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STRONGLY NONLINEAR DEGENERATED ELLIPTIC EQUATION HAVING LOWER ORDER TERM IN WEIGHTED ORLICZ–SOBOLEV SPACES AND L^1 DATA

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Abstract. The paper deals with the existence of solutions to a nonlinear degenerated equation in divergence form having lower order term. This problem is associated to elliptic operators in the framework of weighted Orlicz–Sobolev spaces and L^1 data.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let M, P be two N-functions such that $P \ll M$. Moreover, let $\overline{M}$, $\overline{P}$ be the complementary functions of M and P, respectively. In this article, we prove the

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existence of solutions for nonlinear degenerate elliptic equations of the form

$$A(u) + g(x, u, \nabla u) = f, \tag{1.1}$$

where $A(u) = -\operatorname{div}(\rho(x)a(x, u, \nabla u)) + a_0(x, u, \nabla u)$ is a Leray–Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory's functions satisfying some natural growth conditions with respect to u and ∇u and the degenerate ellipticity condition

$$a_0(x, s, \xi)s + \rho(x)a(x, s, \xi)\xi \geq \lambda_0[M(\lambda_1 s) + \rho(x)M(\lambda_2|\xi|)];$$

g is a nonlinearity which satisfies the natural growth condition

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|)\rho(x))$$

and the classical sign condition

$$g(x, s, \xi).s \geq 0.$$

The source term f is supposed to be in $L^1(\Omega)$. Some model examples of this problem are

$$-\operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega,$$
$$-\operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u \log^{\beta}(1 + |\nabla u|)) + \rho(x)M(|\nabla u|) = f \quad \text{in } \Omega,$$

where $p > 1$, f is a function in $L^1(\Omega)$ and ρ is a given weight function on Ω.

In the non-degenerate case, the equation (1.1) with $f \in W^{-1}E_{\overline{M}}(\Omega)$ was solved in [5]. An existence theorem has been proved by Benkirane & Elmahi and others [6,8,9] with $f \in W^{-1}E_{\overline{M}}(\Omega)$ and $f \in L^1(\Omega)$, respectively. Another work in this direction can be found in [9] in the non-weighted case. So for our nonlinear operator $A(u) = -\operatorname{div}(\rho(x)a(x, u, \nabla u)) + a_0(x, u, \nabla u)$ with coefficients which are singular or degenerated the classic ellipticity conditions are violated and one has to change the classical approach introducing weighted spaces. Note that this type of equations can be applied in physics. A non-standard example of $M(t)$ which occurs in the mechanics of solids and fluids is $M(t) = t\log(1 + t)$. The use of the truncation operator in (1.1) is justified by the fact that, in general, the solution does not belong to $L^{\infty}(\Omega)$ for $f \in L^1(\Omega)$. The aim of this article is to study the existence of a solution to the problem (1.1) in the setting of weighted Orlicz–Sobolev spaces and L^1 data.

This article is organized as follows. In Section 2, we introduce the mathematical preliminaries. In Section 3, we introduce basic assumptions and prove some main lemmas. Section 4 is devoted to the proof of our general existence result.

2 Preliminaries

In this section we present some definitions and well-known facts about N-functions and weighted Orlicz–Sobolev spaces.

2.1 The N-functions

Let $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, *i.e.*, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation:

$$M(t) = \int_0^t m(\tau)\,\mathrm{d}\tau,$$

where $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau)\,\mathrm{d}\tau,$$

where $\overline{m}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = \sup\{s : m(s) \le t\}$. Clearly, $\overline{\overline{M}} = M$. Moreover, for all $s, t \ge 0$ Young's inequality $st \le M(t) + \overline{M}(s)$ holds.

It is well-known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even functions on all $\mathbb{R}$.

The N-function M is said to satisfy the Δ_2-condition everywhere (resp., near infinity) if there exists $k > 0$ (resp., $t_0 > 0$) such that $M(2t) \le kM(t)$ for all $t \ge 0$ (resp., $t \ge t_0$).

2.2 Weighted Orlicz–Sobolev spaces

First of all, we shall work with weighted Orlicz spaces in the following sense. Let Ω be a domain in $\mathbb{R}^N$, and let M be an N-function and ρ be a weight function on Ω, *i.e.*, ρ is measurable and positive a.e. on Ω.

The weighted Orlicz class $K_M(\Omega, \rho)$ (resp., the weighted Orlicz space $L_M(\Omega, \rho)$) is the set of all (equivalence classes modulo equality a.e. in Ω of) real-valued measurable functions u defined in Ω and satisfying

$$m_\rho(u, M) = \int_\Omega M(|u(x)|)\rho(x)\,\mathrm{d}x < \infty,$$

$$\left(\text{resp., } m_\rho\Big(\frac{u}{\lambda}, M\Big) = \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)\rho(x)\,\mathrm{d}x < \infty \text{ for some } \lambda > 0\right).$$

The weighted Orlicz space $L_M(\Omega, \rho)$ is a Banach space under the Luxemburg's norm

$$\|u\|_{M,\rho} = \inf\left\{\lambda > 0 : m_\rho\Big(\frac{u}{\lambda}, M\Big) \le 1\right\}.$$

The closure in $L_M(\Omega, \rho)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega, \rho)$. We have

$$E_M(\Omega, \rho) \subset K_M(\Omega, \rho) \subset L_M(\Omega, \rho).$$

The equality $L_M(\Omega, \rho) = E_M(\Omega, \rho)$ holds if and only if M satisfies the Δ_2-condition for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega,\rho)$ can be identified with $L_{\overline{M}}(\Omega,\rho)$ by means of the pairing

$$\int_\Omega u(x)v(x)\rho(x)\,\mathrm{d}x,$$

where $u \in L_M(\Omega,\rho)$ and $v \in L_{\overline{M}}(\Omega,\rho)$. The dual norm on $L_{\overline{M}}(\Omega,\rho)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$. It gives rise to the so-called Orlicz norm on $L_M(\Omega,\rho)$ defined by

$$\|u\|_{M,\rho} = \sup\left\{\int_\Omega f(x)g(x)\rho(x)\,\mathrm{d}x : m_\rho(g,\overline{M}) \le 1\right\}.$$

The space $L_M(\Omega,\rho)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition for all t or for t large according to whether Ω has infinite measure or not.

We return now to the weighted Orlicz–Sobolev spaces. $W^1L_M(\Omega,\rho)$ (resp., $W^1E_M(\Omega,\rho)$) is the space of all functions u such that $u \in L_M(\Omega)$ (resp., $u \in E_M(\Omega)$) and its distributional derivatives up to order 1 lie in $L_M(\Omega,\rho)$ (resp., in $E_M(\Omega,\rho)$). It is a Banach space under the norm

$$\|u\|_{1,M,\rho} = \|u\|_M + \|\nabla u\|_{M,\rho}.$$

Thus, $W^1L_M(\Omega,\rho)$ and $W^1E_M(\Omega,\rho)$ can be identified with subspaces of $\prod L_{M,\rho} = L_M \times \prod L_M(\Omega,\rho)$. Note that we have the weak topologies $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and $\sigma(\prod L_{M,\rho}, \prod L_{\overline{M},\rho})$.

The space $W_0^1E_M(\Omega,\rho)$ (resp., $W_0^1L_M(\Omega,\rho)$) is defined as the closure of $D(\Omega)$ in $W^1E_M(\Omega,\rho)$ (resp., $W^1L_M(\Omega,\rho)$) in the norm (resp., in the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$).

Let $W^{-1}L_{\overline{M}}(\Omega,\rho)$ (resp., $W^{-1}E_{\overline{M}}(\Omega,\rho)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega,\rho)$ (resp., $E_{\overline{M}}(\Omega,\rho)$). It is a Banach space under the usual quotient norm (see [4]). If the open set Ω has the segment property, then the space $C_0^\infty(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$.

3 Basic assumptions and fundamental lemmas

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, M, P be two N-functions such that $P \ll M$, $\overline{M}, \overline{P}$ be the complementary functions of M, P, respectively, $A\colon D(A) \subset W_0^1L_M(\Omega,\rho) \to W^{-1}L_{\overline{M}}(\Omega\rho)$ be a mapping (not everywhere defined) given by $A(u) = -\operatorname{div}(\rho(x)a(x,u,\nabla u)) + a_0(x,u,\nabla u)$, where $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ and $a_0\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are Carathéodory's functions. The following lemmas will be applied to the truncation operators and concern operators of the Nemytskii type in Orlicz spaces.

Lemma 1 *Let $f_n, f \in L^1(\Omega)$ be such that*:

1. $f_n \ge 0$ *a.e. in* Ω;
2. $f_n \to f$ *a.e. in* Ω;
3. $\int_\Omega f_n(x)\,\mathrm{d}x \to \int_\Omega f_n(x)\,\mathrm{d}x$.

Then $f_n \to f$ strongly in $L^1(\Omega)$.

Lemma 2 *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega, \rho)$ (resp., $W^1E_M(\Omega, \rho)$). Then $F(u) \in W^1L_M(\Omega, \rho)$ (resp., $W^1E_M(\Omega, \rho)$). Moreover, if the set of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

We will also use the following technical lemmas.

Lemma 3 *If a sequence u_n converges a.e. to u and if u_n remains bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \to u$ for $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 4 *If a sequence u_n converges a.e. to u and if u_n remains bounded in $L_M(\Omega, \rho)$, then $u \in L_M(\Omega, \rho)$ and $u_n \to u$ for $\sigma(L_M(\Omega, \rho), E_{\overline{M}}(\Omega, \rho))$.*

Lemma 5 *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N-functions such that $Q \ll P$, and let F be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|F(x, s)| \le c(x) + k_1 P^{-1} M(k2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_F defined by $N_F(u)(x) = F(x, u(x))$ is strongly continuous from $P(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

3.1 Compactness results

Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ a weight function, and M an N-function such that the following assumptions (H_1)–(H_3) are satisfied with some real $s > 0$:

(H_1) $(M(t))^{\frac{s}{s+1}}$ is an N-function and $\rho^{-s} \in L^1(\Omega)$;

(H_2) $\displaystyle\int_1^\infty \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}}\, \mathrm{d}M(t) = \infty$;

(H_3) $\displaystyle\lim_{t\to\infty} \frac{1}{M^{-1}(t)} \int_0^{t^{\frac{s+1}{s}}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}}\, \mathrm{d}u = 0.$

Remark 1 *In the particular case where $M(t) = \frac{t^p}{p}$ $(1 < p < \infty)$ the first part of* (H_1) *is satisfied if $s > \frac{1}{p-1}$.*

Theorem 1 (see [2, Theorem 9-5]) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary and M an N-function. Suppose that the assumptions* (H_1)–(H_3) *are satisfied. Then we have the following compact injection: $W^1L_M(\Omega, \rho) \hookrightarrow\hookrightarrow E_M$.*

Theorem 2 (Weighted Poincaré inequality) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ a weight function, and M an N-function. If $u \in W_0^1 L_M(\Omega)$, then*

$$\|u\|_M \leq c\|\nabla u\|_{M,\rho},$$

where c is a positive constant, which implies that $\|\nabla u\|_{M,\rho}$ and $\|u\|_{1,M}$ are equivalent norms on $W_0^1 L_{M,\rho}$.

Proof. Under the assumptions (H_1)–(H_3), the Sobolev conjugate N-function M_s^* of M_s is well-defined by

$$M_s^{*-1} = \int_0^s \frac{M^{-1}(t)}{t^{1+\frac{1}{N}}}\,\mathrm{d}t$$

and we have $W_0^1 L_{M_s} \subset L_{M_s^*}$. And since $M \ll M_s^*$, we have $L_{M_s^*} \subset L_M$. Hence

$$\|u\|_M \leq c_1\|u\|_{M_s^*} \leq c_2\|u\|_{1,M_s},$$

where c_1 and c_2 are two positive constants. Then, by using the Poincaré inequality in the non-weighted Orlicz–Sobolev space, there exists a positive constant c' such that

$$\|u\|_{1,M_s} \leq c'\|\nabla u\|_{M_s}.$$

We will show that

$$\|\nabla u\|_{M_s} \leq c\|\nabla u\|_{M,\rho}.$$

For that we have

$$\begin{aligned}\|v\|_{M_s} \leq \int_\Omega M_s(v(x))\,\mathrm{d}x + 1 &= \int_\Omega M_s(v(x))\frac{1}{\rho(x)}\rho(x)\,\mathrm{d}x + 1\\ &\leq \int_\Omega S(M_s(v(x)))\rho(x)\,\mathrm{d}x + \int_\Omega \frac{1}{\rho(x)}\rho(x)\,\mathrm{d}x + 2\\ &= \int_\Omega M(v(x))\rho(x)\,\mathrm{d}x + \int_\Omega \rho^{-s}(x)\,\mathrm{d}x + 1,\end{aligned}$$

which implies that

$$\|v\|_{M_s} \leq c\|v\|_{M,\rho}$$

for some positive constant c. In fact, if this is not true, then there exists a sequence v_n such that $\|v_n\|_{M_s} \to \infty$ and for n large, $\|v_n\|_{M,\rho} \leq 1$. Hence, for n sufficiently large we get

$$\int_\Omega M(v_n(x))\rho(x)\,\mathrm{d}x \leq \|v_n\|_{M,\rho} \leq 1.$$

Then

$$\begin{aligned}\|v_n\|_{M_s} &\leq \int_\Omega M(v(x))\rho(x)\,\mathrm{d}x + \int_\Omega \rho^{-s}(x)\,\mathrm{d}x + 1\\ &\leq \|v_n\|_{M,\rho} + \int_\Omega \rho^{-s}(x)\,\mathrm{d}x + 1,\end{aligned}$$

which is a contradiction, since the left hand-side tends to infinity while the right hand-side is bounded. Finally, by taking $v = \nabla u$, we conclude the result. □

4 Main results

4.1 Statement of the problem

We begin by introducing some imposed conditions and the formulation of our problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, M, P be two N-functions such that $P \ll M$, $\overline{M}, \overline{P}$ be the complementary functions of M, P, respectively, and let $A\colon D(A) \subset W_0^1 L_M(\Omega, \rho) \to W^{-1} L_{\overline{M}}(\Omega \rho)$ be a mapping (not everywhere defined) given by $A(u) = -\operatorname{div}(\rho(x) a(x, u, \nabla u)) + a_0(x, u, \nabla u)$, where $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory's functions satisfying for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$ the following conditions:

$$|a_0(x, s, \xi)| \le K_0[g_0(x) + \overline{M}^{-1} M(\alpha_2 s) + \overline{M}^{-1}(\rho(x) P(\alpha_1 |\xi|))], \tag{4.1}$$

$$|a(x, s, \xi)| \le C_0(x) + K_1 \overline{P}^{-1}(\rho^{-1} M(\alpha_2 s)) + K_2 \overline{M}^{-1} M(\alpha_1 |\xi|), \tag{4.2}$$

$$[a(x, s, \xi) - a(x, s, \eta)][\xi - \eta] > 0, \tag{4.3}$$

$$a_0(x, s, \xi) s + \rho(x) a(x, s, \xi) \xi \ge \lambda_0 [M(\lambda_1 s) + \rho(x) M(\lambda_2 |\xi|)], \tag{4.4}$$

where $\alpha_1, \alpha_2, K_0, K_1, K_2, \lambda_0, \lambda_1, \lambda_2 > 0$. Let T_k be the truncation operator at height $k \ge 0$ defined by the formula

$$T_k(s) = \max(-k, \min(k, s)) \qquad \text{for all } s \in \mathbb{R} \text{ and for all } k \ge 0.$$

And consider the following nonlinear elliptic problem with Dirichlet boundary condition

$$\begin{aligned} A(u) + g(x, u, \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.5}$$

where $g\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory's function which for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$ satisfies

(G_1) the sign condition: $g(x, s, \xi) s \ge 0$;

(G_2) the growth condition: $|g(x, s, \xi)| \le b(|s|)(c(x) + \rho(x) M(\lambda_2 |\xi|))$;

(G_3) the coercivity condition: $|g(x, s, \xi)| \ge \beta \rho(x) M\big(\frac{|\xi|}{\lambda_3}\big)$ for $|s| \ge \gamma$,

where $b\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function and c is a given non-negative function in L^1.

Finally, we assume that

$$f \in L^1(\Omega). \tag{4.6}$$

Let us first define the entropy solution of our problem.

Definition 1 *An entropy solution of the problem* (4.5) *is a measurable function* $u \in W_0^1 L_M(\Omega, \rho)$ *such that*

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(\varphi - u) \rho(x)\, \mathrm{d}x + \int_\Omega a_0(x, u, \nabla u) T_k(\varphi - u)\, \mathrm{d}x + \int_\Omega g(x, u, \nabla u) T_k(\varphi - u)\, \mathrm{d}x = \int_\Omega f T_k(\varphi - u)\, \mathrm{d}x$$

for all $k \ge 0$ *and* $\varphi \in W_0^1 E_M(\Omega, \rho) \cap L^\infty(\Omega)$.

4.2 Existence theorem

Let us prove the following existence theorem.

Theorem 3 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary. Moreover, assume that* (4.1)–(4.4)*,* (G_1) *and* (G_2) *hold, and that $f \in L^1(\Omega)$. Then there exists at least one entropy solution of the problem* (4.5)*.*

In order to prove this existence result we proceed in several steps. In the first one we use the pseudo-monotonicity to prove an existence result for a variational approximation problem, before we use some tools of compactness to pass to the limit. For this let us show the following intermediate result.

Theorem 4 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary and let M be an N-function. Suppose that the assumptions* (4.1)–(4.4) *are satisfied. Let (u_n) be a sequence such that*

(i) $a(x, T_k(u_n), \nabla T_k(u_n))$ *remains bounded in* $(L_{\overline{M}}(\Omega, \rho))^N$;

(ii) $\displaystyle\int_{\Omega_r} [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)\,\mathrm{d}x \to 0$;

(iii) $u_n \rightharpoonup u$ *weakly in* $W_0^1 L_M(\Omega, \rho)$ *for* $\sigma\left(\prod L_{M,\rho}, \prod E_{\overline{M},\rho}\right)$;

where $\Omega_r = \{x \in \Omega : |\nabla u_n| \leq r\}$ and χ_r denote the characteristic functions of the sets Ω_r. Then

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

Proof. Fix $r > 0$ and let $s > r$. By the monotonicity condition (4.3) we have

$$\int_{\Omega_r} [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)\,\mathrm{d}x \geq 0, \tag{4.7}$$

$$\int_{\Omega_s} [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla u)][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)\,\mathrm{d}x \geq 0. \tag{4.8}$$

Then

$$\int_{\Omega} [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \chi_s \nabla T_k(u))][T_k(u_n) - \chi_s \nabla T_k(u)]\rho(x)\,\mathrm{d}x \geq 0, \tag{4.9}$$

which with the condition (iii) implies that

$$\int_{\Omega} [a(x, u_n, T_k(u_n)) - a(x, u_n, \chi_s \nabla T_k(u))][T_k(u_n) - \chi_s \nabla T_k(u)]\rho(x)\,\mathrm{d}x \to 0.$$

Passing to a subsequence, we have

$$[a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x) \to 0$$

a.e. in Ω_r, for a subsequence still denoted by u_n. Fix $x \in \Omega \backslash R$ with $|R| = 0$. By (4.1) and (4.3) one has

$$\begin{aligned}&[a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x) \geq \\ &\quad \geq \lambda_0 M(\lambda\,|\nabla T_k(u_n)|)\rho(x) - C_3\left[1 + |\nabla T_k(u_n)| + \overline{M}^{-1} M(\alpha_1\,|\nabla T_k(u_n)|)\right] + C_4\end{aligned} \tag{4.10}$$

for some positive constants C_3 and C_4, which implies that $\nabla T_k(u_n)$ is bounded in $\mathbb{R}^N$.

Indeed, suppose that there exists a subsequence denoted again by $\nabla T_k(u_n(x))$ such that $\nabla T_k(u_n(x)) \to \infty$ as $n \to \infty$. Writing (4.10) in the form

$$\begin{aligned}&[a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x) \geq \\ &\quad \leq M(\lambda\,|\nabla u_n|)\left[\lambda_0\rho(x) - C_3\left(\frac{1 + |\nabla T_k(u_n)|}{M(\lambda\,|\nabla T_k(u_n)|)} + \frac{\overline{M}^{-1} M(c_1\,|\nabla T_k(u_n)|}{M(\lambda\,|\nabla T_k(u_n)|)}\right)\right] + C_4,\end{aligned}$$

yields a contradiction, since the right hand-side converges to infinity while the left hand-side tends to zero as $n \to \infty$.

Then, for a subsequence $u_{n_p}(x)$, we have $\nabla T_k(u_{n_p})(x) \to \xi \in \mathbb{R}^N$, and

$$[a(x, u_{n_p}, \nabla T_k(u_{n_p})) - a(x, u_{n_p}, \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)$$

tends to $[a(x, u, \xi) - a(x, u, \nabla T_k(u))][\xi - \nabla T_k(u)]\rho(x)$ as $n_p \to \infty$. Hence

$$[a(x, u, \xi) - a(x, u, \nabla T_k(u))][\xi - \nabla T_k(u)]\rho(x) = 0.$$

Consequently, we get that $\nabla T_k(u) = \xi$, and thus $\nabla T_k(u_n(x)) \to \nabla T_k(u(x))$. Since n and k are arbitrary we can construct a subsequence such that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{4.11}$$

This completes the proof. $\square$

4.3 Proof of the existence Theorem 3

As already mentioned, we will divide the proof of Theorem 3 into several steps.

Step 1. Variational approximated problem

Let f_n be a smooth function which converges to $f \in L^1(\Omega)$ such that $\|f_n\|_{L^1(\Omega)} \leq c_0$. And let us define

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n} g(x, s, \xi)}.$$

We consider the approximated problems

$$\begin{aligned}&u_n \in W_0^1 L_M(\Omega, \Omega), \\ &-\operatorname{div}(\rho(x) a(x, u_n, \nabla u_n)) + a_0(x, u_n, \nabla u_n) + g_n(x, u_n, \nabla u_n) = f_n.\end{aligned} \tag{4.12}$$

For a fixed n let us define the operator G_n by

$$\langle G_n u, v\rangle = \int_\Omega g_n(x, u_n, \nabla u_n) v \,\mathrm{d}x.$$

For our purposes, we can show as in [3] that the operator $T_n = A + G_n$ is finitely continuous, pseudo-monotone and coercive, and by using the theory of pseudo-monotone operators (see [11, 12]) the problem (4.5) has at least one solution u_n in $W_0^1 L_M(\Omega, \rho)$.

Step 2. A priori estimate

For $k > 0$, by taking $T_k(u_n)$ as a test function in (4.12), one has

$$\int_\Omega a(x, u_n, \nabla T_k(u_n))\nabla T_k(u_n)\rho(x)\,\mathrm{d}x + \int_\Omega a_0(x, u_n, \nabla u_n) T_k(u_n)\,\mathrm{d}x \\ + \int_\Omega g(x, u_n, \nabla u_n) T_k(u_n)\,\mathrm{d}x = \langle f_n, T_k(u_n)\rangle.$$

Since u_n and $T_k(u_n)$ have the same sign, by the sign condition (G_1) we have

$$g(x, u_n, \nabla u_n) T_k(u_n) \geq 0.$$

In view of the degenerate ellipticity condition and the fact that

$$\|f_n\|_{L^1(\Omega)} \leq c_0,$$

we get

$$\int_{\{|u_n|<k\}} M(\lambda_2 |\nabla T_k(u_n)|)\rho(x)\,\mathrm{d}x + \int_{\{|u_n|<k\}} M(\lambda_1 |u_n|)\,\mathrm{d}x \\ + \int_{\{|u_n|>k\}} |g(x, u_n, \nabla u_n)|\,\mathrm{d}x \leq \frac{c_0 k}{\lambda_0}.$$

By the condition (G_3) we deduce that

$$\lambda_0 \int_{\{|u_n|<k\}} M(\lambda_2 |\nabla T_k(u_n)|)\rho(x)\,\mathrm{d}x + \beta k \int_{\{|u_n|>k\}} M\left(\frac{\nabla |u_n|}{\lambda_3}\right)\mathrm{d}x \leq c_0 k. \tag{4.13}$$

Therefore

$$\int_\Omega M\left(\frac{|\nabla u_n|}{\lambda}\right)\rho(x)\,\mathrm{d}x \leq C', \tag{4.14}$$

where $\lambda = \max(\frac{1}{\lambda_2}, \lambda_3)$ and $C' = \frac{c_0 k}{\min(\lambda_0, \beta k)}$.

Thanks to Theorem 2, u_n is bounded in $W_0^1 L_M(\Omega, \rho)$. Then by Lemma 3, there exists some measurable function u such that

$$u_n \to u \text{ almost everywhere in } \Omega. \tag{4.15}$$

Then

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^1 L_M(\Omega, \rho) \text{ for } \sigma\left(\prod L_{M,\rho}, \prod E_{\overline{M},\rho}\right). \tag{4.16}$$

And by Theorem 1 we deduce that

$$u_n \to u \quad \text{strongly in } E_M(\Omega). \tag{4.17}$$

Step 3. Boundedness of $a(x, u_n, \nabla T_k(u_n))$ and $a_0(x, u_n, \nabla u_n)$

In this step we will shows that $a(x, u_n, \nabla T_k(u_n))$ remains bounded in $(L_{\overline{M}}(\Omega, \rho))^N$. We will use the Orlicz norm. For that let $\psi \in (L_M(\Omega))^N$ with $\|\psi\|_M \leq 1$. In fact, by the monotonicity condition (4.3) we have

$$\int_\Omega [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \psi)][\nabla T_k(u_n) - \psi]\rho(x)\,\mathrm{d}x \geq 0.$$

So that

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla T_k(u_n))\psi\rho(x)\,\mathrm{d}x \leq &\int_\Omega a(x, u_n, \nabla T_k(u_n))\nabla T_k(u_n)\rho(x)\,\mathrm{d}x \\ &- \int_\Omega a(x, u_n, \psi)\nabla T_k(u_n)\rho(x)\,\mathrm{d}x \\ &+ \int_\Omega a(x, u_n, \psi)\psi\rho(x)\,\mathrm{d}x.\end{aligned}$$

By (4.7) we have

$$\int_\Omega a(x, u_n, \nabla T_k(u_n))\nabla T_k(u_n)\rho(x)\,\mathrm{d}x \leq c_0 k.$$

To estimate the second and third term we use Young's inequality. Hence

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla T_k(u_n))\psi\rho(x)\,\mathrm{d}x \leq &c_0 k + 2\int_\Omega \overline{M}\left(\frac{|a(x, u_n, \psi|}{r}\right)\rho(x)\,\mathrm{d}x \\ &+ \int_\Omega M(r|\nabla T_k(u_n)|)\rho(x)\,\mathrm{d}x + \int_\Omega M(r|\psi|)\rho(x)\,\mathrm{d}x,\end{aligned}$$

where $r > 0$. Using the growth condition (4.2) and the fact that $P \ll M$ we conclude for r large and ε small that

$$\begin{aligned}\int_\Omega \overline{M}\left(\frac{|a(x, u_n, \psi|)}{r}\right)\rho(x)\,\mathrm{d}x \leq &\frac{1}{r}\int_\Omega \overline{M}(C_0(x))\rho(x)\,\mathrm{d}x + \frac{\varepsilon K_1}{r}\int_\Omega M(\alpha_2 T_k(u_n))\,\mathrm{d}x \\ &+ \frac{K_2}{r}\int_\Omega M(\alpha_1|\psi|)\,\mathrm{d}x + K'_\varepsilon\end{aligned}$$

for some positive constant K_ε. Since u_n is bounded in $W_0^1 L_M(\Omega, \rho)$ and ψ is bounded in $(L_M(\Omega))^N$, we get $\int_\Omega \overline{M}(|a(x, u_n, \psi)|)\rho(x)\,\mathrm{d}x \leq C_k$ for all $\psi \in (L_M(\Omega,))^N$ with $\|\psi\|_{M,} \leq 1$. Therefore, we deduce that $a(x, u_n, \nabla T_k(u_n))$ remains bounded in $(L_{\overline{M}}(\Omega, \rho))^N$.

Now, let us prove that $a_0(x, u_n, \nabla u_n)$ is bounded in $(L_{\overline{M}}(\Omega))$. First, using the growth condition (4.1) it follows that for λ large

$$\begin{aligned}\int_\Omega \overline{M}\left(\frac{|a_0(x, u_n, \nabla u_n|}{\lambda}\right)\mathrm{d}x \leq &\frac{1}{2}\int_\Omega \overline{M}\left(\frac{2}{\lambda}\right)K_0|g_0(x)|\,\mathrm{d}x \\ &+ \frac{K_0}{\lambda}\int_\Omega M(\alpha_2 u_n)\,\mathrm{d}x \\ &+ \frac{K_0}{\lambda}\int_\Omega P(\alpha_1|\nabla u_n|)\rho(x)\,\mathrm{d}x.\end{aligned}$$

Since $P \ll M$, for λ large and ε small we get

$$\int_\Omega \overline{M}\left(\frac{a(x,u_n,\nabla u_n)}{\lambda}\right)\mathrm{d}x \leq \frac{1}{2}\int_\Omega \overline{M}\left(\frac{2}{\lambda}\right)K_0|g_0(x)|\,\mathrm{d}x + \frac{K_0}{\lambda}\int_\Omega M(\alpha_2 u_n)\,\mathrm{d}x + \frac{K_0}{\lambda}\int_\Omega M(\varepsilon|\nabla u_n|)\rho(x)\,\mathrm{d}x + \frac{k_\varepsilon}{\lambda} \leq 1$$

for some positive constant k_ε, which implies that $a_0(x,u_n,\nabla u_n)$ is bounded in $L_{\overline{M}}(\Omega)$.

Step 4. Almost everywhere convergence of the gradient

In this step we prove that $\nabla u_n \to \nabla u$ a.e. in Ω for a subsequence.

Let $\varphi(s) = se^{\gamma s^2}$ and $\gamma = (\frac{b(k)}{\lambda_0})^2$. It is well-known that

$$\varphi^{'}(s) - \frac{b(k)}{\lambda_0}|\varphi(s)| > \frac{1}{2} \quad \text{for all } s \in \mathbb{R}$$

(see [13]). For $k > 0$ and fixed n we take $v_n = \varphi_\gamma(z_n)$ with $z_n = T_k(u_n) - T_k(u)$ as a test function in (4.12). One has

$$\langle Bu_n, v_n\rangle + \int_\Omega g(x,u_n,\nabla u_n)v_n\,\mathrm{d}x = \langle f_n, v_n\rangle - \int_\Omega a_0(x,u_n,\nabla u_n)v_n\,\mathrm{d}x, \tag{4.18}$$

where

$$\langle Bu_n, v_n\rangle = \int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(z_n)\rho(x)\,\mathrm{d}x.$$

Since $v_n \in W_0^1E_M(\Omega,\rho) \cap L^\infty(\Omega)$, and $v_n \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$, and $f_n \to f$ strongly in $L^1(\Omega)$, and $a_0(x,u_n,\nabla(u_n))$ is bounded in $(L_{\overline{M}}(\Omega,\rho))^N$, then $\langle f_n, v_n\rangle \to 0$ and $\int_\Omega a_0(x,u_n,\nabla u_n)v_n\,\mathrm{d}x \to 0$.

We can see that $T_k(u_n) - T_k(u)$ and u has the same sign on $\{x \in \Omega : |u_n(x)| > k\}$. Then

$$\int_{\{x\in\Omega:|u_n(x)|>k\}} g(x,u_n,\nabla u_n)v_n\,\mathrm{d}x \geq 0.$$

One has

$$\langle Bu_n, v_n\rangle + \int_{\{x\in\Omega:|u_n(x)|\leq k\}} g(x,u_n,\nabla u_n)v_n\,\mathrm{d}x \leq \varepsilon_1(n), \tag{4.19}$$

where in the sequel $\varepsilon_i(n)$, $i = 1, 2, \ldots$, are sequences of real numbers which converge to zero as n tends to infinity.

Let $\Omega_r = \{x \in \Omega : |\nabla T_k(u_n)| \leq r\}$ and let χ_r denote the characteristic functions of the sets Ω_r. We have $\Omega_r \subset \Omega_{r+1}$ and $|\Omega_r \backslash \Omega_{r+1}| \to 0$ as $r \to +\infty$. Fix $r > 0$ and let $s > r$. By the monotonicity condition (4.3) we have

$$\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)\,\mathrm{d}x \geq 0, \tag{4.20}$$

$$\int_{\Omega_s} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)\,\mathrm{d}x \geq 0. \tag{4.21}$$

Then

$$\int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\chi_s \nabla T_k(u))][\nabla T_k(u_n) - \chi_s \nabla T_k(u)]\rho(x)\,\mathrm{d}x \geq 0. \tag{4.22}$$

On the other hand we have

$$\begin{aligned}\langle Bu_n, v_n\rangle = &\int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)\,\mathrm{d}x\\ &- \int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla T_k(u_n)]\nabla T_k(u)\rho(x)\,\mathrm{d}x\\ &- \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\rho(x)\,\mathrm{d}x\\ &+ \int_\Omega a(x,u_n,\nabla T_k(u_n)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)\,\mathrm{d}x,\end{aligned}$$

which can be written as

$$\langle Bu_n, v_n\rangle = I_n - I_n^1 - I_n^2 + I_n^3, \tag{4.23}$$

where

$$\begin{aligned} I_n &= \int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)\,\mathrm{d}x.\\ I_n^1 &= \int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla T_k(u_n)]\nabla T_k(u)\rho(x)\,\mathrm{d}x.\\ I_n^2 &= \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\rho(x)\,\mathrm{d}x.\\ I_n^3 &= \int_\Omega a(x,u_n,\nabla T_k(u_n)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)\,\mathrm{d}x.\end{aligned}$$

Now, we will use the following lemma which is proved in the Appendix.

Lemma 6 *For the integrals I_n^1, I_n^2, I_n^3 defined above we have*:

(i) $I_n^1 \to 0$;

(ii) $I_n^2 \to \int_{\Omega\setminus\Omega_s} h\nabla T_k(u)\rho(x)\,\mathrm{d}x$;

(iii) $I_n^3 \to \int_{\Omega\setminus\Omega_s} a(x,u,0)\nabla T_k(u)\rho(x)\,\mathrm{d}x$.

Using (4.23) and Lemma 6 we get

$$\langle Bu_n, v_n\rangle = I_n + \int_{\Omega\setminus\Omega_s} (h - a(x,u,0))\nabla T_k(u)\rho(x)\,\mathrm{d}x + \varepsilon_2(n). \tag{4.24}$$

On the other hand, thanks to the growth condition (G_2) we have

$$\begin{aligned}\left|\int_{\{x\in\Omega:|u_n(x)|\le k\}} g_n(x,u_n,\nabla u_n)v_n\,\mathrm{d}x\right| &\\ \le \int_{\{x\in\Omega:|u_n(x)|\le k\}} & b(k)\left(c(x)+M(\lambda_2\,|\nabla u_n|)\rho(x)\right)|v_n|\;\mathrm{d}x\\ &\le b(k)\int_\Omega c(x)\,|v_n|\;\mathrm{d}x+b(k)\int_\Omega M(\lambda_2\,|\nabla T_k u_n|)\,|v_n|\,\rho(x)\,\mathrm{d}x\\ &\le b(k)\int_\Omega M(\lambda_2\,|\nabla T_k u_n|)\,|v_n|\,\rho(x)\,\mathrm{d}x+\varepsilon_3(n).\end{aligned}$$

And by the coercivity condition (4.4) we can see that

$$\begin{aligned}\lambda_0\int_\Omega M(\lambda_2\,|\nabla T_k u_n|)\,|v_n|\,\rho(x)\,\mathrm{d}x &\le \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)\,|v_n|\,\rho(x)\,\mathrm{d}x\\ &\quad+\int_\Omega a_0(x,u_n,\nabla T_k(u_n))T_k(u_n)\,|v_n|\,\rho(x)\,\mathrm{d}x.\end{aligned}$$

Hence we can estimate

$$\begin{aligned}&\left|\int_{\{x\in\Omega:|u_n(x)|\le k\}} g_n(x,u_n,\nabla u_n)v_n\,\mathrm{d}x\right|\\ &\quad\le \frac{b(k)}{\lambda_0}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_s\,|v_n|\,\rho(x)\,\mathrm{d}x\\ &\quad+\frac{b(k)}{\lambda_0}\int_\Omega \left[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)\right]\left[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\right]|v_n|\,\rho(x)\,\mathrm{d}x\\ &\quad+\frac{b(k)}{\lambda_0}\int_\Omega a(x,u_n,\nabla T_k(u)\chi_s)\left[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\right]|v_n|\,\rho(x)\,\mathrm{d}x+\varepsilon_4(n)\\ &\quad+\frac{b(k)}{\lambda_0}\int_\Omega a_0(x,u_n,\nabla T_k(u_n))T_k(u_n)\,|v_n|\,\rho(x)\,\mathrm{d}x.\end{aligned}$$

The first term on the right-hand side above tends to 0, since $a(x,u_n,\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega,\rho))^N$ and $\nabla T_k(u)\chi_s\,|v_n|\,\rho(x)\to 0$ a.e. in Ω. We have that $a(x,u_n,\nabla T_k(u_n))\,|v_n|\,\rho(x)$ tends to 0 in $(E_{\overline{M}}(\Omega,\rho))^N$ and that $\nabla T_k(u_n)-\nabla T_k(u)\chi_s$ is bounded in $(L_M(\Omega,\rho))^N$. Then the third term tends to 0. The last term tends to 0 as above. Hence we get

$$\begin{aligned}&\left|\int_{\{x\in\Omega:|u_n(x)|\le k\}} g_n(x,u_n,\nabla u_n)v_n\,\mathrm{d}x\right|\\ &\quad\le \varepsilon_5(n)+\frac{kb(k)}{\lambda_0}\int_\Omega\left[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)\right]\times\\ &\qquad\qquad\times\left[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\right]|v_n|\,\rho(x)\,\mathrm{d}x.\end{aligned}\tag{4.25}$$

By (4.24) and (4.25) we deduce that

$$\begin{aligned}&\int_\Omega[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)]\times\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\left(\varphi'(z_n)-\frac{b(k)}{\lambda_0}\right)|\varphi(z_n)|\rho(x)\,\mathrm{d}x\\ &\quad\le\int_{\Omega\setminus\Omega_s}(h-a(x,u,0))\nabla T_k(u)\rho(x)+\varepsilon_6(n).\end{aligned}$$

Since

$$\varphi'(z_n) - \frac{b(k)}{\lambda_0}|\varphi(z_n)| \geq \frac{1}{2},$$

we obtain

$$\int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]$$
$$\leq 2\varepsilon_6(n) + 2\int_{\Omega\backslash\Omega_s}(h - a(x,u,0))\nabla T_k(u)\rho(x).$$

This implies that

$$\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]$$
$$\leq 2\varepsilon_6(n) + 2\int_{\Omega\backslash\Omega_s}(h - a(x,u,0))\nabla T_k(u)\rho(x).$$

Finally, by passing to the limit with n and letting $s \to \infty$, since $\mathrm{meas}(\Omega\backslash\Omega_s) \to 0$, we get

$$\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)\,\mathrm{d}x \to 0. \tag{4.26}$$

So, by Theorem 4, we get

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{4.27}$$

Step 5. Strong convergence of $g_n(x,u_n,\nabla u_n)$

In this step we prove that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega).$$

To this purpose let us show that $g_n(x,u_n,\nabla u_n)$ is equi-integrable. First, we take $T_{l+1}(u_n) - T_l(u_n)$ as a test function in (4.12) and we get

$$\int_{\{|u_n|>l+1\}} |g_n(x,u_n,\nabla u_n)|\,\mathrm{d}x \leq \int_{\{|u_n|>l\}} |f_n(x)|\,\mathrm{d}x.$$

Let ε be fixed. Then there exists $l_\varepsilon \geq 1$ such that

$$\int_{\{|u_n>l+1|\}} |g_n(x,u_n,\nabla u_n)|\,\mathrm{d}x \leq \frac{\varepsilon}{2}. \tag{4.28}$$

Let E be a measurable subset of Ω. Then we have

$$\int_E |g_n(x,u_n,\nabla u_n)|\,\mathrm{d}x \leq \int_E b(l_\varepsilon)(c(x) + K\rho(x)M(\lambda_2\nabla T_{l(\varepsilon)}(u_n)))\,\mathrm{d}x$$
$$+ \int_{\{|u_n|>l(\varepsilon)\}} |g_n(x,u_n,\nabla u_n)|\,\mathrm{d}x.$$

In view of (4.17) there exists $\eta(\varepsilon)$ such that

$$\int_E b(l_\varepsilon)(c(x) + K\rho(x)M(\lambda_2\nabla T_{l(\varepsilon)}(u_n)))\,\mathrm{d}x \leq \frac{\varepsilon}{2} \tag{4.29}$$

for all E such that $\mathrm{meas}(E) < \eta(\varepsilon)$.

Finally, by combining (4.28) and (4.29) one has

$$\int_E |g_n(x, u_n, \nabla u_n)| \,\mathrm{d}x < \varepsilon$$

for all E such that $\mathrm{meas}(E) < \eta(\varepsilon)$. And thanks to Vitali's theorem we conclude that

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{strongly in} \quad L^1(\Omega). \tag{4.30}$$

Step 6. Passage to the limit

Going back to the approximate problems (4.12) and taking $v \in D(\Omega)$ as a test function we have

$$\int_\Omega a(x, u, \nabla u)\nabla v \rho(x) \,\mathrm{d}x + \int_\Omega a_0(x, u, \nabla u) v \,\mathrm{d}x + \int_\Omega g(x, u, \nabla u) v \,\mathrm{d}x = \langle f, v\rangle$$

for all $v \in W_0^1 E_M(\Omega, \rho) \cap L^\infty(\Omega)$. We have

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{strongly in} \quad L^1(\Omega).$$

Moreover, by (4.17) and (4.27) we deduce that $a(x, u_n, \nabla u_n) \to a(x, u, \nabla u)$ a.e. in Ω and $a_0(x, u_n, \nabla u_n) \to a_0(x, u, \nabla u)$ a.e. in Ω. Moreover, Lemmas 3 and 4 imply that $a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u)$ weakly in $(L_M(\Omega, \rho))^N$ for $\sigma(\prod L_{\overline{M}}(\Omega, \rho), \prod E_M(\Omega, \rho))$, and $a_0(x, u_n, \nabla u_n) \to a_0(x, u, \nabla u)$ weakly in $(L_M(\Omega))$ for $\sigma(L_{\overline{M}}(\Omega), E_M(\Omega))$. On the other hand, $f_n \to f$ strongly in $L^1(\Omega)$.

Finally, by using (4.27) and passing to the limit in the sequence of approximate problems (4.12), we obtain the existence result.

5 Appendix – the proof of Lemma 6

(i) First, we can write

$$I_n^1 = \int_\Omega [a(x, u_n, \nabla(u_n)) - a(x, u, 0)]\chi_{G_n}\chi_s \nabla T_k(u)\rho(x) \,\mathrm{d}x,$$

where $G_n = \{x \in \Omega : |u_n(x))| > k\}$. We have

$$M(|\chi_{G_n}\chi_s \nabla T_k(u)|)\rho(x) \le M(|\nabla T_k(u)\chi_s|)\rho(x) \in L^1(\Omega).$$

And we have $u_n(x) \to u(x)$ a.e. in Ω, hence if $|u(x)| < k$, then for n large $|u_n(x)| < k$, which implies that

$$|\nabla T_k(u)|\chi_{G_n}\chi_s \to 0 \text{ a.e. in } \Omega.$$

By the Lebesgue theorem we deduce that

$$M(|\nabla T_k(u)|\chi_{G_n}\chi_s)\rho(x) \to 0.$$

Thus, $\chi_s\chi_{G_n}\nabla T_k(u) \to 0$ strongly in $(E_M(\Omega, \rho))^N$ and since $a(x, u_n, \nabla(u_n))$ and $a(x, u, 0)$ are bounded in $(L_{\overline{M}}(\Omega, \rho))^N$, we see that $I_n^1 \to 0$.

(ii) We have that $a(x, u_n, \nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega, \rho))^N$. Then there exist $h \in L_{\overline{M}}(\Omega, \rho)^N$ and a subsequence, also denoted by $a(x, u_n, \nabla T_k((u_n)))$, such that $a(x, u_n, \nabla T_k((u_n))) \rightharpoonup h$ weakly in $(L_{\overline{M}}(\Omega, \rho))^N$. Passing to the limit in n, yields

$$I_n^2 \to \int_{\Omega \setminus \Omega_s} h \nabla T_k(u) \rho(x) \, \mathrm{d}x.$$

(iii) By (4.17) we have that $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $(L_M(\Omega, \rho))^N$ for $\sigma(\prod L_M(\Omega, \rho), \prod E_{\overline{M}}(\Omega, \rho))$. Therefore, $a(x, u_n, \nabla T_k(u_n)\chi_s) \to a(x, u_n, \nabla T_k(u)\chi_s)$ strongly in $(E_{\overline{M}}(\Omega, \rho))^N$. Then

$$I_n^3 \to \int_\Omega a(x, u, \nabla T_k(u)\chi_s)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\rho(x) \, \mathrm{d}x,$$

which shows our claim and ends the proof of Lemma 6.

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