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POSITIVE PSEUDO ALMOST PERIODIC SOLUTIONS FOR A HEMATOPOIESIS MODEL

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Abstract. This paper is concerned with the existence and stability of pseudo almost periodic solutions to a hematopoiesis model $x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)}{1+x^n(t-\tau_i(t))}$, $t \in \mathbb{R}$. We consider the case of a being pseudo almost periodic without the assumption of $\inf_{t\in\mathbb{R}} a(t) > 0$.

Keywords: Almost periodic, ergodic, hematopoiesis, pseudo almost periodic.

2010 Mathematics Subject Classification: 34K14.

1 Introduction

In [8], Mackey and Glass proposed the following nonlinear delay differential equation

$$h'(t) = -\alpha h(t) + \frac{\beta}{1 + h^n(t-\tau)} \tag{1.1}$$

as an appropriate model of hematopoiesis that describes the process of production of all types of blood cells generated by a remarkable self-regulated system that is responsive to the demands put upon it. In medical terms, $h(t)$ denotes the density of mature cells in blood circulation at time t and τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstream. It is assumed that the cells are lost from the circulation at a rate α, and the flux of the cells into the circulation from the stem cell compartment depends on the density of mature cells at the previous time $t - \tau$.

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In this paper, we consider the following hematopoiesis model:

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}, \quad t \in \mathbb{R}, \tag{1.2}$$

where $n > 0$, k is a positive integer, $a\colon \mathbb{R} \to \mathbb{R}$ is continuous, and $b_i, \tau_i\colon \mathbb{R} \to \mathbb{R}^+$ are all continuous functions for $i = 1, 2, \ldots, k$.

Recently, the existence of almost periodic solutions or pseudo almost periodic solutions for equation (1.2) and its variants have attracted much attention (see, *e.g.*, [1, 3, 6, 7, 9, 12, 13, 14] and references therein). Stimulated by these works, we aim to make further study on this topic. As one will see, there are two differences of our work from many earlier works on almost periodic type solutions to equation (1.2). The first difference is that we do not assume that $\inf_{t\in\mathbb{R}} a(t) > 0$ (we even do not assume that a is non-negative), which is assumed in many earlier results. In fact, this idea is stimulated by the interesting work [10], where the author Shao replaced the usual assumption $\inf_{t\in\mathbb{R}} a(t) > 0$ with some other assumptions, which allow $a(t)$ to be negative for some $t \in \mathbb{R}$. In this paper, we simplify the assumptions on a in [10] to $M(a) := \lim_{T\to+\infty} \frac{1}{T} \int_0^T a(t)\,\mathrm{d}t > 0$. The second difference is that we investigate the existence and stability of pseudo almost periodic solutions to equation (1.2) with $a(t)$ being pseudo almost periodic. In fact, to the best of our knowledge, it seems that until now there is no result concerning pseudo almost periodic solutions to equation (1.2) with $a(t)$ being pseudo almost periodic. So, we think it will be of interest for some colleagues to investigate the existence and stability of pseudo almost periodic solutions to equation (1.2) with $a(t)$ being pseudo almost periodic. That is the main motivation of this paper.

Throughout the rest of this paper, for every bounded function $f\colon \mathbb{R} \to \mathbb{R}$, we denote

$$f^+ = \sup_{t\in\mathbb{R}} f(t), \quad f^- = \inf_{t\in\mathbb{R}} f(t),$$

where $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ is the set of non-negative real numbers. We let $\tau = \max_{1\le i\le k} \tau_i^+$.

Next, let us recall some definitions and basic properties of almost periodic functions and pseudo almost periodic functions. For more details, we refer the reader to [4, 5, 11].

Definition 1 *A set $E \subset \mathbb{R}$ is called relatively dense if there exists a number $l > 0$ such that*

$$[a, a+l] \cap E \neq \emptyset$$

for every $a \in \mathbb{R}$.

Definition 2 *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$ the set*

$$P_\varepsilon := \left\{ \tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| < \varepsilon \right\}$$

is relatively dense. We denote the set of all such functions by $AP(\mathbb{R}, \mathbb{R})$.

Recall that for every $f \in AP(\mathbb{R}, \mathbb{R})$, the limit

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(t)\,\mathrm{d}t$$

exists. Throughout the rest of this paper, we denote

$$M(f)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t)\,\mathrm{d}t,\quad f\in AP(\mathbb{R},\mathbb{R}).$$

Also, by $PAP_0(\mathbb{R},\mathbb{R})$ we denote the set of all bounded and continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ with

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|\,\mathrm{d}t=0.$$

Definition 3 *A bounded and continuous function $f\colon\mathbb{R}\to\mathbb{R}$ is called pseudo almost periodic if there exist $g\in AP(\mathbb{R},\mathbb{R})$ and $h\in PAP_0(\mathbb{R},\mathbb{R})$ such that*

$$f(t)=g(t)+h(t),\quad t\in\mathbb{R}.$$

We denote the set of all such functions by $PAP(\mathbb{R},\mathbb{R})$. Moreover, we denote by $PAP(\mathbb{R},\mathbb{R}^+)$ the set of all non-negative pseudo almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 4 *A bounded and continuous function $f\colon\mathbb{R}\to\mathbb{R}$ is called ergodic if the limit*

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}f(t+s)\,\mathrm{d}s$$

exists uniformly with respect to $t\in\mathbb{R}$.

Lemma 1 *Let $f,g\in PAP(\mathbb{R},\mathbb{R})$. Then the following assertions hold*:

(a) $PAP(\mathbb{R},\mathbb{R})$ *is a Banach space under the norm* $\|f\|=\sup_{t\in\mathbb{R}}|f(t)|$;

(b) $f\cdot g\in PAP(\mathbb{R},\mathbb{R})$;

(c) $f/g\in PAP(\mathbb{R},\mathbb{R})$ *provided that* $\inf_{t\in\mathbb{R}}|g(t)|>0$.

2 Main results

We first present some lemmas, which are of importance in proving our main theorems.

Lemma 2 *Let $a\in PAP(\mathbb{R},\mathbb{R})$ be ergodic with $M(a)>0$. Then for every $\alpha\in(0,M(a))$, there exists $T_0>0$ such that for all $s,t\in\mathbb{R}$ with $s\le t$, there holds*

$$a^+(s-t)\le\int_t^s a(u)\,\mathrm{d}u\le\alpha(T_0+s-t).$$

Proof. It is clear that

$$a^+(s-t)\le\int_t^s a(u)\,\mathrm{d}u.$$

The second inequality has been proved in [2]. For the reader's convenience, we give the proof here.

Since $a \in PAP(\mathbb{R},\mathbb{R})$ is ergodic with $M(a) > \alpha > 0$, it follows from [11, p. 208, Lemma 1.5] that there exists $T > 0$ such that

$$\int_t^s a(u)\,\mathrm{d}u < \alpha(s-t)$$

for all $s, t \in \mathbb{R}$ with $s - t < -T$. On the other hand, we have

$$\int_t^s a(u)\,\mathrm{d}u \le \|a\| \cdot T \le \|a\| \cdot T + \alpha(T + s - t) = \alpha\left[T \cdot \left(\frac{\|a\|}{\alpha} + 1\right) + s - t\right]$$

for all $s, t \in \mathbb{R}$ with $-T \le s - t \le 0$. Then, taking $T_0 = T \cdot \left(\frac{\|a\|}{\alpha} + 1\right)$, the conclusion follows. □

Lemma 3 ([11, p. 214, Lemma 1.9]) *Let $a \in PAP(\mathbb{R},\mathbb{R})$ be ergodic with $M(a) > 0$ and $f \in PAP(\mathbb{R},\mathbb{R})$. Then the equation*

$$x'(t) = -a(t)x(t) + f(t) \tag{2.1}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^t e^{-\int_s^t a(r)\,\mathrm{d}r} f(s)\,\mathrm{d}s. \tag{2.2}$$

The following lemma is a direct corollary of [11, Theorem 5.11].

Lemma 4 *Let $x \in PAP(\mathbb{R},\mathbb{R})$ be uniformly continuous on $\mathbb{R}$ and let $\tau \in PAP(\mathbb{R},\mathbb{R})$. Then $x(\cdot - \tau(\cdot)) \in PAP(\mathbb{R},\mathbb{R})$.*

Let $x(t,\varphi)$ be the solution of equation (1.2) with the initial condition $x(t) = \varphi(t),\ t \in [-\tau, 0]$, and by $[0, \eta_\varphi)$ denote the (right) maximal interval of existence of $x(t,\varphi)$.

Lemma 5 *Let $a \in PAP(\mathbb{R},\mathbb{R})$ be ergodic with $M(a) > \alpha > 0$ and let $\inf_{t\in[-\tau,0]} \varphi(t) \ge 0$. Then, $\eta_\varphi = +\infty$ and $x(t,\varphi)$ is non-negative on $[0, +\infty)$.*

Proof. For simplicity, we denote $x(t) = x(t,\varphi)$ if there is no confusion. From (1.2), we have

$$x(t) = e^{-\int_0^t a(u)\,\mathrm{d}u} x(0) + \int_0^t e^{-\int_s^t a(u)\,\mathrm{d}u} \sum_{i=1}^k \frac{b_i(s)}{1 + x^n(s - \tau_i(s))}\,\mathrm{d}s.$$

In view of the fact that $\inf_{t\in[-\tau,0]} \varphi(t) \ge 0$, we obtain that $x(t)$ is non-negative on $[0, \eta_\varphi)$. It suffices to show that $x(t)$ is bounded on $[0, \eta_\varphi)$.

It follows from Lemma 2 that

$$\begin{aligned}
|x(t)| &\le |e^{-\int_0^t a(u)\,\mathrm{d}u} x(0)| + \left|\int_0^t e^{-\int_s^t a(u)\,\mathrm{d}u} \sum_{i=1}^k \frac{b_i(s)}{1 + x^n(s - \tau_i(s))}\,\mathrm{d}s\right| \\
&\le e^{\alpha T_0} e^{-\alpha t} x(0) + \int_0^t e^{\alpha T_0} e^{-\alpha(t-s)} \sum_{i=1}^k b_i^+\,\mathrm{d}s \\
&= e^{\alpha T_0} e^{-\alpha t} x(0) + \frac{1}{\alpha} \cdot e^{\alpha T_0} \sum_{i=1}^k b_i^+ (1 - e^{-\alpha t}) \\
&\le e^{\alpha T_0} x(0) + \frac{1}{\alpha} \cdot e^{\alpha T_0} \sum_{i=1}^k b_i^+, \quad t \in [0, \eta_\varphi).
\end{aligned}$$

The conclusion then follows. □

Lemma 6 *Let $a \in PAP(\mathbb{R},\mathbb{R})$ be ergodic with $M(a) > \alpha > 0$ and let $\inf_{t\in[-\tau,0]}\varphi(t) \geq 0$. Suppose that there exist two positive constants κ and M such that*

$$M > \kappa, \quad -\alpha M + e^{\alpha T_0}\sum_{i=1}^{k} b_i^{+} < 0, \quad -a^{+}\kappa + \sum_{i=1}^{k}\frac{b_i^{-}}{1+M^n} > 0,$$

where T_0 is defined in Lemma 2. Then there exists $t_\varphi > 0$ such that $\kappa < x(t) = x(t,\varphi) < M$ for all $t \geq t_\varphi$.

Proof. Letting $p = \alpha M - e^{\alpha T_0}\sum_{i=1}^{k} b_i^{+}$, by Lemma 2 and Lemma 5, we have

$$\begin{aligned}
x(t) &= e^{-\int_0^t a(u)\,\mathrm{d}u}x(0) + \int_0^t e^{-\int_s^t a(u)\,\mathrm{d}u}\sum_{i=1}^{k}\frac{b_i(s)}{1+x^n(s-\tau_i(s))}\,\mathrm{d}s\\
&\leq e^{\alpha T_0}e^{-\alpha t}x(0) + \int_0^t e^{\alpha T_0}e^{-\alpha(t-s)}\sum_{i=1}^{k} b_i^{+}\,\mathrm{d}s\\
&= e^{\alpha T_0}e^{-\alpha t}x(0) + \int_0^t (\alpha M - p)e^{-\alpha(t-s)}\,\mathrm{d}s\\
&= e^{\alpha T_0}e^{-\alpha t}x(0) + \Big(M - \frac{p}{\alpha}\Big)(1-e^{-\alpha t})\\
&:= A(t), \quad t \in [0,+\infty),
\end{aligned}$$

Noting that $\lim_{t\to+\infty} A(t) = M - \frac{p}{\alpha} < M$, we know that there exists $t_0 \in [0,+\infty)$ such that

$$0 \leq x(t) < M \text{ for all } t \in [t_0,+\infty). \tag{2.3}$$

Letting $q = a^{+} - \kappa + \sum_{i=1}^{k}\frac{b_i^{-}}{1+M^n}$, again by Lemma 2 and Lemma 5, we have

$$\begin{aligned}
x(t) &= e^{-\int_{t_0+\tau}^t a(u)\,\mathrm{d}u}x(t_0+\tau) + \int_{t_0+\tau}^t e^{-\int_s^t a(u)\,\mathrm{d}u}\sum_{i=1}^{k}\frac{b_i(s)}{1+x^n(s-\tau_i(s))}\,\mathrm{d}s\\
&\geq e^{-\int_{t_0+\tau}^t a^{+}\,\mathrm{d}u}x(t_0+\tau) + \int_{t_0+\tau}^t e^{-\int_s^t a^{+}\,\mathrm{d}u}\sum_{i=1}^{k}\frac{b_i^{-}}{1+M^n}\,\mathrm{d}s\\
&= e^{-\int_{t_0+\tau}^t a^{+}\,\mathrm{d}u}x(t_0+\tau) + \int_{t_0+\tau}^t (a^{+}\kappa + q)e^{-a^{+}(t-s)}\,\mathrm{d}s\\
&= e^{-\int_{t_0+\tau}^t a^{+}\,\mathrm{d}u}x(t_0+\tau) + \Big(\kappa + \frac{q}{a^{+}}\Big)\Big(1 - e^{a^{+}(t_0+\tau-t)}\Big)\\
&:= B(t), \quad t \in [t_0+\tau,+\infty),
\end{aligned}$$

which, together with the fact that $\lim_{t\to+\infty} B(t) = \kappa + \frac{q}{a^{+}} > \kappa$, implies that there exists $t_\varphi > t_0+\tau$ such that $\kappa < x(t) < M$ for all $t \geq t_\varphi$. □

Before presenting our existence theorem, we list some assumptions:

(H0) $a \in PAP(\mathbb{R}, \mathbb{R})$ is ergodic with $M(a) > 0$, and $b_i, \tau_i \in PAP(\mathbb{R}, \mathbb{R}^+)$ with $b_i^- > 0$ for all $i = 1, 2, \ldots, k$.

(H1) There exist $\alpha \in (0, M(a))$ and two positive constants $M, \kappa > 0$ such that

$$M > \kappa, \quad -\alpha M + e^{\alpha T_0} \sum_{i=1}^{k} b_i^+ < 0, \quad -a^+\kappa + \sum_{i=1}^{k} \frac{b_i^-}{1 + M^n} > 0,$$

where T_0 is defined in Lemma 2.

(H2) $\frac{e^{\alpha T_0} \sum_{i=1}^{k} b_i^+}{\alpha} < \frac{4\kappa}{n}$.

Theorem 1 *Under the assumptions* (H0)–(H2) *equation* (1.2) *has a unique pseudo almost periodic solution in*

$$\Omega = \{\varphi \in PAP(\mathbb{R}, \mathbb{R}) \text{ is uniformly continuous on } \mathbb{R} : \kappa \leq \varphi(t) \leq M, \forall t \in \mathbb{R}\}.$$

Proof. Fix $\varphi \in \Omega$. Let us consider the following differential equation

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)}{1 + \varphi^n(t - \tau_i(t))}. \tag{2.4}$$

It follows from Lemma 4 that $\varphi(\cdot - \tau_i(\cdot)) \in PAP(\mathbb{R}, \mathbb{R})$. Then, by Lemma 1, we infer that

$$\sum_{i=1}^{k} \frac{b_i(\cdot)}{1 + \varphi^n(\cdot - \tau_i(\cdot))} \in PAP(\mathbb{R}, \mathbb{R}).$$

By Lemma 3, equation (2.4) has a unique pseudo almost periodic solution given by

$$x^{\varphi}(t) = \int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} \frac{b_i(s)}{1 + \varphi^n(s - \tau_i(s))}\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{2.5}$$

Now we define a mapping T on Ω by

$$(T\varphi)(t) = x^{\varphi}(t), \quad \varphi \in \Omega,\ t \in \mathbb{R}.$$

Next, we show that $T(\Omega) \subset \Omega$. For every $t \in \mathbb{R}$ and $\varphi \in \Omega$, by Lemma 2, we have

$$\begin{aligned}
(T\varphi)(t) &= x^{\varphi}(t) \\
&= \int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} \frac{b_i(s)}{1 + \varphi^n(s - \tau_i(s))}\,\mathrm{d}s \\
&\leq \int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} b_i(s)\,\mathrm{d}s \\
&\leq \int_{-\infty}^{t} e^{\alpha T_0} e^{-\alpha(t-s)} \sum_{i=1}^{k} b_i^+\,\mathrm{d}s \\
&\leq \int_{-\infty}^{t} \alpha M e^{-\alpha(t-s)}\,\mathrm{d}s \\
&= M.
\end{aligned}$$

Moreover, for every $t \in \mathbb{R}$ and $\varphi \in \Omega$, we have

$$\begin{aligned}(T\varphi)(t) &= x^{\varphi}(t)\\ &= \int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} \frac{b_i(s)}{1+\varphi^n(s-\tau_i(s))}\,\mathrm{d}s\\ &\geq \int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} \frac{b_i(s)}{1+M^n}\,\mathrm{d}s\\ &\geq \int_{-\infty}^{t} e^{-a^+(t-s)} \sum_{i=1}^{k} \frac{b_i^-}{1+M^n}\,\mathrm{d}s\\ &\geq \int_{-\infty}^{t} a^+\kappa e^{-a^+(t-s)}\,\mathrm{d}s\\ &= \kappa.\end{aligned}$$

On the other hand, it is not difficult to see that $(x^{\varphi}(t))'$ is bounded for all $t \in \mathbb{R}$. Therefore, $x^{\varphi}(t) \in PAP(\mathbb{R},\mathbb{R})$ is uniformly continuous on $\mathbb{R}$. Thus, T is a self-mapping from Ω to Ω.

Next, let us show that T is a contraction mapping. By a direct calculation, one can show that

$$\left|\frac{1}{1+x^n} - \frac{1}{1+y^n}\right| \leq \frac{n}{4\kappa}\cdot|x-y| \tag{2.6}$$

for all $x, y \geq \kappa$. By using (2.6) and Lemma 2, we conclude for every $\varphi, \psi \in \Omega$,

$$\begin{aligned}||T\varphi - T\psi|| &= \sup_{t\in\mathbb{R}} |(T\varphi)(t) - (T\psi)(t)|\\ &= \sup_{t\in\mathbb{R}} \left|\int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} b_i(s)\left[\frac{1}{1+\varphi^n(s-\tau_i(s))} - \frac{1}{1+\psi^n(s-\tau_i(s))}\right]\mathrm{d}s\right|\\ &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r} \sum_{i=1}^{k} b_i(s)\cdot\frac{n}{4\kappa}\cdot|\varphi(s-\tau_i(s)) - \psi(s-\tau_i(s))|\,\mathrm{d}s\\ &\leq \frac{n\sum_{i=1}^{k} b_i^+}{4\kappa}\cdot||\varphi-\psi||\cdot\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t a(r)\,\mathrm{d}r}\,\mathrm{d}s\\ &\leq \frac{ne^{\alpha T_0}\sum_{i=1}^{k} b_i^+}{\alpha\cdot 4\kappa}\cdot||\varphi-\psi||.\end{aligned}$$

By (H2), T is a contraction. Thus, T has a unique fixed point in Ω, *i.e.*, equation (1.2) has a unique pseudo almost periodic solution in Ω. □

Remark 1 *Compared with most earlier results concerning almost periodic solutions to equation* (1.2)*, in Theorem* 1 *we do not assume that* $a^- > 0$ (*see also Remark* 2).

Next, let us study the exponential stability of the pseudo almost periodic solution of (1.2).

Theorem 2 *Suppose that* (H0)–(H2) *are satisfied,* $x(t)$ *is the unique pseudo almost periodic solution of* (1.2) *in* Ω*, and* $y(t) = y(t,\varphi)$ *is an arbitrary solution of* (1.2) *with* $\inf_{t\in[-\tau,0]}\varphi(t) \geq 0$*. Then,* $y(t)$ *converges exponentially to* $x(t)$ *as* $t \to +\infty$*.*

Proof. By (H2), there holds

$$\frac{e^{\alpha T_0}\sum_{i=1}^{k} b_i^+}{\alpha} < \frac{4\kappa}{n}.$$

Thus, there exists $\lambda \in (0,\alpha)$ such that

$$\lambda - \alpha + e^{\alpha T_0}\sum_{i=1}^{k} b_i^+ \cdot \frac{n}{4\kappa} e^{\lambda\tau} < 0. \tag{2.7}$$

According to Lemma 6, there exists $t_\varphi > 0$ such that

$$\kappa < y(t) < M \quad \text{for all } t \geq t_\varphi. \tag{2.8}$$

Let

$$t_1 = t_\varphi + \tau, \quad E = \sup_{t\in[-\tau,t_1]} |x(t) - y(t)|, \quad z(t) = x(t) - y(t);$$

we claim that for every $\varepsilon > 0$ there holds

$$|z(t)| \leq (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t}, \quad t \in [-\tau, +\infty). \tag{2.9}$$

It is obvious that

$$|z(t_1)| < E + \varepsilon, \quad |z(t)| < (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t}, \quad t \in [-\tau, t_1].$$

It suffices to prove that

$$|z(t)| \leq (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t}, \quad t \in (t_1, +\infty).$$

Otherwise, for some $\varepsilon > 0$,

$$\{t > t_1 : |z(t)| > (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t}\} \neq \emptyset.$$

Let

$$t_2 = \inf\{t > t_1 : |z(t)| > (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t}\}.$$

Then, $t_2 > t_1$ and

$$|z(t_2)| = (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t_2}, \quad |z(t)| \leq (e^{\alpha T_0} + 1)(E+\varepsilon)e^{\lambda t_1} e^{-\lambda t}, \quad t \in [-\tau, t_2).$$

Combining this with (2.6), (2.7), (2.8) and Lemma 2, we conclude that

$$
\begin{aligned}
|z(t_2)| &= \Bigg| e^{-\int_{t_1}^{t_2} a(u)\,\mathrm{d}u} z(t_1) \\
&\qquad + \int_{t_1}^{t_2} e^{-\int_s^{t_2} a(u)\,\mathrm{d}u} \cdot \sum_{i=1}^{k} b_i(s) \cdot \left[\frac{1}{1+x^n(s-\tau_i(s))} - \frac{1}{1+y^n(s-\tau_i(s))} \right] \mathrm{d}s \Bigg| \\
&\le e^{\alpha T_0} e^{-\alpha(t_2-t_1)} |z(t_1)| + \int_{t_1}^{t_2} e^{\alpha T_0} e^{-\alpha(t_2-s)} \cdot \sum_{i=1}^{k} b_i^+ \frac{n}{4\kappa} |z(s-\tau_i(s))|\,\mathrm{d}s \\
&\le e^{\alpha T_0} e^{-\alpha(t_2-t_1)} (E+\varepsilon) \\
&\qquad + \frac{n}{4\kappa} e^{\alpha T_0} \int_{t_1}^{t_2} e^{-\alpha(t_2-s)} \sum_{i=1}^{k} b_i^+ \cdot (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda(s-\tau_i(s))}\,\mathrm{d}s \\
&\le e^{\alpha T_0} e^{-\alpha(t_2-t_1)} (E+\varepsilon) \\
&\qquad + \frac{n}{4\kappa} e^{\alpha T_0} e^{\lambda\tau} \int_{t_1}^{t_2} e^{-\alpha(t_2-s)} \sum_{i=1}^{k} b_i^+ \cdot (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda s}\,\mathrm{d}s \\
&\le (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda t_2} \Bigg[\frac{e^{\alpha T_0}}{e^{\alpha T_0}+1} e^{(\alpha-\lambda)(t_1-t_2)} \\
&\qquad + \int_{t_1}^{t_2} \frac{n}{4\kappa} e^{\alpha T_0} e^{\lambda\tau} \sum_{i=1}^{k} b_i^+ \cdot e^{-\alpha(t_2-s)} e^{\lambda(t_2-s)}\,\mathrm{d}s \Bigg] \\
&\le (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda t_2} \left[\frac{e^{\alpha T_0}}{e^{\alpha T_0}+1} e^{(\alpha-\lambda)(t_1-t_2)} + \int_{t_1}^{t_2} (\alpha-\lambda) e^{(\alpha-\lambda)(s-t_2)}\,\mathrm{d}s \right] \\
&= (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda t_2} \left[\frac{e^{\alpha T_0}}{e^{\alpha T_0}+1} e^{(\alpha-\lambda)(t_1-t_2)} + 1 - e^{(\alpha-\lambda)(t_1-t_2)} \right] \\
&= (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda t_2} \left[1 - e^{(\alpha-\lambda)(t_1-t_2)} \left(1 - \frac{e^{\alpha T_0}}{e^{\alpha T_0}+1} \right) \right] \\
&< (e^{\alpha T_0}+1)(E+\varepsilon) e^{\lambda t_1} e^{-\lambda t_2},
\end{aligned}
$$

which is a contradiction. Thus, for every $\varepsilon > 0$, (2.9) holds. By the arbitrariness of ε, we conclude that

$$|z(t)| \le (e^{\alpha T_0}+1) E e^{\lambda t_1} e^{-\lambda t}, \quad t \in [-\tau, +\infty).$$

This completes the proof. □

Next, we give two examples to illustrate our main results.

First, we give the definition of an important function $\omega(t)$. For $n = 1, 2, \ldots$ and $0 \le i < n$, let $a_1 = 0$, $a_{n+1} = a_n + n + n^2$ and intervals $I_n^i = [a_n + i,\ a_n + i + 1]$. Choose a non-negative, continuous function g on $[0,\ 1]$ such that $g(0) = g(1) = 0$ and

$$\int_0^1 g(t)\,\mathrm{d}t = 1.$$

Define the function ω on $\mathbb{R}$ by

$$\omega(t) = \begin{cases} g[t-(a_n+i)], & t \in I_n^i \text{ for some } n \text{ and even } i, \\ -g[t-(a_n+i)], & t \in I_n^i \text{ for some } n \text{ and odd } i, \\ 0, & t \in \mathbb{R}^+ \setminus \bigcup\{I_n^i : n = 1,2,\ldots, 0 \le i < n\}, \\ \omega(-t), & t < 0. \end{cases}$$

From [11, p. 211, Example 1.7], we know that $\omega \in PAP_0(\mathbb{R},\mathbb{R})$ is ergodic. Moreover, $\omega \notin C_0(\mathbb{R},\mathbb{R})$. Let $\mu(t) = \frac{\omega(t)}{\|g\|}$, $t \in \mathbb{R}$, which will be used in the following examples.

Example 1 *Let* $n = k = 1$, $a(t) = 1 + \frac{\sin 20t + \sin 20\pi t}{2} + \frac{1}{10}\mu(t)$, *and*

$$b_1(t) = \frac{30 + \sin^2 t + \sin^2 \sqrt{2}t}{100}, \quad \tau_1(t) = \cos^2 t + \cos^2 \sqrt{2}t + \frac{1}{1+t^2}.$$

It is easy to see that (H0) *holds. For all* $t, s \in \mathbb{R}$ *with* $s \le t$, *we have*

$$\int_t^s a(u)\,\mathrm{d}u \le (s-t) + \frac{1}{40}\cdot\left(2+\frac{2}{\pi}\right) + \frac{1}{10}(t-s) \le \frac{9}{10}\Big[(s-t)+\frac{1}{12}\Big].$$

So, we can choose $\alpha = \frac{9}{10}$ *and* $T_0 = \frac{1}{12}$. *By a direct calculation, we can see that*

$$e^{\alpha T_0} = e^{\frac{3}{40}}, \quad b_1^+ = \frac{32}{100}, \quad b_1^- = \frac{30}{100}, \quad a^+ = 2.1.$$

Let $\kappa = 0.1,\ M = 0.4$. *We have*

$$-\alpha M + e^{\alpha T_0}\sum_{i=1}^k b_i^+ = -0.0151 < 0, \quad -a^+\kappa + \sum_{i=1}^k \frac{b_i^-}{1+M^n} = 0.0043 > 0,$$

which yields that (H1) *holds. Moreover,*

$$\frac{e^{\alpha T_0}\sum_{i=1}^k b_i^+}{\alpha} = 0.3832 < 0.4 = \frac{4\kappa}{n},$$

which means that (H2) *holds. Thus, by Theorem* 1 *and Theorem* 2, *equation* (1.2) *has a unique pseudo almost periodic solution* $x(t)$ *in* Ω, *and* $x(t)$ *is exponentially stable.*

Example 2 *Let* $n = k = 2$, $a(t) = 10 + \left(\sin\frac{3}{\ln 1.05}t + \sin\frac{3\pi}{\ln 1.05}t\right) + \frac{1}{10}\mu(t)$,

$$b_1(t) = \frac{7 + \sin^2 t + \sin^2\sqrt{2}t}{10}, \quad b_2(t) = \frac{7 + \cos^2 t + \cos^2\sqrt{2}t}{10}$$

and

$$\tau_1(t) = \cos^2 t + \cos^2\sqrt{2}t + e^{-t^2}, \quad \tau_2(t) = \sin^2 t + \sin^2\sqrt{2}t + e^{-t^2}.$$

It is easy to see that (H0) *holds. For all* $t, s \in \mathbb{R}$ *with* $s \le t$, *we have*

$$\begin{aligned}\int_t^s a(u)\,\mathrm{d}u &\le 10(s-t) + \frac{\ln 1.05}{3}\left(2+\frac{2}{\pi}\right) + \frac{1}{10}(t-s) \le 9.9(s-t) + \ln 1.05 \\ &= 9.9\Big[(s-t) + \frac{1}{9.9}\ln 1.05\Big].\end{aligned}$$

So, we can choose $\alpha = 9.9$ *and* $T_0 = \frac{1}{9.9} \ln 1.05$. *By a direct calculation, we see that*

$$e^{\alpha T_0} = 1.05, \quad b_1^+ = b_2^+ = 0.9, \quad b_1^- = b_2^- = 0.7, \quad a^+ = 12.1.$$

Let $\kappa = 0.1$, $M = 0.2$; *thus, we have*

$$-\alpha M + e^{\alpha T_0} \sum_{i=1}^{k} b_i^+ = -0.09 < 0, \quad -a^+ \kappa + \sum_{i=1}^{k} \frac{b_i^-}{1 + M^n} = 0.1362 > 0,$$

which yields that (H1) *holds. Moreover,*

$$\frac{e^{\alpha T_0} \sum_{i=1}^{k} b_i^+}{\alpha} = 0.1910 < 0.2 = \frac{4\kappa}{n},$$

which means that (H2) *holds. Thus, by Theorem* 1 *and Theorem* 2, *equation* (1.2) *has a unique pseudo almost periodic solution* $x(t)$ *in* Ω, *and* $x(t)$ *is exponentially stable.*

Remark 2 *It is easy to see that* $a^- < 0$ *in Example* 1. *So, many earlier results, which require* $a^- > 0$, *cannot be applied to Example* 1.

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