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# ENTROPY SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS IN ORLICZ–SOBOLEV SPACES, WITHOUT SIGN CONDITION AND $L^1$ DATA

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**Abstract.** We prove the existence of an entropy solution for the obstacle parabolic problem associated to the equation:

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + g(x,t,u,Du) = f \quad \text{in} \quad \Omega \times (0,T),$$

where $b(u)$ is an unbounded function on $u$ and where $-\operatorname{div}(a(x,t,u,Du))$ is a Leray–Lions operator in Orlicz–Sobolev spaces. The critical growth condition on $g$ is with respect to $Du$, no growth with respect to $u$ and no sign conditions are assumed. The data $f$ belongs to $L^1(\Omega \times (0,T))$.

**Keywords:** Nonlinear parabolic unilateral, entropy solutions, Orlicz–Sobolev spaces, sign condition.

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# 1 Introduction

In the present paper we prove the existence of entropy solutions for a class of nonlinear parabolic unilateral problems of the type:

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \dfrac{\partial b(u)}{\partial t} + Au + g(x,t,u,Du) = f & \text{in } \Omega \times (0,T), \\ b(u)(t=0) = b(u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

In problem (1.1), $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, $T$ is a positive real number and $Q = \Omega \times (0,T)$, while the data $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, $b$ is a strictly increasing $C^1$-function. Let $M$ and $P$ be two N-functions such that $P \ll M$ (for definitions see Section 2). The differential operator $A\colon D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ is defined by $Au = -\operatorname{div}(a(x,t,u,\nabla u))$, where $a$ is a Carathéodory function such that

$$|a(x,t,s,\xi)| \leq \beta[h(x,t) + k_1\overline{M}^{-1}P(k_2|s|) + k_3\overline{M}^{-1}M(k_4|\xi|)],$$

where $h(x,t) \in E_{\overline{M}}(Q)$, $c \geq 0$ and $\beta$, $k_i > 0$ $(i = 1,2,3)$ are given real numbers.

Let $g$ be a Carathéodory function such that the growth condition

$$g(x,t,s,\xi) \leq \gamma(x,t) + \rho(s)M(|\xi|) \tag{1.2}$$

is satisfied, where $\rho\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous non-decreasing function which belongs to $L^1(\mathbb{R})$ and $\gamma(x,t)$ is a given non-negative function in $L^1(Q)$. The function $\psi \in W_0^{1,x}E_M(Q) \cap L^\infty(Q)$.

Under these assumptions, the above problem does not admit, in general, a weak solution since the field $a(x,t,u,Du)$ does not belong to $(L^1_{loc}(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of entropy solutions. This notion was introduced by Bénilan *et al.* [4] for the study of nonlinear elliptic problems.

Note that Dall'aglio–Orsina [18] and Porretta [19] proved the existence of solutions for the problem (1.1) without obstacle with the function $b$ equal to the identity, *i.e.*, $b(u) \equiv u$ and the nonlinearity $g$ satisfying the following 'natural' growth condition (of order $p$):

$$|g(x,t,s,\xi)| \leq b(s)\,(|\xi|^p + c(x,t)) \tag{1.3}$$

and the classical sign condition

$$g(x,t,s,\xi)s \geq 0. \tag{1.4}$$

It is our purpose, in this paper, to prove the existence of a unilateral entropy solution for the problem (1.1) in the setting of the Orlicz–Sobolev spaces without the sign condition (1.4) and without the following coercivity condition

$$|g(x,t,s,\xi)| \geq \beta|\xi|^p \quad \text{for} \quad |s| \geq \gamma.$$

The nonlinearity term $g$ has to fulfil only a weaker condition than (1.3) (see assumption (1.2)). This condition is a growth condition with respect to $Du$; we do not assume any growth conditions with respect to $u$. The case where $g(x,t,u,Du) = \operatorname{div}(\phi(u))$ was studied by H. Redwane in the classical

Sobolev spaces $W^{1,p}(\Omega)$ and Orlicz–Sobolev spaces, and the assumptions for the parabolic part are inspired by [24, 25].

The aim of our work is to investigate the relationship between the obstacle problem (1.1) and some penalized sequence of approximate equations.

This result generalizes an analogous one due to Boccardo–Gallouët [12], see also [13, 14, 19].

A large number of papers was devoted to the study of the existence of renormalized solutions of parabolic problems with rough data under various assumptions and in different contexts: for a review on classical results, see [2, 3, 5, 6, 7, 8, 10, 11, 16].

The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. In Section 3 we make precise all the assumptions on $b$, $a$, $g$, $f$ and $u_0$, the definition of an entropy solution of (1.1) and we establish the existence of such a solution (Theorem 3.5).

## 2 Mathematical preliminaries

**2.1** Let $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, that is, $M$ is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, $M$ admits the representation: $M(t) = \int_0^t a(\tau)\,\mathrm{d}\tau$, where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right-continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to $M$ is defined by $\overline{M}(t) = \int_0^t \overline{a}(\tau)\,\mathrm{d}\tau$, where $\overline{a}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \le t\}$ (see [1, 23]).

The N-function $M$ is said to satisfy the $\Delta_2$-condition if, for some $k > 0$,

$$M(2t) \le k\,M(t) \quad \text{for all} \quad t \ge 0.$$

When this inequality holds only for $t \ge t_0 > 0$, $M$ is said to satisfy the $\Delta_2$-condition near infinity.

Let $P$ and $Q$ be two N-functions; $P \ll Q$ means that $P$ grows essentially less rapidly than $Q$, that is, for each $\varepsilon > 0$, $P(t)/Q(\varepsilon\, t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim\limits_{t\to\infty} (Q^{-1}(t)/P^{-1}(t)) = 0$.

We will extend these N-functions into even functions on all $\mathbb{R}$.

**2.2** Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions $u$ on $\Omega$ such that

$$\int_\Omega M(u(x))\,\mathrm{d}x < +\infty \qquad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\,\mathrm{d}x < +\infty \text{ for some } \lambda > 0\right).$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\,\mathrm{d}x \le 1\right\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M$ satisfies the $\Delta_2$-condition for all $t$ or for $t$ large, according to whether $\Omega$ has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ through the pairing $\int_\Omega u(x)v(x)\,\mathrm{d}x$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if $M$ and $\overline{M}$ satisfy the $\Delta_2$-condition for all $t$ or for $t$ large, according to whether $\Omega$ has infinite measure or not.

**2.3** We now turn to the Orlicz–Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) \mathrm{d}x \to 0 \quad \text{for all} \quad |\alpha| \le 1.$$

This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If $M$ satisfies the $\Delta_2$-condition on $\mathbb{R}^+$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

**2.4** Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order less than or equal to 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set $\Omega$ has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence, and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (cf. [21]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well-defined.

**2.5** Let now $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times (0,T)$. Let $M$ be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by $D_x^\alpha$ the distributional derivative on $Q$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz–Sobolev spaces of order 1 are defined as follows

$$W^{1,x}L_M(Q) = \{u \in L_M(Q) :\ D_x^\alpha u \in L_M(Q),\ \forall\, |\alpha| \le 1\},$$
$$W^{1,x}E_M(Q) = \{u \in E_M(Q) :\ D_x^\alpha u \in E_M(Q),\ \forall\, |\alpha| \le 1\}.$$

The latter space is a subspace of the former one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{M,Q}.$$

We can easily show that they form a complementary system when $\Omega$ satisfies the segment property. These spaces are considered as subspaces of the product space $\prod L_M(Q)$ which have as many copies as there are $\alpha$-order derivatives, $|\alpha| \le 1$. We shall also consider the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q)$, then the function $t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$, then the concerned function is a $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore, the following continuous imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0,T;W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable; if $u \in W^{1,x}L_M(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|D_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \le 1$.

**2.6** The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [20] that when $\Omega$ has the segment property, then each element $u$ of the closure of $\mathcal{D}(Q)$ with respect of the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q)$, of some subsequence $(u_n) \subset \mathcal{D}(Q)$ for the modular convergence, *i.e.*, there exists some $\lambda > 0$ such that for all $|\alpha| \le 1$,

$$\int_Q M\left(\frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that $(u_n)$ converges to $u$ in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\prod L_M, \prod L_{\overline{M}})$. Consequently, $\overline{\mathcal{D}(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\prod L_M, \prod L_{\overline{M}})}$. This space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \prod E_M(Q)$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$, *i.e.*, there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has

$$\sum_{|\alpha| \le 1} \|D_x^\alpha u\|_{M,Q} \le C \sum_{|\alpha| = 1} \|D_x^\alpha u\|_{M,Q}.$$

Thus, both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}$$

with $F$ being the dual space of $W_0^{1,x}E_M(Q)$. It is also, except for an isomorphism, the quotient of $\prod L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that

$$W^{-1,x}L_{\overline{M}}(Q) = \left\{ f = \sum_{|\alpha| \le 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \right\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \le 1} \|f_\alpha\|_{\overline{M},Q},$$

where the $\inf$ is taken over all possible decompositions

$$f = \sum_{|\alpha| \le 1} D_x^\alpha f_\alpha \quad \text{with} \quad f_\alpha \in L_{\overline{M}}(Q).$$

The space $F_0$ is then given by

$$F_0 = \left\{ f = \sum_{|\alpha| \le 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

**Definition 2.1** *We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ for the modular convergence if we can write*

$$u_n = \sum_{|\alpha| \le 1} D_x^\alpha u_n^\alpha + u_n^0 \quad \text{and} \quad u = \sum_{|\alpha| \le 1} D_x^\alpha u^\alpha + u^0$$

*with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence for all $|\alpha| \le 1$ and $u_n^0 \to u^0$ strongly in $L^1(Q)$.*

In this point, we are in position to recall the following abstract lemmas, which will be applied to the truncation operators.

**Lemma 2.2 (cf. [21])** *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let $M$ be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : \ u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : \ u(x) \in D\}. \end{cases}$$

**Lemma 2.3 (cf. [21])** *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. We suppose that the set of discontinuity points of $F'$ is finite. Let $M$ be an N-function. Then the mapping $F\colon W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

**Remark 2.4** *We can easily check, using Lemma 2.2, that each uniformly Lipschitzian mapping $F$, with $F(0) = 0$, acts in inhomogeneous Orlicz–Sobolev spaces of order 1: $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.*

In order to deal with the time derivative, we introduce a time mollification of a function $u \in L_M(Q)$. Thus, we define, for all $\mu > 0$ and all $(x,t) \in Q$

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s) \exp(\mu(s-t))\, \mathrm{d}s,$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$ is the zero extension of $u$. The following lemma is fundamental in the sequel.

**Lemma 2.5 (cf. [16])**

1) *If $u \in L_M(Q)$, then $u_\mu \to u$ in $L_M(Q)$ for the modular convergence, as $\mu \to +\infty$.*
2) *If $u \in W^{1,x}L_M(Q)$, then $u_\mu \to u$ in $W^{1,x}L_M(Q)$ for the modular convergence.*
3) *If $u \in W^{1,x}L_M(Q)$, then $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$.*

**Lemma 2.6 (cf. [16])** *Let $M$ be an $N$-function and let $u_n$ be a sequence in $W^{1,x}L_M(Q)$ such that $u_n$ converges to $u$ weakly in $W^{1,x}L_M(Q)$ for $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\frac{\partial u_n}{\partial t} = h_n + k_n$ in $\mathcal{D}'(Q)$ with $(h_n)_n$ bounded in $W^{-1,x}L_{\overline{M}}(Q)$ and $(k_n)_n$ bounded in the space $\mathcal{M}(Q)$. Then, $u_n$ converges to $u$ strongly in $L^1_{loc}(Q)$.*

*If further, $u_n \in W_0^{1,x}L_{\overline{M}}(Q)$, then $u_n$ converges to $u$ strongly in $L^1(Q)$.*

**Lemma 2.7 (cf. [17])** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with the segment property. Then*

$$\left\{u \in W_0^{1,x}L_{\overline{M}}(Q):\ \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)\right\} \subset C([0,T], L^1(\Omega)).$$

**Remark 2.8**

1) *Note that Lemma 2.6 generalizes the result of Corollary 4 due to J. Simon (see [27]).*

2) *Let us mention that the trace result of Lemma 2.7 generalizes the following classical result due to J.-L. Lions (see [26]).*

3) *Let us mention that the following trace result holds true: $\mathcal{D}(Q)$ is dense in the space $\left\{u \in W_0^{1,x}L_{\overline{M}}(Q) \cap L^2(Q) : \dfrac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)\right\}$ for the modular convergence (see [17]). The trace result generalizes the following classical result,* i.e.,

$$\left\{u \in L^2(0,T,H_0^1(\Omega)):\ \frac{\partial u}{\partial t} \in L^2(0,T,H^{-1}(\Omega))\right\} \subset C([0,T], L^2(\Omega)).$$

**Proposition 2.9 (cf. [16])** *Assume that $(u_n)_n$ is a bounded sequence in $W_0^{1,x}L_M(Q)$ such that $\frac{\partial u_n}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$. Then $u_n$ is relatively compact in $L^1(Q)$.*

We end this section by recalling the following approximation theorem, that will be needed in the sequel to prove the existence of solutions for parabolic inequalities.

**Theorem 2.10** *Let $\psi \in W_0^{1,x}E_M(Q) \cap L^\infty(Q)$ and consider the convex set*

$$\mathcal{K}_\psi = \left\{v \in W_0^{1,x}L_M(Q):\ v \geq \psi \text{ a.e. in } Q\right\}.$$

*Then for every $u \in \mathcal{K}_\psi \cap L^\infty(Q)$ such that $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$, there exists $v_j \in \mathcal{K}_\psi \cap \mathcal{D}(\overline{Q})$ such that*

$$\begin{aligned} v_j &\to u \quad \text{in} \quad W^{1,x}L_M(Q), \\ \frac{\partial v_j}{\partial t} &\to \frac{\partial u}{\partial t} \quad \text{in} \quad W^{-1,x}L_{\overline{M}}(Q) + L^1(Q), \end{aligned}$$

*for the modular convergence.*

*Proof.* Immediate, by using [17, Theorem 3] and the approximation techniques from [22]. □

**Remark 2.11** *The result is still true for $\psi \in W^{1,x}E_M(Q) \cap L^\infty(Q)$, when $\Omega$ is more regular, see [22].*

# 3 Assumptions and statement of main results

## 3.1 Basic assumptions

Throughout this paper, we assume that the following assumptions hold true: $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property, $T > 0$ is given and we set $Q = \Omega \times (0, T)$. Let $M$ and $P$ be two N-functions such that $P \ll M$.

$$\begin{gathered} b\colon \mathbb{R} \to \mathbb{R} \text{ is a strictly increasing } C^1\text{-function with } b(0) = 0 \text{ and such that} \\ 0 < b_0 \leq b'(s) \leq b_1 \qquad \forall s \in \mathbb{R}, \end{gathered} \tag{3.1}$$

where $b_1$ and $b_2$ are given real numbers.

The differential operator $A\colon D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ is defined by

$$Au = -\operatorname{div}(a(x, t, u, \nabla u)),$$

where

$$a\colon Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function} \tag{3.2}$$

which for almost every $(x, t) \in Q$ and for every $s \in \mathbb{R}$, $\xi \neq \xi' \in \mathbb{R}^N$ satisfies

$$|a(x, t, s, \xi)| \leq \beta[h(x, t) + k_1 \overline{M}^{-1} P(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|)], \tag{3.3}$$

$$[a(x, t, s, \xi) - a(x, t, s, \xi')][\xi - \xi'] > 0, \tag{3.4}$$

$$a(x, t, s, \xi)\xi \geq \alpha M(|\xi|), \tag{3.5}$$

where $h(x, t) \in E_{\overline{M}}(Q)$, $c \geq 0$ and $\alpha$, $\beta$, $k_i > 0$ $(i = 1,\ 2,\ 3,\ 4)$, are given real numbers.

Furthermore, let $g\colon Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$g(x, t, s, \xi) \leq \gamma(x, t) + \rho(s) M(|\xi|) \tag{3.6}$$

is satisfied, where $\rho\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous non-decreasing function which belongs to $L^1(\mathbb{R})$ and $\gamma(x, t)$ is a given non-negative function in $L^1(Q)$.

$$f \in L^1(Q) \quad \text{and} \quad u_0 \in L^1(\Omega). \tag{3.7}$$

For all $t \in \mathbb{R}$ and $k \geq 0$, we define

$$T_k(t) = \begin{cases} t, & \text{if } |t| \leq k, \\ k\dfrac{t}{|t|}, & \text{if } |t| > k. \end{cases}$$

## 3.2 Some intermediates results

This subsection is devoted to introduce some basic technical lemmas and results that will be needed throughout this paper. For some details concerning their related contents, the reader can consult [6, 9, 10, 15] for instance.

**Lemma 3.1 (cf. [6])** *Let $(f_n)_n$, $f$ and $\gamma \in L^1(\Omega)$ be such that*

(i) $f_n \geq \gamma$ *a.e. in* $\Omega$;

(ii) $f_n \to f$ *a.e. in* $\Omega$;

(iii) $\displaystyle\int_\Omega f_n(x)\,\mathrm{d}x \to \int_\Omega f(x)\,\mathrm{d}x$.

*Then $f_n \to f$ strongly in $L^1(\Omega)$.*

**Lemma 3.2** *Assume that assumptions* (3.2)–(3.5) *are satisfied, and let $(z_n)_n$ be a sequence in $W_0^{1,x}L_M(Q)$ such that*

(i) $z_n \rightharpoonup z$ *in* $W_0^{1,x}L_M(Q)$ *for* $\sigma(\prod L_M, \prod E_{\overline{M}})$;

(ii) $(a(x,t,z_n,\nabla z_n))_n$ *is bounded in* $(L_{\overline{M}}(Q))^N$;

(iii) $\displaystyle\int_Q [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,\mathrm{d}x\,\mathrm{d}t \to 0$, *as* $n, s \to \infty$,

*where $\chi_s$ is the characteristic function of $Q_s = \{(x,t) \in Q,\ |\nabla z| \leq s\})$. Then*

1) $M(|\nabla z_n|) \to M(|\nabla z|)$ *in* $L^1(Q)$;

2) $\displaystyle\lim_{n\to\infty} \int_Q a(x,t,z_n,\nabla z_n)\nabla z_n\,\mathrm{d}x\,\mathrm{d}t = \int_Q a(x,t,z,\nabla z)\nabla z\,\mathrm{d}x\,\mathrm{d}t$;

3) $\nabla z_n \to \nabla z$ *a.e. in* $Q$.

*Proof.* Fix $r > 0$ and let $s > r$ one has

$$\begin{aligned}
0 &\leq \int_{Q_r} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)][\nabla z_n - \nabla z]\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_{Q_s} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)][\nabla z_n - \nabla z]\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_s} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_{Q} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which with (iii) implies that

$$\lim_{n\to\infty} \int_{Q_r} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)][\nabla z_n - \nabla z]\,\mathrm{d}x\,\mathrm{d}t = 0.$$

So, following the same argument as in [21], one can show that

$$\nabla z_n \to \nabla z \quad \text{a.e. in} \quad Q. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}\int_Q a(x,t,z_n,\nabla z_n)\nabla z_n \,\mathrm{d}x = \int_Q &[a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)] \\ &\qquad\times [\nabla z_n - \nabla z\chi_s]\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_Q a(x,t,z_n,\nabla z\chi_s)(\nabla z_n - \nabla z\chi_s)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_Q a(x,t,z_n,\nabla z_n)\nabla z\chi_s \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.9}$$

Since $(a(x,t,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(Q))^N$, by (3.8), we obtain that $a(x,t,z_n,\nabla z_n)$ converges to $a(x,t,z,\nabla z)$ weakly in $(L_{\overline{M}}(Q))^N$ for $\sigma(\prod L_{\overline{M}}, \prod E_M)$, which implies that

$$\int_Q a(x,t,z_n,\nabla z_n)\nabla z\chi_s \,\mathrm{d}x\,\mathrm{d}t \to \int_Q a(x,t,z,\nabla z)\nabla z\chi_s \,\mathrm{d}x\,\mathrm{d}t \quad \text{as} \quad n\to\infty. \tag{3.10}$$

Letting also $s \to \infty$, one has

$$\int_Q a(x,t,z,\nabla z)\nabla z\chi_s \,\mathrm{d}x\,\mathrm{d}t \to \int_Q a(x,t,z,\nabla z)\nabla z \,\mathrm{d}x\,\mathrm{d}t. \tag{3.11}$$

On the other side, it is easy to see that the second term on the right-hand side of (3.9) tends to 0 as $n\to\infty$.

Consequently, from (iii), (3.10) and (3.11) we have

$$\lim_{n\to\infty}\int_Q a(x,t,z_n,\nabla z_n)\nabla z_n \,\mathrm{d}x\,\mathrm{d}t = \int_Q a(x,t,z,\nabla z)\nabla z \,\mathrm{d}x\,\mathrm{d}t$$

and by virtue of (3.4), Lemma 3.1 and Vitali's Theorem, one can deduce that

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad \text{in} \quad L^1(Q),$$

which completes the proof. □

**Remark 3.3** *It is interesting to note that the condition* (ii) *in Lemma 3.2 is not necessary in the case where the $N$-function $M$ satisfies the $\Delta_2-$condition.*

### 3.3 The principal result

We now give the definition of an entropy solution of (1.1).

**Definition 3.4** *A real-valued function $u$ defined on $Q$ is a unilateral entropy solution of problem* (1.1) *if*

$$T_k(u) \in W_0^{1,x}L_M(Q) \quad \textit{and} \quad u \geq \psi \quad \textit{a.e. in} \quad Q,$$
$$g(x,t,u,\nabla u) \in L^1(Q),$$

*and for all* $v \in W_0^{1,x}L_M(Q) \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q)$ *such that* $v \geq b(\psi)$ *a.e. in* $Q$ *and* $\forall\, k > 0$, $\tau \in (0,T)$

$$\begin{aligned}\int_\Omega \overline{T}_k(b(u(\tau)) - v(\tau))\,\mathrm{d}x + \int_0^\tau \Big\langle \frac{\partial v}{\partial t}, T_k(b(u) - v)\Big\rangle\,\mathrm{d}t \\ + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(b(u)-v)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} g(x,t,u,\nabla u) T_k(b(u)-v)\,\mathrm{d}x\,\mathrm{d}t \\ \leq \int_{Q_\tau} f T_k(b(u)-v)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega B_k(b(u_0) - v(0))\,\mathrm{d}x,\end{aligned} \tag{3.12}$$

*where* $\overline{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s$.

The aim of the present work is to prove the following

**Theorem 3.5** *Under assumptions* (3.1)–(3.7)*, there exists at least one unilateral entropy solution of problem* (1.1).

*Proof.* The proof is divided into 5 steps. In Step 1, we introduce an approximate problem. In Step 2, we establish a few a priori estimates which allow us to prove that the approximate solutions $u_n$ converge to $u$ a.e. in $Q$. In Step 3, we define a time regularization of the field $T_k(u)$, establish the boundedness of the sequence $(a(x,t,u_n,\nabla u_n))_n$ in $(L_{\overline{M}}(Q))^N$, and prove that $u_n$ satisfies (3.30). In this step, using some techniques, we also prove the modular convergence of $T_k(u_n)$ to $T_k(u)$ in $W_0^{1,x}L_M(Q)$, which allows us to control the parabolic contribution that arises in the monotonicity method when passing to the limit. Step 4 is devoted to prove the equi-integrability of the nonlinearities $g$. At last, in Step 5, we pass to the limit which is the final step to prove Theorem 3.5.

**Step 1. The approximate problem.** Let us introduce the following regularization of the data:

$$f_n \in \mathcal{D}(Q)\text{: } \|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)} \text{ and } f_n \to f \text{ in } L^1(Q) \text{ as } n \to +\infty, \tag{3.13}$$

$$u_{0n} \in \mathcal{D}(\Omega)\text{: } \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \text{ and } u_{0n} \to u_0 \text{ in } L^1(\Omega) \text{ as } n \to +\infty, \tag{3.14}$$

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1 + \frac{1}{n}|g(x,t,s,\xi)|}.$$

Note that $g_n(x,t,s,\xi)$ satisfies the following conditions

$$|g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)| \quad \text{and} \quad |g_n(x,t,s,\xi)| \leq n.$$

Let us now consider the following regularized approximate problem

$$(P_n)\begin{cases} u_n \in W_0^{1,x}L_M(Q), \\ \displaystyle\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, b(u_n) - v\Big\rangle\,\mathrm{d}t + \int_Q a(x,t,u_n,\nabla u_n)\nabla(b(u_n) - v)\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle\quad + \int_Q g_n(x,t,u_n,\nabla u_n)(b(u_n) - v)\,\mathrm{d}x\,\mathrm{d}t - n\int_Q m(T_n(u_n - \psi)^-)(b(u_n) - v)\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle\quad = \int_Q f_n(b(u_n) - v)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all} \quad v \in W_0^{1,x}L_M(Q) \cap L^\infty(Q). \end{cases}$$

**Remark 3.6** *Note that, thanks to [15], there exists at least one solution $u_n$ of the approximate problem $(P_n)$.*

**Step 2. A priori estimates.** The estimates derived in this step rely on standard techniques for problems of the type $(P_n)$.

**Proposition 3.7** *Assume that* (3.1)–(3.7) *hold true and let $u_n$ be a solution of the approximate problem $(P_n)$. Then for all $k > 0$, we have*

$$\|T_k(u_n)\|_{W_0^{1,x}L_M(Q)} \leq Ck \quad \textit{for all} \quad n \in \mathbb{N}, \tag{3.15}$$

*where $C$ is a constant independent of $n$.*

*Proof.* Let $v = b(u_n) - \exp(G(u_n))T_k(u_n - T_h(u_n))$, where $G(r) = \int_0^r \frac{\rho(s)}{\alpha}\,\mathrm{d}s$ and $h \geq \|\psi\|_\infty$ (the function $\rho$ appears in (3.6)). Choosing $v$ as a test function in the approximate problem $(P_n)$, we get

$$\begin{aligned}
\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\Big\rangle \,\mathrm{d}t \\
+ \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - T_h(u_n))\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_Q a(x,t,u_n,\nabla u_n)\nabla\left(\exp(G(u_n))\,T_k(u_n - T_h(u_n)\right)\mathrm{d}x\,\mathrm{d}t \\
+ \int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
- n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
= \int_Q f_n\exp(G(u_n))T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Which gives

$$\begin{aligned}
\int_Q \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\Big\rangle \,\mathrm{d}t \\
+ \int_{\{h\leq|u_n|\leq h+k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\frac{\rho(u_n)}{\alpha}\exp(G(u_n)T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
- n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
= \int_Q f_n\exp(G(u_n))T_k(u_n - T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using the coercivity condition (3.5) and (3.6), we obtain

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\Big\rangle \,\mathrm{d}t \\ + \int_{\{h\le |u_n|\le h+k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\ - n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n-T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_Q [|f_n + \gamma(x,t)]\exp(G(u_n))T_k(u_n-T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\Big\rangle \,\mathrm{d}t \\ = \int_\Omega B_{k,h}(u_n(T))\,\mathrm{d}x - \int_\Omega B_{k,h}(u_{0n})\,\mathrm{d}x,\end{aligned}$$

where $B_{k,h}(r) = \int_0^r b'(s)T_k(s-T_h(s))\exp(G(s))\,\mathrm{d}s$. Thus, due to the definition of $B_{k,h}$ and since $|G(u_n)| \le \frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}$, we have

$$0 \le \int_\Omega B_{k,h}(u_{0n})\,\mathrm{d}x \le k\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)\|b(u_{0n})\|_{L^1(\Omega)} \equiv Ck.$$

So, using again the coercivity condition (3.5) and the fact that $\int_\Omega B_{k,h}(u_n(T))\,\mathrm{d}x \ge 0$, then (3.16) becomes, for all $n \in \mathbb{N}$

$$\begin{aligned}\alpha\int_{\{h\le |u_n|\le h+k\}} M(|\nabla u_n|)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\ - n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n-T_h(u_n))\,\mathrm{d}x\,\mathrm{d}t \le Ck.\end{aligned}$$

Thus

$$- n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))\frac{T_k(u_n-T_h(u_n))}{k}\,\mathrm{d}x\,\mathrm{d}t \le C.$$

And since

$$- n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n-T_h(u_n))\,\mathrm{d}x \ge 0$$

as well as

$$\exp(G(-\infty)) \le \exp(G(u_n)) \le \exp(G(+\infty)) \quad \text{and} \quad \exp(|G(\pm\infty)|) \le \exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right),$$

we deduce thanks to Fatou's Lemma, as $k \to 0$, that

$$n\int_Q m(T_n(u_n-\psi)^-)\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{3.17}$$

Now, using $v = b(u_n) - \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}$ as a test function in the approximate problem $(P_n)$ with $\tau \in (0,T)$, we get

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)} \Big\rangle \,\mathrm{d}t \\
&\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla\Big(\exp(G(u_n))T_k(u_n)^+\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad\quad - n\int_{Q_\tau} m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad\quad\quad = \int_{Q_\tau} f_n \exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)} \Big\rangle \,\mathrm{d}t \\
&\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)^+\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla u_n\frac{\rho(u_n)}{\alpha}\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad\quad - n\int_{Q_\tau} m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad\quad\quad \le \int_{Q_\tau} |g_n(x,t,u_n,\nabla u_n)|\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad\quad\quad\quad + \int_{Q_\tau} |f_n|\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Let $\varphi_k(r) = \int_0^r b'(s)T_k(s)^+\exp(G(s))\,\mathrm{d}s$. We have $|\varphi_k(r)| \le k\exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\Big)|b(r)|$. Then

$$\int_0^\tau \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n)^+ \Big\rangle \,\mathrm{d}t = \int_\Omega \varphi_k(u_n(\tau))\,\mathrm{d}x - \int_\Omega \varphi_k(u_{0n})\,\mathrm{d}x.$$

Then

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)} \Big\rangle \,\mathrm{d}t \\
&\quad\quad \ge \int_\Omega \varphi_k(u_n(\tau))\,\mathrm{d}x - k\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)\|b(u_{0n})\|_{L^1(\Omega)}.
\end{aligned}$$

Which gives

$$\begin{aligned}
&\int_\Omega \varphi_k(u_n(\tau))\,\mathrm{d}x + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)^+ \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla u_n \frac{\rho(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - n\int_{Q_\tau} m(T_n(u_n-\psi)^-) \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad \le \int_{Q_\tau} |g_n(x,t,u_n,\nabla u_n)| \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \int_{Q_\tau} |f_n| \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t + k\ \exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha}\right) \|b(u_{0n})\|_{L^1(\Omega)}.
\end{aligned} \tag{3.18}$$

Moreover, using the fact that $\varphi_k(u_n(\tau)) \ge 0$, then (3.18) becomes

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)^+ \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - n\int_{Q_\tau} m(T_n(u_n-\psi)^-) \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_{Q_\tau} (|f_n| + \gamma(x,t)) \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t + k\ \exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right) \|u_{0n}\|_{L^1(\Omega)},
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_{\{0\le u_n\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - n\int_{Q_\tau} m(T_n(u_n-\psi)^-) \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \le k\ \exp\left(\frac{\|\rho\|_{L^1(Q)}}{\alpha}\right) \Big(\|f\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big) \equiv Ck.
\end{aligned} \tag{3.19}$$

Thanks to (3.17), we have

$$\begin{aligned}
&\left| -n\int_{Q_\tau} m(T_n(u_n-\psi)^-) \exp(G(u_n)) T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad\qquad \le k \exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right) n\int_{Q_\tau} m(T_n(u_n-\psi)^-)\,\mathrm{d}x\,\mathrm{d}t \equiv Ck.
\end{aligned}$$

Therefore, (3.19) becomes

$$\int_{\{0\le u_n\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \le Ck.$$

Now, since

$$\exp(G(-\infty)) \le \exp(G(u_n)) \le \exp(G(+\infty)) \quad \text{and} \quad \exp(|G(\pm\infty)|) \le \exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right),$$

we get

$$\int_{\{0\le u_n\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t \le Ck. \tag{3.20}$$

Thus, by (3.5), we have

$$\int_{\{0\le u_n\le k\}} M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le Ck. \tag{3.21}$$

Similarly, taking $v = b(u_n) + \exp(G(u_n))T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in the approximate problem $(P_n)$, we obtain

$$\int_{\{-k\le u_n\le 0\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t \le Ck, \tag{3.22}$$

and then

$$\int_{\{-k\le u_n\le 0\}} M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le Ck. \tag{3.23}$$

Combining (3.21) and (3.23), we deduce that

$$\int_Q M(|\nabla T_k(u_n)|)\,\mathrm{d}x\,\mathrm{d}t = \int_{\{|u_n|\le k\}} M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le Ck. \tag{3.24}$$

Hence, the inequality (3.24) give the desired estimate (3.15). □

**Proposition 3.8** *Assume that* (3.1)–(3.7) *hold true and let* $u_n$ *be a solution of the approximate problem* $(P_n)$*. Then for all* $k > h > 0$ *there exists a constant* $C$ (*which does not depend on the* $n$, $k$ *and* $h$) *such that*

$$\int_Q M(|\nabla T_k(u_n - T_h(u_n))|)\,\mathrm{d}x\,\mathrm{d}t \le Ck. \tag{3.25}$$

*Proof.* Let $k > h > 0$. By using $v = b(u_n) - \eta\exp(G(u_n))T_k(u_n - T_h(u_n))^+\chi_{(0,\tau)}$, with $\tau \in (0,T)$, as a test function in the approximate problem $(P_n)$, we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))^+\chi_{(0,\tau)}\Big\rangle\,\mathrm{d}t \\
&\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - T_h(u_n))^+ \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla u_n \frac{\rho(u_n)}{\alpha}\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad - n\int_{Q_\tau} m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \le \int_{Q_\tau} |g_n(x,t,u_n,\nabla u_n)|\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad\quad + \int_{Q_\tau} |f_n|\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which yields, thanks to (3.6) and (3.17),

$$\begin{aligned}
&\int_\Omega B^+_{k,h}(u_n(\tau))\,\mathrm{d}x + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - T_h(u_n))^+\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_{Q_\tau} \Big(|f_n| + \gamma(x,t)\Big)\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + k\exp\left(\frac{\rho(u_n)}{\alpha}\right)\Big(\|b(u_{0n})\|_{L^1(\Omega)} + C\Big),
\end{aligned}$$

where $B^+_{k,h}(r) = \int_0^r b'(s)\exp(G(s))T_k(s - T_h(s))^+ \,\mathrm{d}s$. Then, since $B^+_{k,h} > 0$, we have

$$\begin{aligned}\int_{\{h\le u_n\le h+k\}} & a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\ & \le k\exp\left(\frac{\rho(u_n)}{\alpha}\right)\Big(\|f\|_{L^1(Q)} + \gamma(x,t) + \|u_{0n}\|_{L^1(\Omega)} + C\Big) \equiv Ck.\end{aligned}$$

Therefore, and by the coercivity condition (3.5), we get

$$\int_{\{h\le u_n\le h+k\}} M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le Ck, \tag{3.26}$$

where $C$ is a positive constant not depending on $n$, $k$ and $h$.

On the other hand, if we consider the test function $v = b(u_n) + \exp(-G(u_n))T_k(u_n - T_h(u_n))^-\chi_{(0,\tau)}$ in the approximate problem $(P_n)$ and reason as in (3.26), we get

$$\int_{\{-h-k\le u_n\le -h\}} M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le Ck. \tag{3.27}$$

From the inequalities (3.26) and (3.27) follows the estimate (3.25). □

**Proposition 3.9** *Assume that* (3.1)–(3.7) *hold true and let* $u_n$ *be a solution of the approximate problem* $(P_n)$*. Then there exists a measurable function* $u$ *such that for all* $k > 0$*, we have* (*for a subsequence still denoted by* $u_n$)*,*

1) $u_n \to u$ *a.e. in* $Q$;
2) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^{1,x}L_M(Q)$ *for* $\sigma(\prod L_M, \prod E_{\overline{M}})$;
3) $T_k(u_n) \to T_k(u)$ *strongly in* $E_M(Q)$ *and a.e. in* $Q$.

*Proof.* Let $k > h > 0$ be large enough. Thanks to [21, Lemma 5.7], there exist two positive constants $C_1$ and $C_2$ such that

$$\int_Q M(C_1|T_k(u_n - T_h(u_n))|)\,\mathrm{d}x\,\mathrm{d}t \le C_2\int_Q M(|\nabla T_k(u_n - T_h(u_n))|)\,\mathrm{d}x\,\mathrm{d}t.$$

Then, by Proposition 3.8 we deduce that

$$\begin{aligned}M(C_1k)\,\mathrm{meas}(\{|u_n - T_h(u_n)| > k\}) &= \int_{\{|u_n - T_h(u_n)|>k\}} M(C_1|T_k(u_n - T_h(u_n))|)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C_2\int_Q M(|\nabla T_k(u_n - T_h(u_n))|)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C_3k.\end{aligned}$$

Hence

$$\mathrm{meas}(\{|u_n - T_h(u_n)| > k\}) \le \frac{C_3k}{M(C_1k)}, \qquad \forall\, n\in\mathbb{N},\ \forall\, k > h > 0.$$

Finally, we have $\forall\, n\in\mathbb{N},\ \forall\, k > h > 0$

$$\mathrm{meas}(\{|u_n| > k\}) \le \mathrm{meas}(\{|u_n - T_h(u_n)| > k - h\}) \le \frac{(k-h)C_3}{M((k-h)C_1)}.$$

Letting $k$ to infinity, we deduce that

$$\text{meas}(\{|u_n| > k\}) \to 0 \quad \text{as} \quad k \to \infty.$$

For every $\lambda > 0$, we have

$$\begin{aligned}\text{meas}(\{|u_n - u_m| > \lambda\}) \leq \text{meas}(\{|u_n| > k\}) + \text{meas}(\{|u_m| > k\}) \\ + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \lambda\}).\end{aligned} \tag{3.28}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$, there exists some $v_k \in W_0^{1,x}L_M(Q)$ such that

$$\begin{aligned}&T_k(u_n) \rightharpoonup v_k \quad \text{weakly in} \quad W_0^{1,x}L_M(Q) \text{ for } \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}),\\ &T_k(u_n) \to v_k \quad \text{strongly in} \quad E_M(Q) \text{ and a.e. in } Q.\end{aligned}$$

Therefore, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in $Q$.

Let $\varepsilon > 0$. By (3.28) there exists some $k(\varepsilon) > 0$ such that

$$\text{meas}(\{|u_n - u_m| > \lambda\}) \leq \varepsilon \quad \text{for all} \quad n,\ m \geq h_0(k(\varepsilon), \lambda).$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure in $Q$, thus it converges almost everywhere to some measurable function $u$. Then

$$\begin{aligned}&T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^{1,x}L_M(Q) \text{ for } \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}),\\ &T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad E_M(Q) \text{ and a.e. in } Q.\end{aligned}$$

This completes the proof of Proposition 3.9. □

**Step 3. Almost everywhere convergence of the gradients.** Since $T_k(u) \in W_0^{1,x}L_M(Q)$, then there exists a sequence $(\alpha_j^k)_j \in \mathcal{D}(Q)$ such that $\alpha_j^k \to T_k(u)$ for the modular convergence in $W_0^{1,x}L_M(Q)$. In the sequel and throughout the paper, $\chi_{j,s}$ and $\chi_s$ will denote, respectively, the characteristic functions of the sets:

$$Q^{j,s} = \{(x,t) \in Q : |\nabla T_k(\alpha_j^k)| \leq s\} \quad \text{and} \quad Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \leq s\}. \tag{3.29}$$

We will introduce the following function of one real variable $s$, which is define as

$$h_{\widetilde{m}}(s) = \begin{cases} 1, & \text{if } |s| \leq \widetilde{m},\\ 0, & \text{if } |s| \geq \widetilde{m} + 1,\\ \widetilde{m} + 1 - |s|, & \text{if } \widetilde{m} \leq |s| \leq \widetilde{m} + 1,\end{cases}$$

where $\widetilde{m}$ is a non-negative real parameter with $\widetilde{m} > k$.

**Proposition 3.10** *Assume that assumptions* (3.2)–(3.6) *hold true and let $u_n$ be a solution of the approximate problem* ($P_n$). *Then for all $k > 0$*

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \textit{ strongly in } L^1(Q) \textit{ as } n \textit{ tends to infinity.}$$

*Proof.* In order to prove the modular convergence of the truncation $T_k(u_n)$, we shall show the following assertions:
Assertion (i)

Boundedness of the sequence $(a(x,t,u_n,\nabla u_n))_n$ in $(L_{\overline{M}}(Q))^N$.

Assertion (ii)

$$\lim_{\widetilde{m}\to\infty}\limsup_{n\to\infty}\int_{\{\widetilde{m}\le|u_n|\le\widetilde{m}+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.30}$$

Assertion (iii)

$T_k(u_n)\to T_k(u)$ modular convergence in $W_0^{1,x}L_M(Q)$.

Proof of Assertion (i). Let $\varphi\in(E_M(Q))^N$ with $\|\varphi\|_{M,Q}=1$. Using the assumption (3.4), one has

$$\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\varphi)][\nabla T_k(u_n)-\varphi]\,\mathrm{d}x\,\mathrm{d}t\ge 0,$$

which gives

$$\begin{aligned}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\varphi\,\mathrm{d}x\,\mathrm{d}t\le&\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t\\&-\int_Q a(x,t,T_k(u_n),\varphi)[\nabla T_k(u_n)-\varphi]\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

On the one hand, by (3.20) and (3.22), we have

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t\le Ck.$$

On the other hand, for $\lambda$ large enough, and thanks to (3.2), we get

$$\int_Q \overline{M}(a(x,t,T_k(u_n),\varphi))\,\mathrm{d}x\,\mathrm{d}t\le\beta\left(\int_Q\overline{M}\left(\frac{h(x,t)}{\lambda}\right)\mathrm{d}x\,\mathrm{d}t+\frac{1}{\lambda}\int_Q M(k_4|\varphi|)\,\mathrm{d}x\,\mathrm{d}t+C\right)\le C,$$

hence $(a(x,t,T_k(u_n),\varphi))_n$ is bounded in $(L_{\overline{M}}(Q))^N$.

At present, since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$, it obviously follows that

$$\int_Q a(x,t,T_k(u_n),\varphi)[\nabla T_k(u_n)-\varphi]\,\mathrm{d}x\,\mathrm{d}t\le C.$$

So, by using the dual norm, we conclude that $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is a bounded sequence in $(L_{\overline{M}}(Q))^N$. Thus, up to a subsequence

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup h_k\quad\text{in}\quad(L_{\overline{M}}(Q))^N\quad\text{for}\quad\sigma(\textstyle\prod L_{\overline{M}},\prod E_M)$$

for some $h_k\in(L_{\overline{M}}(Q))^N$.

Proof of Assertion (ii). Taking $v = b(u_n) + \exp(-G(u_n))T_1(u_n - T_{\widetilde{m}}(u_n))^-$ as a test function in $(P_n)$, we get

$$\begin{aligned}
-\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(-G(u_n))T_1(u_n - T_{\widetilde{m}}(u_n))^- \Big\rangle \,\mathrm{d}t \\
+ \int_{\{-(\widetilde{m}+1)\le u_n \le -\widetilde{m}\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(-G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
+ n\int_Q m(T_n(u_n-\psi)^-)\exp(-G(u_n))T_1(u_n - T_{\widetilde{m}}(u_n))^-\,\mathrm{d}x\,\mathrm{d}t \\
\le \int_Q |\gamma(x,t)|\exp(-G(u_n))T_1(u_n - T_{\widetilde{m}}(u_n))^-\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_Q |f_n| T_1(u_n - T_{\widetilde{m}}(u_n))^- dx\,\mathrm{d}t.
\end{aligned}$$

By setting $\beta_{\widetilde{m}}(r) = -\int_0^r b'(s)T_1(s - T_{\widetilde{m}}(s))^- \exp(-G(s))\,\mathrm{d}s$, and using the fact that

$$+ n\int_Q m(T_n(u_n-\psi)^-)\exp(-G(u_n))T_1(u_n - T_{\widetilde{m}}(u_n))^-\,\mathrm{d}x\,\mathrm{d}t \ge 0,$$

we obtain

$$\begin{aligned}
\int_\Omega \beta_{\widetilde{m}}(u_n(T))\,\mathrm{d}x + \int_{\{-(\widetilde{m}+1)\le u_n \le -\widetilde{m}\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(-G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
\le \exp\left(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha}\right)\Bigg(\int_{\{|u_n|>\widetilde{m}\}} |f_n|\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\{|u_n|>\widetilde{m}\}} |\gamma|\,\mathrm{d}x\,\mathrm{d}t + \int_{\{|b(u_{0n})|>\widetilde{m}\}} |b(u_{0n})|\,\mathrm{d}x\Bigg).
\end{aligned}$$

Since $\beta_{\widetilde{m}}(r) \ge 0$, $\gamma \in L^1(Q)$ and by using (3.13) and (3.14), then Lebesgue's Theorem, we deduce that

$$\lim_{\widetilde{m}\to\infty}\lim_{n\to\infty}\int_{\{-(\widetilde{m}+1)\le u_n \le -\widetilde{m}\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.31}$$

On the other hand, if we set $v = b(u_n) - \exp(G(u_n))T_1(u_n - T_{\widetilde{m}}(u_n))^+$ as a test function in the approximate problem $(P_n)$ and reason as in the proof of (3.31), we deduce that

$$\lim_{\widetilde{m}\to\infty}\lim_{n\to\infty}\int_{\{\widetilde{m}\le u_n \le \widetilde{m}+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.32}$$

Thus (3.30) follows from (3.31) and (3.32).

Proof of Assertion (iii). Taking $v = b(u_n) - \exp(G(u_n))(T_k(u_n) - T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)$ as a test function in the approximate problem $(P_n)$, we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)\Big\rangle \,\mathrm{d}t \\
&\quad + \int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu \geq 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(\alpha_j^k)_\mu)\exp(G(u_n))h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\{\widetilde{m}\leq u_n\leq \widetilde{m}+1\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_Q \gamma(x,t)\exp(G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_Q f_n \exp(G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.33}$$

Observe that

$$\begin{aligned}
&-\int_{\{\widetilde{m}\leq u_n\leq \widetilde{m}+1\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \leq 2k\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_{\{\widetilde{m}\leq u_n\leq \widetilde{m}+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Thus, thanks to Assertion (ii), the third term of (3.33) tends to zero as $n$ and $\widetilde{m}$ tends to infinity, and by Lebesgue's Theorem, we deduce that the right-hand side converges to zero as $n$, $j$ and $\mu$ tend to infinity. Indeed, since

$$\begin{gathered}
(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ \rightharpoonup (T_k(u)-T_k(\alpha_j^k)_\mu)^+ \text{ weakly in } (E_M(Q))^N \text{ as } n\to\infty, \\
(T_k(u)-T_k(\alpha_j^k)_\mu)^+ \rightharpoonup (T_k(u)-T_k(u)_\mu)^+ \text{ weakly in } (E_M(Q))^N \text{ as } j\to\infty,
\end{gathered}$$

and

$$(T_k(u)-T_k(u)_\mu)^+ \rightharpoonup 0 \text{ weakly in } (E_M(Q))^N \text{ as } \mu\to\infty.$$

So, it's easy to see that

$$\left|-n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t\right| \to 0,$$

as $n,\ j$ and $\mu\to\infty$. Let $\varepsilon(n,\widetilde{m},j,\mu)>0$ be a positive sequence such that

$$\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{\widetilde{m}\to\infty}\lim_{n\to\infty}\varepsilon(n,\widetilde{m},j,\mu)=0.$$

Therefore, (3.33) becomes

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)^+ h_{\widetilde{m}}(u_n)\Big\rangle \,dt \\
&\qquad\qquad + \int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu \geq 0\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n) \\
&\qquad\qquad\qquad\qquad \times \nabla(T_k(u_n)-T_k(\alpha_j^k)_\mu)h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon(n,\widetilde{m},j,\mu).
\end{aligned} \tag{3.34}$$

Thanks to [25, Lemma 3.2], we deduce that

$$\int_0^T \Big\langle \frac{\partial b(u_n)}{\partial t}, \exp(G(u_n)) h_{\widetilde{m}}(u_n)(T_k(u_n) - T_k(\alpha_j^k))^+ \Big\rangle \, \mathrm{d}t \geq \varepsilon(n, j, \mu).$$

On the other hand, the second term of left-hand side of (3.34) reads as

$$\begin{aligned}
&\int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \geq 0\}} \exp(G(u_n)) a(x, t, u_n, \nabla u_n) \\
&\qquad\qquad \times \nabla(T_k(u_n) - T_k(\alpha_j^k)_\mu) h_{\widetilde{m}}(u_n) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \geq 0\}} \exp(G(u_n)) a(x, t, T_k(u_n), \nabla T_k(u_n)) \\
&\qquad\qquad \times \nabla(T_k(u_n) - T_k(\alpha_j^k)_\mu) h_{\widetilde{m}}(u_n) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \geq 0;\ |u_n| > k\}} \exp(G(u_n)) a(x, t, u_n, \nabla u_n) \\
&\qquad\qquad \times \nabla T_k(\alpha_j^k)_\mu h_{\widetilde{m}}(u_n) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\left| \int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \geq 0;\ |u_n| > k\}} \exp(G(u_n)) a(x, t, u_n, \nabla u_n) \nabla T_k(\alpha_j^k)_\mu h_{\widetilde{m}}(u_n) \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad \leq C \int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \geq 0;\ |u_n| > k\}} |a(x, t, T_{\widetilde{m}+1}(u_n), \nabla T_{\widetilde{m}+1}(u_n))| |\nabla T_k(\alpha_j^k)_\mu| \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{3.35}$$

On the one hand, since $(|a(x, t, T_{\widetilde{m}+1}(u_n), \nabla T_{\widetilde{m}+1}(u_n))|)_n$ is bounded in $(L_{\overline{M}}(Q))^N$, we get for a subsequence that

$$|a(x, t, T_{\widetilde{m}+1}(u_n), \nabla T_{\widetilde{m}+1}(u_n))| \rightharpoonup h_{\widetilde{m}+1} \quad \text{weakly in} \quad (L_{\overline{M}}(Q))^N$$

for $\sigma(\prod L_{\overline{M}}(Q), \prod E_M(Q))$. Since $|\nabla T_k(\alpha_j^k)_\mu \chi_{\{u_n > k\}}|$ converges to $|\nabla T_k(\alpha_j^k)_\mu \chi_{\{u > k\}}|$ strongly in $(E_M(Q))^N$, so by tending $n$ to infinity, we get

$$\int_{\{|u_n| > k\}} |a(x, t, T_{\widetilde{m}+1}(u_n), \nabla T_{\widetilde{m}+1}(u_n))| |\nabla T_k(\alpha_j^k)_\mu| \, \mathrm{d}x \, \mathrm{d}t \to \int_{\{|u| > k\}} h_{\widetilde{m}+1} |\nabla T_k(\alpha_j^k)_\mu| \, \mathrm{d}x \, \mathrm{d}t.$$

Using now, the modular convergence of $\nabla T_k(\alpha_j^k)_\mu$ to $\nabla T_k(u)$ as $j$ and $\mu$ tend to infinity, we get

$$\int_{\{|u| > k\}} h_{\widetilde{m}+1} |\nabla T_k(\alpha_j^k)_\mu| \, \mathrm{d}x \, \mathrm{d}t \to \int_{\{|u| > k\}} h_{\widetilde{m}+1} |\nabla T_k(u)| \, \mathrm{d}x \, \mathrm{d}t.$$

Therefore, since $\nabla T_k(u) = 0$ in $\{|u| > k\}$, we deduce that

$$\int_{\{|u_n| > k\}} |a(x, t, T_{\widetilde{m}+1}(u_n), \nabla T_{\widetilde{m}+1}(u_n))| |\nabla T_k(\alpha_j^k)_\mu| \, \mathrm{d}x \, \mathrm{d}t = \varepsilon(n, j, \mu).$$

Combining this with (3.35), we get

$$\begin{aligned}
&\int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \geq 0\}} \exp(G(u_n)) a(x, t, T_k(u_n), \nabla T_k(u_n)) \\
&\qquad\qquad \times \nabla(T_k(u_n) - T_k(\alpha_j^k)_\mu) h_{\widetilde{m}}(u_n) \, \mathrm{d}x \, \mathrm{d}t \leq \varepsilon(n, j, \mu).
\end{aligned} \tag{3.36}$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} \exp(G(u_n))a(x,t,T_k(u_n),\nabla T_k(u_n)) \\
&\qquad\qquad \times \nabla(T_k(u_n)-T_k(\alpha_j^k)_\mu)h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} \big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\big] \\
&\qquad\qquad \times [\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j]\exp(G(u_n))h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} a(x,t,T_k(u_n),T_k(\alpha_j^k)_\mu\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j] \\
&\qquad\qquad \times \exp(G(u_n))h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - C\int_{Q\setminus Q^{j,s}} |a(x,t,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(\alpha_j^k)_\mu|h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{3.37}
$$

where $\chi_s^j$ denotes the characteristics function of the subset $Q^{j,s}$ defined as in (3.29).

For the third term on the right-hand side of (3.37), and by tending $n$, $j$, $\widetilde{m}$ and $\mu$ to infinity, it obviously follows that

$$
\begin{aligned}
&-C\int_{Q\setminus Q^{j,s}} |a(x,t,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(\alpha_j^k)_\mu|h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad = -C\int_{Q\setminus Q^s} h_k|\nabla T_k(u)|\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n,j,\widetilde{m},\mu).
\end{aligned}
\tag{3.38}
$$

For what concerns the second term on the right-hand side of (3.37), we can write,

$$
\begin{aligned}
&\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j] \\
&\qquad\qquad\qquad \times \exp(G(u_n))h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\nabla T_k(u_n) \\
&\qquad\qquad\qquad \times \exp(G(u_n))h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\nabla T_k(\alpha_j^k)_\mu\chi_s^j \\
&\qquad\qquad\qquad \times \exp(G(u_n))h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.39}
$$

Starting with the first term of the last equality, since

$$
\begin{aligned}
&\exp(G(u_n))a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)h_{\widetilde{m}}(u_n)\chi_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} \\
&\qquad\qquad \to \exp(G(u))a(x,t,T_k(u),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)h_{\widetilde{m}}(u)\chi_{\{T_k(u)-T_k(\alpha_j^k)_\mu\geq 0\}}
\end{aligned}
$$

strongly in $(E_{\overline{M}}(Q))^N$ as $n$ tends to infinity, and since $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $(L_M(Q))^N$ for $\sigma(\prod L_M, \prod E_{\overline{M}}(Q))$, by Proposition 3.9 we deduce that

$$
\begin{aligned}
&\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}} \exp(G(u_n))a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\nabla T_k(u_n)h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{\{T_k(u)-T_k(\alpha_j^k)_\mu\geq 0\}} \exp(G(u))a(x,t,T_k(u),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\nabla T_k(u)h_{\widetilde{m}}(u)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n),
\end{aligned}
$$

and by letting $j,\ s$ and $\mu$ to infinity, one easily has

$$\begin{aligned}\int_{\{T_k(u)-T_k(\alpha_j^k)_\mu\geq 0\}}&\exp(G(u))a(x,t,T_k(u),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\nabla T_k(u)h_{\widetilde{m}}(u)\,\mathrm{d}x\,\mathrm{d}t\\&=\int_Q\exp(G(u))a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)h_{\widetilde{m}}(u)\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n,j,s,\mu).\end{aligned}$$

In the same way, for the second term on the right-hand side of (3.39), we have

$$\begin{aligned}-\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}}&\exp(G(u_n))a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\nabla T_k(\alpha_j^k)_\mu\chi_s^j h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_Q\exp(G(u))a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)h_{\widetilde{m}}(u)\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n,j,s,\mu).\end{aligned}$$

Adding the two last equalities, we get

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}}&\exp(G(u_n))a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j)\\&\times[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j]h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t=\varepsilon(n,j,s,\mu).\end{aligned}\tag{3.40}$$

Combining (3.36)–(3.38) and (3.40), we then conclude

$$\begin{aligned}\exp(G(-\infty))\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\geq 0\}}&[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j]\\&\times[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j]h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t\\\leq C\int_{Q\setminus Q^s}&h_{\widetilde{m}+1}|\nabla T_k(u)|\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n,j,s,\mu).\end{aligned}\tag{3.41}$$

Now, taking $v=b(u_n)+\exp(-G(u_n))(T_k(u_n)-T_k(\alpha_j^k)_\mu)h_{\widetilde{m}}(u_n)$ as a test function in the approximate problem $(P_n)$ and reasoning as in (3.41) it is possible to conclude that

$$\begin{aligned}\exp(G(-\infty))\int_{\{T_k(u_n)-T_k(\alpha_j^k)_\mu\leq 0\}}&[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j]\\&\times[\nabla(T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j]h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t\\\leq C\int_{Q\setminus Q^s}&h_{\widetilde{m}+1}|\nabla T_k(u)|\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n,j,s,\mu).\end{aligned}\tag{3.42}$$

Finally, by (3.41) and (3.42), we get

$$\begin{aligned}\exp(G(-\infty))\int_Q&[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu\chi_s^j]\\&\times[\nabla(T_k(u_n)-\nabla T_k(\alpha_j^k)_\mu\chi_s^j]h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t\\\leq C\int_{Q\setminus Q^s}&h_{\widetilde{m}+1}|\nabla T_k(u)|\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n,j,s,\mu).\end{aligned}\tag{3.43}$$

On the other hand

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)_\mu \chi_s] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu \chi_s^j)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_s^j] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_Q a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu \chi_s^j)[\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_s^j] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)_\mu \chi_s] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(\alpha_j^k)_\mu \chi_s^j) - \nabla T_k(u)_\mu \chi_s] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t;
\end{aligned} \tag{3.44}$$

it is easy to see that each integral on the right-hand side of (3.44) has the form $\varepsilon(n,j,\mu)$ or $\varepsilon(n,j,s,\mu)$, which implies that

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)_\mu \chi_s^j)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_s^j] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon(n,j,s,\mu).
\end{aligned} \tag{3.45}$$

Furthermore, using (3.43) and (3.45), we have

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_{\widetilde{m}})\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le C \int_{Q \setminus Q^s} h_k |\nabla T_k(u)|\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n,j,s,\mu).
\end{aligned} \tag{3.46}$$

Now, we remark that

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_{\widetilde{m}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_{\widetilde{m}}(u_n))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.47}$$

Since $1 - h_{\widetilde{m}}(u_n) = 0$ in $\{(x,t) : \in |u_n(x,t)| \leq \widetilde{m}\}$ and since $\{|u_n| \leq k\} \subset \{|u_n| \leq \widetilde{m}\}$, for $\widetilde{m}$ large enough the second term on the right-hand side of (3.47) can be written as follows

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_{\widetilde{m}}(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s(1 - h_{\widetilde{m}}(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u_n)(1 - h_{\widetilde{m}}(u_n))\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.48}$$

Because $(a(x,t,T_k(u_n),\nabla T_k(u_n))_n$ is bounded in $(L_{\overline{M}}(Q))^N$ uniformly in $n$ and $\nabla T_k(u)\chi_s(1 - h_{\widetilde{m}}(u_n))$ converges strongly to zero in $(E_M(Q))^N$, the first term on the right-hand side of (3.48) converges to zero as $n$ goes to infinity.

The second term converges also to zero, because

$$a(x,t,T_k(u_n),\nabla T_k(u)\chi_s) \to a(x,t,T_k(u),\nabla T_k(u)\chi_s) \quad \text{strongly in} \quad (L_{\overline{M}}(Q))^N$$

and

$$\nabla T_k(u_n)(1 - h_{\widetilde{m}}(u_n)) \rightharpoonup \nabla T_k(u)(1 - h_{\widetilde{m}}(u)) \quad \text{weakly in} \quad (E_M(Q))^N.$$

Finally, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_{\widetilde{m}}(u_n))\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned} \tag{3.49}$$

Combining (3.46), (3.47) and (3.49), we get

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leq C\int_{Q\setminus Q^s} h_{\widetilde{m}+1}|\nabla T_k(u)|\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n,j,\widetilde{m},s,\mu).
\end{aligned}$$

Letting $n,\ j,\ \widetilde{m},\ s$ and $\mu$ to infinity, we deduce that

$$\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t \tag{3.50}$$

converges to zero. Consequently, by Lemma 3.2 we deduce that

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \qquad \text{strongly in} \qquad L^1(Q).$$

This ends the proof of Proposition 3.10. □

**Proposition 3.11** *Let $u_n$ be a solution of the approximate problem $(P_n)$. Then*

$$u \geq \psi \quad \textit{a.e. in} \quad Q.$$

*Proof.* Thanks to (3.17), we can write $\int_Q m(T_n(u_n-\psi)^-)\,\mathrm{d}x\,\mathrm{d}t \le \frac{C}{n}$. And by using Fatou's Lemma as $n\to\infty$, we have that $\int_Q m((u-\psi)^-)\,\mathrm{d}x\,\mathrm{d}t$ converges to zero. Now, by using the fact that $m(z)=0$ is equivalent to $z=0$, we get $(u-\psi)^-=0$ a.e. in $Q$. Consequently we conclude that $u\ge\psi$ a.e. in $Q$. $\square$

**Step 4. Equi-integrability of the nonlinearities.** First, note that thanks to (3.50), we obtain that $\nabla u_n$ converges to $\nabla u$ a.e. in $Q$ (for a subsequence).

Now, we will show that

$$g(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u) \quad \text{strongly in} \quad L^1(Q).$$

Considering $v=b(u_n)-\exp(G(u_n))\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,\mathrm{d}s$ as a test function in the approximate problem $(P_n)$, we obtain

$$\begin{aligned}
\int_\Omega \widetilde{B}_h(u(T))\,\mathrm{d}x &+ \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\rho(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\frac{\rho(u_n)}{\alpha}\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad- n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\quad\le \int_Q |f_n|\exp(G(u_n))\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega \widetilde{B}_h(b(u_{0n}))\,\mathrm{d}x,
\end{aligned}$$

where $\widetilde{B}_h(r)=\int_0^r b'(s)\exp(G(s))\int_0^s\rho(\tau)\chi_{\{\tau>h\}}\,\mathrm{d}\tau\,\mathrm{d}s\ge 0$, which implies that

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\rho(u_n)\chi_{\{u_n>h\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\left(\int_h^\infty\rho(s)\,\mathrm{d}s\right)\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|b(u_{0n})\|_{L^1(\Omega)}+C\right).
\end{aligned}$$

Using the coercivity condition (3.5) and since $\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,\mathrm{d}s\le\int_h^\infty\rho(s)\,\mathrm{d}s$, we get

$$\int_{\{u_n>h\}} M(|\nabla u_n|)\rho(u_n)\,\mathrm{d}x\,\mathrm{d}t\le C\int_h^\infty\rho(s)\,\mathrm{d}s.$$

And since $\rho\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}\rho(u_n)M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.51}$$

Similarly, let $v=b(u_n)-\exp(-G(u_n))\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}\,\mathrm{d}s$ as a test function in the approximate problem $(P_n)$, we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}}\rho(u_n)M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.52}$$

Consequently, combining (3.51) and (3.52), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}\rho(u_n)M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t=0,$$

which, for $h$ large enough and for a subset $E$ of $Q$, implies that

$$\begin{aligned}\lim_{\mathrm{meas}(E)\to 0}&\int_E\rho(u_n)M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t\\ &\le\max_{|u_n|<h}(\rho(s))\lim_{\mathrm{meas}(E)\to 0}\int_E M(|\nabla T_h(u_n)|)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\lim_{\mathrm{meas}(E)\to 0}\int_{E\cap\{|u_n|>h\}}\rho(u_n)M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

So, we conclude that $\rho(u_n)M(|\nabla u_n|)$ is equi-integrable, which implies that

$$\rho(u_n)M(|\nabla u_n|)\to\rho(u)M(|\nabla u|)\quad\text{in}\quad L^1(Q).$$

Consequently, using (3.6) and Vitali's Theorem, we conclude the equi-integrability of the nonlinearities.

**Step 5. Passage to the limit.** Let $\phi\in K_{b(\psi)}\cap\mathcal{D}(\overline{Q})$ and $\tau\in(0,T)$. Choosing now $v=b(u_n)-T_k(b(u_n)-\phi)\xi_{(0,\tau)}$ as a test function in $(P_n)$, we get

$$\begin{aligned}\int_0^T&\Big\langle\frac{\partial b(u_n)}{\partial t},T_k(b(u_n)-\phi)\xi_{(0,\tau)}\Big\rangle\,\mathrm{d}t+\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\nabla T_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q_\tau}g_n(x,t,u_n,\nabla u_n)T_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t-n\int_{Q_\tau}m(T_n(u_n-\psi)^-)T_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_{Q_\tau}f_nT_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\phi\in K_{b(\psi)}\cap\mathcal{D}(\overline{Q})$, we have $-n\int_{Q_\tau}m(T_n(u_n-\psi)^-)T_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t\ge 0$, and

$$\begin{aligned}\int_{Q_\tau}&a(x,t,u_n,\nabla u_n)\nabla T_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\big(b'(T_{k+\|\phi\|_\infty}(u_n))\nabla T_{k+\|\phi\|_\infty}(u_n)-\nabla\phi\big)\chi_{\{|b(u_n)-\phi|<k\}}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_{Q_\tau}a(x,t,u,\nabla u)\big(b'(T_{k+\|\phi\|_\infty}(u))\nabla T_{k+\|\phi\|_\infty}(u)-\nabla\phi\big)\chi_{\{|b(u)-\phi|<k\}}\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n)\\ &=\int_{Q_\tau}a(x,t,u,\nabla u)\nabla T_k(b(u)-\phi)\,\mathrm{d}x\,\mathrm{d}t+\varepsilon(n).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_\Omega&\overline{T_k}(b(u_n(\tau))-\phi(\tau))\,\mathrm{d}x+\int_0^\tau\Big\langle\frac{\partial\phi}{\partial t},T_k(b(u_n)-\phi)\Big\rangle\,\mathrm{d}t\\ &+\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\big(b'(T_{k+\|\phi\|_\infty}(u_n))\nabla T_{k+\|\phi\|_\infty}(u_n)-\nabla\phi\big)\chi_{\{|b(u_n)-\phi|<k\}}\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q_\tau}g_n(x,t,u_n,\nabla u_n)T_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &\le\int_{Q_\tau}f_nT_k(b(u_n)-\phi)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\overline{T_k}(b(u_{0n})-\phi(0))\,\mathrm{d}x.\end{aligned}$$

Hence, by passing to the limit, we obtain

$$\begin{aligned}\int_\Omega \overline{T_k}(b(u(\tau)) - \phi(\tau))\,\mathrm{d}x &+ \int_0^\tau \Big\langle \frac{\partial \phi}{\partial t}, T_k(b(u) - \phi)\Big\rangle\,\mathrm{d}s \\ &+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(b(u) - \phi)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} g(x,t,u,\nabla u)T_k(b(u) - \phi)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_{Q_\tau} f T_k(b(u) - \phi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \overline{T_k}(b(u_0) - \phi(0))\,\mathrm{d}x.\end{aligned}$$

Now, since for every $v \in K_{b(\psi)} \cap L^\infty(Q)$, there exists $\phi_j \in K_{b(\psi)} \cap \mathcal{D}(\overline{Q})$ such that $v_j$ converges to $v$ for the modular convergence in $W_0^{1,x}L_M(Q)$ and $\frac{\partial v_j}{\partial t}$ converges to $\frac{\partial v}{\partial t}$ for the modular convergence in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$. Then we conclude that $u$ satisfies (3.12).

As a conclusion of Step 1 to Step 5, the proof of Theorem 3.5 is complete. □

## References


[1] R. Adams, *Sobolev spaces*, Academic Press, New York, (1975).

[2] R. Aboulaich, B. Achchab, D. Meskine, A. Souissi, *Parabolic inequalities with nonstandard growths and $L^1$ data*, Boundary Value Problems **2006** (2006), Article ID 29286, pp. 1–18.

[3] E. Azroul, H. Redwane, M. Rhoudaf, *Existence of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces*, Portugaliae Mathematica **66**, no. 1 (2009), pp. 29–63.

[4] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.-L. Vazquez, *An $L^1$-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **22** (1995), pp. 241–273.

[5] A. Benkirane, J. Bennouna, *Existence and uniqueness of solution of unilateral problems with $L^1$ data in Orlicz spaces*, Italian Journal of Pure and Applied Mathematics **16** (2004), pp. 87–102.

[6] A. Benkirane, A. Elmahi, *A strongly nonlinear elliptic equation having natural growth terms and $L^1$ data*, Nonlinear Analysis: Theory, Methods & Applications **39** (2000), pp. 403–411.

[7] D. Blanchard, *Truncations and monotonicity methods for parabolic equations*, Nonlinear Analysis: Theory, Methods & Applications **21** (1993), pp. 725–743.

[8] D. Blanchard, F. Murat, *Renormalized solutions of nonlinear parabolic problems with $L^1$ data, existence and uniqueness*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **127**, no. 1 (1997), pp. 1137–1152.

[9] D. Blanchard, F. Murat, H. Redwane, *Existence et unicité de la solution reormalisée d'un problème parabolique assez général*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics **329**, no. 7 (1999), pp. 575–580.

[10] D. Blanchard, F. Murat, H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems*, Journal of Differential Equations **177** (2001), pp. 331–374.

[11] D. Blanchard, H. Redwane, *Renormalized solutions of nonlinear parabolic evolution problems*, Journal de Mathématiques Pures et Appliquées **77** (1998), pp. 117–151.

[12] L. Boccardo, T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, Journal of Functional Analysis **87** (1989), pp. 149–169.

[13] L. Boccardo, F. Murat, *Strongly nonlinear Cauchy problems with gradient dependent lower order nonlinearity* in *Recent advances in nonlinear elliptic and parabolic problems*, pp. 247-254, Pitman Research Notes in Mathematics Series **208**, Longman Sci. Tech., Harlow, 1989.

[14] L. Boccardo, F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Analysis: Theory, Methods & Applications **19**, no. 6 (1992), pp. 581–597.

[15] A. Elmahi, D. Meskine, *Parabolic equations in Orlicz spaces*, Journal of the London Mathematical Society **72**, no. 2 (2005), pp. 410–428.

[16] A. Elmahi, D. Meskine, *Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces*, Nonlinear Analysis: Theory, Methods & Applications **60**, no. 1 (2005), pp. 1–35.

[17] A. Elmahi, D. Meskine, *Strongly nonlinear parabolic equations with natural growth terms and $L^1$ data in Orlicz spaces*, Portugaliae Mathematica. Nova Série **62**, no. 2 (2005), pp. 143–183.

[18] A. Dall'Aglio, L. Orsina, *Nonlinear parabolic equations with natural growth conditions and $L^1$ data*, Nonlinear Analysis: Theory, Methods & Applications **27** (1996), pp. 59–73.

[19] A. Porretta, *Nonlinear equations with natural growth terms and measure data*, Electronic Journal of Differential Equataions: Conference 09, 2002, Proceedings of the 2002-Fez Conference on Partial Differential Equations, Fez, Morocco, June 6–8, 2002, (2002), pp. 183–202.

[20] J. P. Gossez, *Some approximation properties in Orlicz–Sobolev*, Studia Mathematica **74** (1982), pp. 17–24.

[21] J. P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Transactions of the American Mathematical Society **190** (1974), pp. 163–205.

[22] J. P. Gossez, V. Mustonen, *Variational inequalities in Orlicz–Sobolev spaces*, Nonlinear Analysis: Theory, Methods & Applications **11** (1987), pp. 379–392.

[23] M. Krasnosel'skii, Ya. Rutickii, *Convex functions and Orlicz spaces*, Noordhoff, Groningen, (1969).

[24] H. Redwane, *Existence results for a class of parabolic equations in Orlicz spaces*, Electronic Journal of Qualitative Theory of Differential Equations **2010**, no. 2 (2010), pp. 1–19.

[25] H. Redwane, *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities*, Advances in Dynamical Systems and Applications **2**, no. 2 (2007), pp. 241–264.

[26] J.-L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod et Gauthier-Villars, Paris, (1969).

[27] J. Simon, *Compact set in $L^p(0,T;B)$*, Annali di Matematica Pura ed Applicata **146** (1987), pp. 65–96.