpha h(t+\tau, x(t+\tau)) - (-A)^\alpha h(t, x(t))\right|\right|^2 < \frac{\epsilon}{\left(K_{1-\alpha}\Gamma(\alpha)\right)^2 \times C^2 \times \delta^{-2q\alpha}}
$$

for $t \in \mathbb{R}$.

Now, using the Cauchy–Schwarz inequality, we see that

$$
\begin{aligned}
&E\left|\left|\Psi_1 x(t+\tau) - \Psi_1 x(t)\right|\right|^2 \\
&\quad = E\left|\left|\int_{-\infty}^{t+\tau} (t+\tau-s)^{q-1} AE^{-1}\mathcal{T}_q(t+\tau-s)h(s+\tau,x(s+\tau))\right.\right. \\
&\qquad \left.\left. - \int_{-\infty}^{t} (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s,x(s))\,\mathrm{d}s\right|\right|^2 \\
&\quad = E\left|\left|\int_{-\infty}^{\xi} (\xi-s)^{q-1} AE^{-1}\mathcal{T}_q(\xi-s)h(s+\tau,x(s+\tau))\right.\right. \\
&\qquad \left.\left. - \int_{-\infty}^{t} (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)h(s,x(s))\,\mathrm{d}s\right|\right|^2 \\
&\quad = E\left|\left|\int_{-\infty}^{t} (t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)\left[h(s+\tau,x(s+\tau)) - h(s,x(s))\right]\mathrm{d}s\right|\right|^2 \\
&\quad = E\left|\left|\int_{-\infty}^{t} (t-s)^{q-1} (-A)^{1-\alpha} E^{-1}\mathcal{T}_q(t-s)\right.\right. \\
&\qquad \left.\left. \times \left[(-A)^\alpha h(s+\tau,x(s+\tau)) - (-A)^\alpha h(s,x(s))\right]\,\mathrm{d}s\right|\right|^2 \\
&\quad \leq \left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2 \times E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{q\alpha-1}\right. \\
&\qquad \left. \times \left|\left|(-A)^\alpha h(s+\tau,x(s+\tau)) - (-A)^\alpha h(s,x(s))\right|\right|\,\mathrm{d}s\right)^2
\end{aligned}
$$

$$\leq \left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2 \times \left(\int_{-\infty}^{t} e^{-\delta(t-s)} (t-s)^{q\alpha-1} \, \mathrm{d}s\right)$$
$$\times \left(\int_{-\infty}^{t} e^{-\delta(t-s)} (t-s)^{q\alpha-1} E\,||(-A)^{\alpha} h(s+\tau, x(s+\tau)) - (-A)^{\alpha} h(s, x(s))||^2 \, \mathrm{d}s\right)$$
$$\leq \left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2 \times \left(\int_{-\infty}^{t} e^{-\delta(t-s)} (t-s)^{q\alpha-1} \, \mathrm{d}s\right)^2$$
$$\times \sup_{t\in\mathbb{R}} E\,||(-A)^{\alpha} h(t+\tau, x(t+\tau)) - (-A)^{\alpha} h(t, x(t))||^2$$
$$\leq \left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2 \times \frac{\epsilon}{(K_{1-\alpha}\Gamma(\alpha))^2 \times C^2 \times \delta^{-2q\alpha}}$$
$$\times \left(\int_{-\infty}^{t} e^{-\delta(t-s)} (t-s)^{q\alpha-1} \, \mathrm{d}s\right)^2$$
$$\leq \left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 \times C^2 \times \frac{\epsilon}{(K_{1-\alpha}\Gamma(\alpha))^2 \times C^2 \times \delta^{-2q\alpha}} \times \delta^{-2q\alpha} \left[\Gamma(q\alpha)\right]^2 = \epsilon.$$

Hence, $\Psi_1 x(\cdot)$ is quadratic mean almost periodic.

Similarly, by using Lemma 5, one can easily see that $s \to \sigma(s, x(s))$ is quadratic mean almost periodic. Therefore, it follows from Definition 5 that for any $\epsilon > 0$ there exists $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains at least a number τ with the property that

$$E\,||\sigma(t+\tau, x(t+\tau)) - \sigma(t, x(t))||^2 \leq \frac{\delta^{2q(1-\alpha)}\epsilon}{\operatorname{Tr} Q \times C^2 \times (K_{\alpha}\Gamma(1-\alpha))^2}$$

for each $t \in \mathbb{R}$. Now, let us prove that $\Psi_2 x(t)$ is quadratic mean almost periodic. We adopt the techniques developed in [2]. Let $\tilde{w}(t) = w(t+\tau) - w(\tau)$ for each $t \in \mathbb{R}$. Note that $\tilde{w}$ is also a Brownian motion and has the same distribution as w.

Now, we consider

$$E\,||\Psi_2 x(t+\tau) - \Psi_2 x(t)||^2$$
$$= E\left|\left|\int_{-\infty}^{t+\tau} (t+\tau-s)^{q-1} E^{-1}\mathcal{T}_q(t+\tau-s)\sigma(s, x(s)) \, \mathrm{d}w(s)\right.\right.$$
$$\left.\left. - \int_{-\infty}^{t} (t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\sigma(s, x(s)) \, \mathrm{d}w(s)\right|\right|^2$$
$$= E\left|\left|\int_{-\infty}^{t} (t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\left[\sigma(s+\tau, x(s+\tau)) - \sigma(s, x(s))\right] \mathrm{d}\tilde{w}(s)\right|\right|^2.$$

Thus, using an estimate on Itô integral established by Ichikawa [19], we obtain that

$$E\,||\Psi_2 x(t+\tau) - \Psi_2 x(t)||^2$$
$$= E\left|\left|\int_{-\infty}^{t} (t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\left[\sigma(s+\tau, x(s+\tau)) - \sigma(s, x(s))\right] \mathrm{d}\tilde{w}(s)\right|\right|^2$$

$$
\begin{aligned}
&\leq \operatorname{Tr} QE\left[\int_{-\infty}^{t}\left|\left|(-A)^{\alpha}(t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\left[\sigma(s+\tau, x(s+\tau))-\sigma(s, x(s))\right]\right|\right|^2 \mathrm{d}s\right]\\
&\leq \operatorname{Tr} QE\Bigg[\int_{-\infty}^{t}\left|\left|(-A)^{\alpha}(t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\right|\right|^2\\
&\qquad \times ||\sigma(s+\tau, x(s+\tau))-\sigma(s, x(s))||_{L_2^0}^2 \ \mathrm{d}s\Bigg]\\
&\leq \operatorname{Tr} Q \times C^2 \times \left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2 \times \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{q(1-\alpha)-1}\ \mathrm{d}s\right)\\
&\qquad \times \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{q(1-\alpha)-1} E\,||\sigma(s+\tau, x(s+\tau))-\sigma(s, x(s))||_{L_2^0}^2\ \mathrm{d}s\right)\\
&\leq \operatorname{Tr} Q \times C^2 \times \left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2 \times \left(\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{q(1-\alpha)-1}\ \mathrm{d}s\right)^2\\
&\qquad \times \sup_{t\in\mathbb{R}} E\,||\sigma(s+\tau, x(s+\tau))-\sigma(s, x(s))||_{L_2^0}^2\\
&\leq \operatorname{Tr} Q \times C^2 \times \left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2 \times \delta^{-2q(1-\alpha)}\left[\Gamma(q(1-\alpha))\right]^2\\
&\qquad \times \frac{\delta^{2q(1-\alpha)}\epsilon}{\operatorname{Tr} Q \times C^2 \times (K_{\alpha}\Gamma(1-\alpha))^2} = \epsilon.
\end{aligned}
$$

Thus, $\Psi_2 x(\cdot)$ is quadratic mean almost periodic. And in view of the above, it is clear that Ψ maps $AP(\mathbb{R}, L^2(\Omega, X))$ into itself.

Now, the remaining task is to prove that Ψ is a strict contraction on $AP(\mathbb{R}, L^2(\Omega, X))$. Indeed, for each $t \in \mathbb{R}$, $x, y \in AP(\mathbb{R}, L^2(\Omega, X))$, we have

$$
\begin{aligned}
&E\,||\Psi x(t)-\Psi y(t)||^2 \leq 3E\left|\left|E^{-1}h(t, x(t))-E^{-1}h(t, y(t))\right|\right|^2\\
&\quad + 3E\left(\left|\left|\int_{-\infty}^{t}(t-s)^{q-1} AE^{-1}\mathcal{T}_q(t-s)\left[h(s, x(s))-h(s, y(s))\right]\ \mathrm{d}s\right|\right|\right)^2\\
&\quad + 3E\left(\left|\left|\int_{-\infty}^{t}(t-s)^{q-1} E^{-1}\mathcal{T}_q(t-s)\left[\sigma(s, x(s))-\sigma(s, y(s))\right]\ \mathrm{d}w(s)\right|\right|\right)^2\\
&\leq 3C^2\left|\left|(-A)^{-\alpha}\right|\right|^2 E\,||(-A)^{\alpha}\left[h(t, x(t))-h(t, y(t))\right]||^2\\
&\quad + 3C^2 E\left(\left|\left|\int_{-\infty}^{t}(t-s)^{q-1}(-A)^{1-\alpha}\mathcal{T}_q(t-s)\left[(-A)^{\alpha}h(s, x(s))-(-A)^{\alpha}h(s, y(s))\right]\mathrm{d}s\right|\right|\right)^2\\
&\quad + 3C^2 \operatorname{Tr} QE\left(\int_{-\infty}^{t}\left|\left|(-A)^{\alpha}(t-s)^{q-1}\mathcal{T}_q(t-s)\left[\sigma(s, x(s))-\sigma(s, y(s))\right]\right|\right|^2\ \mathrm{d}s\right).
\end{aligned}
$$

We first evaluate the first term of the right-hand side as follows

$$
\begin{aligned}
&3C^2\left|\left|(-A)^{-\alpha}\right|\right|^2 E\,||(-A)^{\alpha}\left[h(t, x(t))-h(t, y(t))\right]||^2\\
&\quad\leq 3C^2\left|\left|(-A)^{-\alpha}\right|\right|^2 \sup_{t\in\mathbb{R}} E\,||(-A)^{\alpha}\left[h(t, x(t))-h(t, y(t))\right]||^2\\
&\quad\leq 3C^2\left|\left|(-A)^{-\alpha}\right|\right|^2 K_h \sup_{t\in\mathbb{R}} E\,||x(t)-y(t)||^2.
\end{aligned}
$$

As regards the second term, by the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&3C^2 E\left(\left|\left|\int_{-\infty}^{t}(t-s)^{q-1}(-A)^{1-\alpha}\mathcal{T}_q(t-s)\left[(-A)^{\alpha}h(s,x(s))-(-A)^{\alpha}h(s,y(s))\right]\mathrm{d}s\right|\right|\right)^2\\
&\quad\le 3C^2\left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2\\
&\qquad\times E\left(\int_{-\infty}^{t}e^{-\delta(t-s)}(t-s)^{q\alpha-1}\left|\left|(-A)^{\alpha}h(s,x(s))-(-A)^{\alpha}h(s,y(s))\right|\right|\mathrm{d}s\right)^2\\
&\quad\le 3C^2\left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 E\left[\left(\int_{-\infty}^{t}e^{-\delta(t-s)}(t-s)^{q\alpha-1}\mathrm{d}s\right)\right.\\
&\qquad\times\left.\left(\int_{-\infty}^{t}e^{-\delta(t-s)}(t-s)^{q\alpha-1}\left|\left|(-A)^{\alpha}h(s,x(s))-(-A)^{\alpha}h(s,y(s))\right|\right|\mathrm{d}s\right)\right]\\
&\quad\le 3C^2\left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2\left(\int_{-\infty}^{t}e^{-\delta(t-s)}(t-s)^{q\alpha-1}\mathrm{d}s\right)\\
&\qquad\times\left(\int_{-\infty}^{t}e^{-\delta(t-s)}(t-s)^{q\alpha-1}E\left|\left|(-A)^{\alpha}h(s,x(s))-(-A)^{\alpha}h(s,y(s))\right|\right|\mathrm{d}s\right)\\
&\quad\le 3C^2\left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 K_h\left(\int_{-\infty}^{t}e^{-\delta(t-s)}(t-s)^{q\alpha-1}\mathrm{d}s\right)^2\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2\\
&\quad\le 3C^2\left(\frac{qK_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}\right)^2 K_h\delta^{-2q\alpha}\left[\Gamma(q\alpha)\right]^2\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2\\
&\quad\le 3C^2K_h\delta^{-2q\alpha}\left(K_{1-\alpha}\Gamma(1+\alpha)\right)^2\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2.
\end{aligned}
$$

As regards the third term, we use again the Cauchy-Schwarz inequality and obtain

$$
\begin{aligned}
&3C^2\operatorname{Tr}QE\left(\int_{-\infty}^{t}\left|\left|(-A)^{\alpha}(t-s)^{q-1}\mathcal{T}_q(t-s)\left[\sigma(s,x(s))-\sigma(s,y(s))\right]\right|\right|^2\mathrm{d}s\right)\\
&\quad\le 3C^2\left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2\\
&\qquad\times\operatorname{Tr}Q\int_{-\infty}^{t}e^{-2\delta(t-s)}(t-s)^{2(q(1-\alpha)-1)}E\left|\left|\sigma(s,x(s))-\sigma(s,y(s))\right|\right|^2\mathrm{d}s\\
&\quad\le 3C^2\left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2\\
&\qquad\times\operatorname{Tr}QK_{\sigma}\left(\int_{-\infty}^{t}e^{-2\delta(t-s)}(t-s)^{2(q(1-\alpha)-1)}\mathrm{d}s\right)\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2\\
&\quad\le 3C^2\left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2\operatorname{Tr}QK_{\sigma}(2\delta)^{-2q(1-\alpha)+3}\Gamma(2q(1-\alpha))\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2.
\end{aligned}
$$

Thus, by combining the above three estimates, it follows that for each $t\in\mathbb{R}$ we have

$$
\begin{aligned}
E\left|\left|\Psi x(t)-\Psi y(t)\right|\right|^2\le 3C^2&\left\{\left|\left|(-A)^{-\alpha}\right|\right|^2K_h+K_h\delta^{-2q\alpha}\left(K_{1-\alpha}\Gamma(1+\alpha)\right)^2\right.\\
&\left.+\left(\frac{qK_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\right)^2\operatorname{Tr}QK_{\sigma}(2\delta)^{-2q(1-\alpha)+3}\Gamma(2q(1-\alpha))\right\}\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2,
\end{aligned}
$$

that is,

$$E\left|\left|\Psi x(t)-\Psi y(t)\right|\right|^2\leq L_0\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2.$$

Note that

$$\sup_{t\in\mathbb{R}}E\left|\left|x(t)-y(t)\right|\right|^2\leq\left[\sup_{t\in\mathbb{R}}\left(E\left|\left|x(t)-y(t)\right|\right|^2\right)^{\frac{1}{2}}\right]^2.$$

Thus, it follows that for each $t\in\mathbb{R}$,

$$\left(E\left|\left|\Psi x(t)-\Psi y(t)\right|\right|^2\right)^{\frac{1}{2}}\leq\sqrt{L_0}\left|\left|x-y\right|\right|_{AP(\mathbb{R};L^2(\Omega,X))}.$$

Hence

$$\begin{aligned}\left|\left|\Psi x-\Psi y\right|\right|_{AP(\mathbb{R};L^2(\Omega,X))}&=\sup_{t\in\mathbb{R}}\left(E\left|\left|x(t)-y(t)\right|\right|^2\right)^{\frac{1}{2}}\\&\leq\sqrt{L_0}\left|\left|x-y\right|\right|_{AP(\mathbb{R};L^2(\Omega,X))}.\end{aligned}$$

Since $L_0<1$, it follows that Ψ is a contraction mapping on $AP(\mathbb{R};L^2(\Omega,X))$. The Banach contraction theorem shows that there exists a unique fixed point $x(\cdot)$ for Ψ in $AP(\mathbb{R};L^2(\Omega,X))$ such that $\Psi x=x$. Thus, we conclude that

$$\begin{aligned}x(t)=&\,\mathcal{S}_q(t-a)[x(a)-h(a,x(a))]+E^{-1}h(t,x(t))\\&+\int_a^t(t-s)^{q-1}AE^{-1}\mathcal{T}_q(t-s)h(s,x(s))\,\mathrm{d}s\\&+\int_a^t(t-s)^{q-1}E^{-1}\mathcal{T}_q(t-s)\sigma\left(s,x\left(s\right)\right)\,\mathrm{d}w\left(s\right)\end{aligned}$$

is a mild solution of the problem (1.1) and $x(\cdot)\in AP(\mathbb{R};L^2(\Omega,X))$. The proof is complete. □

4 Example

In this section, we consider a simple example to illustrate our main theorem.

We consider the following stochastic fractional partial differential equation of Sobolev type

$$\begin{aligned}&{}^CD_t^q\left[x(t,z)-x_{zz}(t,z)-h(t,x(t,z))\right]=\frac{\partial^2}{\partial z^2}x(t,z)+\hat{\sigma}(t,x(t,z))\frac{\mathrm{d}\hat{w}(t)}{\mathrm{d}t},\\&\qquad\qquad t\in\mathbb{R},\ z\in[0,\pi],\\&x(t,0)=x(t,\pi)=0,\quad t\in\mathbb{R},\end{aligned}\tag{4.1}$$

where $0<q\leq1$, the function $x(t)(z)=x(t,z)$, $\sigma(t,x(t))(z)=\hat{\sigma}(t,x(t,z))$ and $\hat{w}(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space (Ω,Γ,P).

To write the system (4.1) into the abstract form of (1.1) we consider the space $X=L^2[0,\pi]$ and define the operators $E\colon D(E)\subset X\to X$ and $A\colon D(A)\subset X\to X$ by $Ex=x-x''$ and $Ax=-x''$ where domains $D(E)$ and $D(A)$ are given by

$$\left\{x\in X:x,x''\text{ are absolutely continuous, }x''\in X,\ x(0)=x(\pi)=0\right\}.$$

Then E and A can be written respectively as

$$Ex = \sum_{n=1}^{\infty}(1+n^2)(x,x_n)x_n,\ x \in D(E) \quad \text{and} \quad Ax = \sum_{n=1}^{\infty} -n^2(x,x_n)x_n,\ x \in D(A),$$

where $x_n(s) = \left(\sqrt{2}\right)\sin(ns)$, $n = 1, 2, \ldots$, is the orthogonal set of eigenfunctions of A. Further, for any $x \in X$ we have

$$E^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(x,x_n)x_n, \qquad AE^{-1}x = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(x,x_n)x_n$$

and

$$S(t)x = \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1+n^2}\right)(x,x_n)x_n.$$

It is easy to see that E^{-1} is compact, bounded with $\left|\left|E^{-1}\right|\right| \leq 1$ and AE^{-1} generates the above strongly continuous semigroup $\{S(t) : t \geq 0\}$ on $L^2(\mathbb{R}, X)$ satisfying **(A1)**. Thus, under the assumptions **(A2)**–**(A3)**, once (3.1) holds, an application of Theorem 1 yields that (4.1) has a unique mild solution, which is obviously quadratic mean almost periodic.

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