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# VECTOR VALUED ALMOST PERIODIC RANDOM FUNCTIONS IN PROBABILITY

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**Abstract.** This paper is concerned with Banach space valued almost periodic random functions in probability. We obtain some basic and fundamental properties of such functions.

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## 1 Introduction

The study on almost periodic random functions in probability is initiated in [7]. Since then, several author have made contributions on such almost periodic random functions (see, e.g., [3, 5, 6] and references therein). To the best of our knowledge, it seems that all the known results concerning such functions are devoted to numerical valued random functions. So, in this paper, we aim to study some basic and fundamental properties of Banach space valued almost periodic random functions in probability.

On the other hand, it is needed to note that another kind of almost periodic random functions, which is called $p$-th mean almost periodic random functions, has been introduced and studied by many authors (see, e.g., [1, 4, 8] and references therein). We refer the reader to the monograph [1] for a detailed knowledge on $p$-th mean almost periodic random functions.

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# 2 Main results

Throughout the rest of this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space, and $X$ be a Banach space.

**Definition 2.1** *A random function $f : \mathbb{R} \times \Omega \to X$ is called almost periodic in probability provided that (i) $f$ is continuous in probability on $\mathbb{R}$, i.e., for every $t_0 \in \mathbb{R}$ and $\varepsilon, \eta > 0$, there exists $\delta > 0$ such that for all $t \in \mathbb{R}$ with $|t - t_0| < \delta$, there holds*

$$P\{\omega; \|f(t,\omega) - f(t_0,\omega)\| \geq \varepsilon\} < \eta;$$

*(ii) for every $\varepsilon > 0$ and $\eta > 0$, there exists a number $l(\varepsilon, \eta) > 0$ with the property that every interval of length $l$ contains at least one number $\tau$ such that*

$$P\{\omega; \|f(t+\tau,\omega) - f(t,\omega)\| \geq \varepsilon\} < \eta$$

*for all $t \in \mathbb{R}$. We denote the set of all such functions by $AP\mathcal{R}(\mathbb{R} \times \Omega, X)$.*

**Remark 2.2** The notion of almost periodicity in probability for complex valued random functions has been studied in some earlier works. For a detailed knowledge about complex valued almost periodic random functions, we refer the reader to [5]. In addition, if $f$ is independent of $\omega \in \Omega$, i.e., $f$ is deterministic, then Definition 2.1 equals to the classical definition of vector valued almost periodic functions (cf. [5]). Throughout the rest of this paper, we denote the set of all deterministic almost periodic functions from $\mathbb{R}$ to $X$ by $AP(\mathbb{R}, X)$.

Before we study the properties of almost periodic random functions in probability. We would like first compare almost periodic random functions in probability with $p$-th mean almost periodic random functions.

**Definition 2.3** [1] *Let $p \geq 1$. A random function $f : \mathbb{R} \times \Omega \to X$ is called $p$-th mean almost periodic provided that $\widetilde{f} \in AP(\mathbb{R}, L^p(\Omega, X))$, where*

$$[\widetilde{f}(t)](\omega) = f(t,\omega), \quad t \in \mathbb{R},\ \omega \in \Omega.$$

*For convenience, we denote the set of all such functions by $AP(\mathbb{R}, L^p(\Omega, X))$.*

**Remark 2.4** It is not difficult to show that $AP\mathcal{R}(\mathbb{R} \times \Omega, X) \subset AP(\mathbb{R}, L^p(\Omega, X))$ by using the definitions. However, the contrary is not true since for a random function from $\mathbb{R} \times \Omega$ to $X$, continuity in probability does not necessarily mean $p$-th mean continuity.

**Lemma 2.5** *The following properties hold true:*

*(i) for every $c \in \mathbb{R}$, $cf \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$ provided that $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$;*

*(ii) for every $a \in \mathbb{R}$, $f(\cdot + a, \cdot) \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$ provided that $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$;*

*(iii) $f$ is uniformly continuous in probability on $\mathbb{R}$ provided that $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$;*

*(iv)* *if* $\{f_n\} \subset AP\mathcal{R}(\mathbb{R} \times \Omega, X)$ *is uniformly convergent in probability on* $\mathbb{R}$ *to a random function* $f : \mathbb{R} \times \Omega \to X$*, then* $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$*;*

*(v)* *if* $f : \mathbb{R} \times \Omega \to X$ *is continuous in probability on* $\mathbb{R}$*, then a necessary and sufficient condition for* $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$ *is that* $f$ *is normal in probability, i.e., for every real sequence* $\{s'_n\}$*, there exists a subsequence* $\{s_n\}$ *such that* $\{f(t + s_n, \omega)\}$ *is uniformly convergent in probability on* $\mathbb{R}$*.*

*(vi)* *Let* $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$ *and* $a : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying* $\int_0^{+\infty} a(t)dt < +\infty$*. Then* $F \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$*, where* $F(t,\omega) = \int_0^{+\infty} a(s)f(t - s, \omega)ds$*.*

*Proof.* One can show (i) and (ii) easily by the definition. The proof of (iii) can be deduced from the proof for necessity part of Theorem 2.11. One can also prove (iv) by the definition and some standard $\frac{\varepsilon}{3} - \frac{\eta}{3}$ arguments. So we omit the details. The proof of (v) is similar to that of [5, Theorem 2.20]. As for the proof of (vi), by using (iii), for every $\varepsilon, \eta > 0$, there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta$, there holds

$$P\{\omega; \|F(t_1,\omega) - F(t_2,\omega)\| \geq \int_0^{+\infty} a(t)dt \cdot \varepsilon\} \leq P\{\omega; \|f(t_1,\omega) - f(t_2,\omega)\| \geq \varepsilon\} < \eta.$$

This means that $F$ is uniformly continuous in probability on $\mathbb{R}$. Then, similarly to the above proof, one can verify (ii) of Definition 2.1. So $F \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$. □

**Lemma 2.6** *Let* $f_1, \ldots, f_k \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$*, where* $k$ *is a fixed positive integer. Then, for every* $\varepsilon > 0$ *and* $\eta > 0$*, there exists a number* $l(\varepsilon, \eta) > 0$ *with the property that every interval of length* $l$ *contains at least one common number* $\tau$ *such that*

$$P\{\omega; \|f_i(t + \tau, \omega) - f_i(t,\omega)\| \geq \varepsilon\} < \eta, \quad i = 1, \ldots, k, \ t \in \mathbb{R}.$$

*Proof.* It is well-known that $X^k$ is Banach space under the norm $\|(x_1, \ldots, x_k)\| = \sum_{i=1}^k \|x_i\|$. We define a $X^k$-valued random function:

$$F(t,\omega) = (f_1(t,\omega), \ldots, f_k(t,\omega)), \quad t \in \mathbb{R}, \ \omega \in \Omega.$$

Then, by using (v) of Lemma 2.5, we can get $F \in AP\mathcal{R}(\mathbb{R} \times \Omega, X^k)$, which yields the conclusion by using the definition of $AP\mathcal{R}(\mathbb{R} \times \Omega, X^k)$. □

The following lemma shows that the range of an almost periodic random function has some kind of compactness.

**Lemma 2.7** *Let* $f \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$*. Then for every* $\varepsilon > 0$ *and* $\eta > 0$*, there exists finite random variables* $X_1, \ldots, X_n : \Omega \to X$ *such that for all* $t \in \mathbb{R}$*, there holds*

$$P\{\omega; \|f(t,\omega) - X_i(\omega)\| \geq \varepsilon, \ i = 1, \ldots, n\} < \eta.$$

*Moreover, it follows that* $f$ *is bounded in probability, i.e., for every* $\eta > 0$ *there corresponds a number* $M > 0$ *such that*

$$P\{\omega; \|f(t,\omega)\| \geq M\} < \eta \quad \textit{for all} \quad t \in \mathbb{R}.$$

*Proof.* Let $\varepsilon, \eta > 0$ and $l = l(\varepsilon/2, \eta/2)$ be as in Definition 2.1. Since $f$ is uniformly continuous in probability on $[0, l]$, there exists finite $t_1, \dots, t_n \in [0, l]$ such that for every $t \in [0, l]$, there exists $t_i \in \{t_1, \dots, t_n\}$ satisfying

$$P\{\omega; \|f(t, \omega) - f(t_i, \omega)\| \geq \varepsilon/2\} < \eta/2.$$

Letting $X_i = f(t_i, \cdot)$, we have

$$P\{\omega; \|f(t, \omega) - X_i(\omega)\| \geq \varepsilon/2,\ i = 1, \dots, n\} < \eta/2$$

for all $t \in [0, l]$.

For every $s \in \mathbb{R}$, by Definition 2.1, we can choose $\tau \in [-s, -s + l]$ such that

$$P\{\omega; \|f(s + \tau, \omega) - f(s, \omega)\| \geq \varepsilon/2\} < \eta/2,$$

which yields that

$$\begin{aligned} & P\{\omega; \|f(s, \omega) - X_i(\omega)\| \geq \varepsilon,\ i = 1, \dots, n\} \\ \leq\ & P\{\omega; \|f(s, \omega) - f(s + \tau, \omega)\| \geq \varepsilon/2\} + P\{\omega; \|f(s + \tau, \omega) - X_i(\omega)\| \geq \varepsilon/2,\ i = 1, \dots, n\} \\ <\ & \eta. \end{aligned}$$

It remains to show that $f$ is bounded in probability. Let $\eta > 0$ be fixed. By the above proof, we know that there exists finite random variables $X_1, \dots, X_n : \Omega \to X$ such that for all $t \in \mathbb{R}$, there holds

$$P\{\omega; \|f(t, \omega) - X_i(\omega)\| \geq 1,\ i = 1, \dots, n\} < \eta/2.$$

Since

$$\lim_{M \to +\infty} P\{\omega; \|X_i(\omega)\| \geq M\} = 0, \quad i = 1, \dots, n,$$

there exist $M_1, \dots, M_n > 0$ such that

$$P\{\omega; \|X_i(\omega)\| \geq M_i\} < \frac{\eta}{2n}, \quad i = 1, \dots, n.$$

Then, we have

$$\begin{aligned} & P\{\omega; \|f(t, \omega)\| \geq \max_{1 \leq i \leq n} M_i + 1\} \\ \leq\ & P\{\omega; \|f(t, \omega) - X_i(\omega)\| \geq 1,\ i = 1, \dots, n\} + \sum_{i=1}^{n} P\{\omega; \|X_i(\omega)\| \geq M_i\} \\ <\ & \eta. \end{aligned}$$

This completes the proof. □

**Lemma 2.8** *Let $f, g \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$. Then, the following properties hold true:*

*(i)* $f + g \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$*;*

*(ii)* $f \cdot g \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$ *provided that* $X = \mathbb{R}$*;*

*(iii)* $f/g \in AP\mathcal{R}(\mathbb{R}\times\Omega, X)$ *provided that* $X = \mathbb{R}$ *and for every* $\eta > 0$*, there exists* $m > 0$ *such that*

$$\sup_{t\in\mathbb{R}} P\{\omega; |g(t,\omega)| \le m\} \le \eta.$$

*Proof.* By using (v) of Lemma 2.5, it not difficult to show that (i) holds. Noting that $f$ and $g$ are bounded in probability, again using (v) of Lemma 2.5, one can prove (ii). The proof of (iii) is similar to that of (ii). □

**Lemma 2.9** *Let* $f \in AP\mathcal{R}(\mathbb{R}\times\Omega, X)$*. Then,* $t \mapsto \frac{\partial f(t,\omega)}{\partial t}$ *belongs to* $AP\mathcal{R}(\mathbb{R}\times\Omega, X)$ *if and only if* $t \mapsto \frac{\partial f(t,\omega)}{\partial t}$ *is uniformly continuous in probability on* $\mathbb{R}$*, provided that* $\frac{\partial f(t,\omega)}{\partial t}$ *exists for every* $t\in\mathbb{R}$ *and* $\omega\in\Omega$*.*

*Proof.* We only need to show the if part. Let

$$f_n(t,\omega) = n[f(t+\frac{1}{n},\omega) - f(t,\omega)], \quad t\in\mathbb{R},\ \omega\in\Omega.$$

It follows from (i), (ii) of Lemma 2.5 and (i) of Lemma 2.8 that $f_n \in AP\mathcal{R}(\mathbb{R}\times\Omega, X)$. On the other hand, using the fact that $t \mapsto \frac{\partial f(t,\omega)}{\partial t}$ is uniformly continuous in probability on $\mathbb{R}$, we can conclude that $f_n(t,\omega)$ is uniformly convergent in probability to $\frac{\partial f(t,\omega)}{\partial t}$ on $\mathbb{R}$, which and (iv) of Lemma 2.5 yields that $t \mapsto \frac{\partial f(t,\omega)}{\partial t}$ belongs to $AP\mathcal{R}(\mathbb{R}\times\Omega, X)$. □

**Definition 2.10** *A random function* $f : \mathbb{R}\times X\times\Omega \to X$ *is called uniform almost periodic in probability provided that (i)* $f$ *is continuous in probability on* $\mathbb{R}\times X$*; (ii) for every* $\varepsilon > 0$*,* $\eta > 0$*, and compact subset* $K \subset X$*, there exists a number* $l(\varepsilon,\eta,K) > 0$ *with the property that every interval of length* $l$ *contains at least one number* $\tau$ *such that*

$$P\{\omega; \|f(t+\tau,x,\omega) - f(t,x,\omega)\| \ge \varepsilon\} < \eta$$

*for all* $t\in\mathbb{R}$ *and* $x\in K$*. We denote the set of all such functions by* $AP\mathcal{R}(\mathbb{R}\times X\times\Omega, X)$*.*

**Theorem 2.11** *A necessary and sufficient condition for* $f \in AP\mathcal{R}(\mathbb{R}\times X\times\Omega, X)$ *is that the following two assertions hold:*

*(i) for every compact subset* $K\subset X$*,* $f$ *is uniformly continuous in probability on* $\mathbb{R}\times K$*;*

*(ii) for every* $x\in X$*,* $f(\cdot,x,\cdot)\in AP\mathcal{R}(\mathbb{R}\times\Omega, X)$*.*

*Proof.* "Necessity". Let $f \in AP\mathcal{R}(\mathbb{R}\times X\times\Omega, X)$. Then (ii) is obviously holds. It remains to show (i). Let $\varepsilon,\eta > 0$, $K$ be a compact subset of $\mathbb{R}$, and $l = l(\varepsilon/3,\eta/3,K)$ be as in the Definition 2.10. Since $f$ is uniformly continuous in probability on $[-1,1+l]\times K$, for the above $\varepsilon,\eta > 0$, there exists $\delta\in(0,1)$, such that for all $(t_1,x_1)$, $(t_2,x_2)\in[-1,1+l]\times K$ with $|t_1-t_2| < \delta$ and $|x_1-x_2| < \delta$, there holds

$$P\{\omega; \|f(t_1,x_1,\omega) - f(t_2,x_2,\omega)\| \ge \frac{\varepsilon}{3}\} < \frac{\eta}{3}. \tag{2.1}$$

Now, for every $(t_1, x_1), (t_2, x_2) \in \mathbb{R} \times K$ with $|t_1 - t_2| < \delta$ and $|x_1 - x_2| < \delta$, taking $\tau \in [-t_1, -t_1 + l]$ as in Definition 2.10 such that

$$P\{\omega; \|f(t+\tau, x, \omega) - f(t, x, \omega)\| \geq \frac{\varepsilon}{3}\} < \frac{\eta}{3} \tag{2.2}$$

for all $t \in \mathbb{R}$ and $x \in K$.

Noting that $(t_1 + \tau, x_1), (t_2 + \tau, x_2) \in [-1, 1 + l] \times K$, by (2.1) and (2.2), we get

$$\begin{aligned}
& P\{\omega; \|f(t_2, x_2, \omega) - f(t_1, x_1, \omega)\| \geq \varepsilon\} \\
\leq\ & P\{\omega; \|f(t_2, x_2, \omega) - f(t_2 + \tau, x_2, \omega)\| \geq \frac{\varepsilon}{3}\} + P\{\omega; \|f(t_2 + \tau, x_2, \omega) - f(t_1 + \tau, x_1, \omega)\| \geq \frac{\varepsilon}{3}\} \\
& + P\{\omega; \|f(t_1 + \tau, x_1, \omega) - f(t_1, x_1, \omega)\| \geq \frac{\varepsilon}{3}\} \\
<\ & \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta,
\end{aligned}$$

which means that $f$ is uniformly continuous in probability on $\mathbb{R} \times K$.

"Sufficiency". Let (i) and (ii) hold. It follows from (i) that $f$ is continuous in probability on $\mathbb{R} \times X$. It remains to show that (ii) of Definition 2.10 holds.

Let $\varepsilon, \eta > 0$, $K$ be a compact subset of $\mathbb{R}$. Then, by the uniform continuity of $f$ on $\mathbb{R} \times K$, there exists $\delta > 0$ such that for all $t \in \mathbb{R}$ and $x', x'' \in K$ with $|x' - x''| < \delta$, there holds

$$P\{\omega; \|f(t, x', \omega) - f(t, x'', \omega)\| \geq \frac{\varepsilon}{3}\} < \frac{\eta}{3}. \tag{2.3}$$

In addition, since $K$ is compact, there exists $x_1, \ldots, x_k \in K$ such that for every $x \in K$, there exists $x_{i_x} \in \{x_1, \ldots, x_k\}$ such that $\|x - x_{i_x}\| < \delta$.

On the other hand, since $f(\cdot, x_i, \cdot) \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$, $i = 1, \ldots, k$, by Lemma 2.6, there exists a number $l > 0$ with the property that every interval of length $l$ contains at least one number $\tau$ such that

$$P\{\omega; \|f(t+\tau, x_i, \omega) - f(t, x_i, \omega)\| \geq \varepsilon/3\} < \eta/3 \tag{2.4}$$

for all $t \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$.

Combining (2.3) and (2.4), we conclude that

$$\begin{aligned}
& P\{\omega; \|f(t+\tau, x, \omega) - f(t, x, \omega)\| \geq \varepsilon\} \\
\leq\ & P\{\omega; \|f(t+\tau, x, \omega) - f(t+\tau, x_{i_x}, \omega)\| \geq \varepsilon/3\} + P\{\omega; \|f(t+\tau, x_{i_x}, \omega) - f(t, x_{i_x}, \omega)\| \geq \varepsilon/3\} \\
& + P\{\omega; \|f(t, x_{i_x}, \omega) - f(t, x, \omega)\| \geq \varepsilon/3\} \\
<\ & \eta,
\end{aligned}$$

for all $t \in \mathbb{R}$ and $x \in K$. This means that $f \in AP\mathcal{R}(\mathbb{R} \times X \times \Omega, X)$. □

**Theorem 2.12** *Let $f \in AP\mathcal{R}(\mathbb{R} \times X \times \Omega, X)$ and $x \in AP(\mathbb{R}, X)$. Then $\widetilde{f} \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$, where $\widetilde{f}(t, \omega) = f(t, x(t), \omega)$.*

*Proof.* Let $\varepsilon, \eta > 0$ be fixed, and $K = \overline{\{x(t) : t \in \mathbb{R}\}}$. Then $K \subset X$ is compact, and by Theorem 2.11, $f$ is uniformly continuous in probability on $\mathbb{R} \times K$. Thus, there exists $\delta \in (0, \varepsilon)$ such that for all $x', x'' \in K$ with $\|x' - x''\| < \delta$, there holds

$$P\{\omega; \|f(t, x', \omega) - f(t, x'', \omega)\| \geq \varepsilon\} < \eta, \quad t \in \mathbb{R}. \tag{2.5}$$

In addition, there is finite $x_1, \dots, x_k \in X$ such that for every $t \in \mathbb{R}$, there exists $x_{i_t} \in \{x_1, \dots, x_k\}$ such that

$$\|x(t) - x_{i_t}\| < \delta. \tag{2.6}$$

Noting that $f \in AP\mathcal{R}(\mathbb{R} \times X \times \Omega, X)$ and $x \in AP(\mathbb{R}, X)$, there exists a number $l > 0$ with the property that every interval of length $l$ contains at least one number $\tau$ such that

$$P\{\omega; \|f(t+\tau, x_i, \omega) - f(t, x_i, \omega)\| \geq \varepsilon\} < \eta, \quad t \in \mathbb{R},\ i = 1, \dots, k, \tag{2.7}$$

and

$$\|x(t+\tau) - x(t)\| < \delta, \quad t \in \mathbb{R}. \tag{2.8}$$

Now, combining (2.5)-(2.8), for every $t \in \mathbb{R}$, we have

$$\begin{aligned}
&P\{\omega; \|\widetilde{f}(t+\tau, \omega) - \widetilde{f}(t, \omega)\| \geq 4\varepsilon\} \\
&\quad = \ P\{\omega; \|f(t+\tau, x(t+\tau), \omega) - f(t, x(t), \omega)\| \geq 4\varepsilon\} \\
&\quad \leq \ P\{\omega; \|f(t+\tau, x(t+\tau), \omega) - f(t+\tau, x(t), \omega)\| \geq \varepsilon\} \\
&\qquad\quad + P\{\omega; \|f(t+\tau, x(t), \omega) - f(t, x(t), \omega)\| \geq 3\varepsilon\} \\
&\quad < \ \eta + P\{\omega; \|f(t+\tau, x(t), \omega) - f(t, x(t), \omega)\| \geq 3\varepsilon\} \\
&\quad \leq \ \eta + P\{\omega; \|f(t+\tau, x(t), \omega) - f(t+\tau, x_{i_t}, \omega)\| \geq \varepsilon\} \\
&\qquad\quad + P\{\omega; \|f(t+\tau, x_{i_t}, \omega) - f(t, x_{i_t}, \omega)\| \geq \varepsilon\} + P\{\omega; \|f(t, x_{i_t}, \omega) - f(t, x(t), \omega)\| \geq \varepsilon\} \\
&\quad < \ 4\eta,
\end{aligned}$$

which yields that $\widetilde{f} \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$. □

**Remark 2.13** In the above composition theorem, $x$ is deterministic. For the more general case, i.e., $x \in AP\mathcal{R}(\mathbb{R} \times \Omega, X)$, it seems difficult to obtain a similar composition theorem. We leave it as a problem to the reader.

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