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EXISTENCE OF A SOLUTION FOR HISTORY VALUED NEUTRAL FRACTIONAL DIFFERENTIAL EQUATION WITH A NONLOCAL CONDITION

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Abstract. In this present work, we study the existence of a mild solution of a history valued neutral delay differential equation of fractional order with a nonlocal condition in a Banach space X. The existence results can be obtained by using a differentiable resolvent operator and fixed point theorems. An example is also considered to show the effectiveness of the obtained theory.

Keywords: Fractional calculus, Caputo derivative, Nonlocal condition, Resolvent operator, Neutral fractional differential equation.

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1 Introduction

In recent few decades, researchers have developed great interest in fractional calculus. The fractional order differential equations have played very important role in physics, engineering and many other

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fields of science, like fractal theory, diffusion in porous media and fractional biological neurons. The tool of fractional calculus has been available and applicable to deal with traffic flow, nonlinear oscillation of earthquake, real system characterized by power laws, anomalous diffusion process and many other areas. For more details on fractional calculus, we refer to [1, 2, 3, 4].

The Cauchy problem for several delay problems in a Banach space has received much attention during the past twenty years. The nonlocal condition is more realistic than the classical initial condition in dealing with many physical problems. Concerning the developments in the study of nonlocal problems we refer to [5, 6, 11, 12, 13, 22] and references given therein. We can found few papers in literature for the solvability of the fractional order neutral delay differential equations with nonlocal conditions.

For the initial study of the existence and uniqueness of mild solutions of the fractional order differential equations by virtue of resolvent operator, we refer to [7, 14, 15, 18, 19, 20, 21] and references given therein. For more details of resolvent operators, we refer to [8, 9, 10, 14, 16, 17].

In this work, we consider the following history valued neutral fractional order differential equation

$$ {}^cD_t^q[u(t)+g(t,u(t),u_t)] = Au(t)+f(t,u(t),u_t), \qquad 0<t\le T<\infty, \tag{1.1}$$

$$ u(t) = \phi(t)+h(u)(t), \qquad t\in[-\tau,0], \tag{1.2}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order q, $q\in(0,1)$, $A\colon D(A)\subset X\to X$ is a closed linear operator with dense domain $D(A)$ in a Banach space X and an infinitesimal generator of a resolvent operator $\{S(t)\}_{t\ge 0}$ of linear operators defined on X. For $u\in C([0,T];X)$, $u_t\colon[-\tau,0]\to X$ is defined as $u_t(\theta)=u(t+\theta)$ for $\theta\in[-\tau,0]$. The functions $f,g\colon[0,T]\times X\times C([-\tau,0];X)\to X$, and $h\colon C([-\tau,\ 0];X)\to C([-\tau,0];X)$ are appropriate functions and $\phi\colon[-\tau,\ 0]\to X$ is a given continuous function.

The organization of the article is as follows: In Section 2, we provide some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. In Section 3, the first existence result for the mild solution to (1.1)–(1.2) is obtained by using a differentiable resolvent operator and the Banach fixed point theorem, and the second existence result of the mild solution for the considered nonlocal problem is established with the help of Krasnoselskii's fixed point theorem. An example is also considered at the end of the article.

2 Preliminaries

In this section, we give some definitions, theorems and propositions of fractional calculus and resolvent operators. Let X be a Banach space and $I=[0,T]$, where $0<T<\infty$ and $J=(a,b)$, where $-\infty<a\le b\le+\infty$, $1\le p<\infty$. Let

- $C(I;X)$ be the Banach space of continuous functions $u(t)$ from I to X equipped with the norm $\|u\|_C=\sup_{t\in[0,T]}\|u(t)\|_X$.
- $C^m(I;X)$ be the Banach space of functions $u\colon[0,T]\to X$, which are m-times continuous differentiable function from $[0,T]$ to X equipped with the norm

$$\|u\|_{C^m}=\sup_{t\in[0,T]}\sum_{k=0}^{m}\|u^k(t)\|_X.$$

- $L^p(J;X)$ be the space of all Bochner-measurable functions $u\colon J \to X$ such that $\|u(t)\|_X^p$ is integrable. $L^p(J;X)$ is a Banach space with the norm

$$\|u\|_{L^p(J;X)} = \left(\int_J \|u(s)\|_X^p \,\mathrm{d}s\right)^{1/p}.$$

For a function $u \in C([-\tau,T];X)$, we define the function $\widetilde{h} \in C\big(C([-\tau,0];X), C([-\tau,0];[D(A)])\big)$ such that

$$(\widetilde{h}u)t = \begin{cases} (hu)(t), & t \in [-\tau,0] \\ (hu)(0), & t \in [0,T], \end{cases} \tag{2.1}$$

and $\widetilde{\phi} \in C([-\tau,0];X)$ defined as

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & t \in [-\tau,0] \\ \phi(0), & t \in [0,T]. \end{cases} \tag{2.2}$$

Definition 2.1 *The Riemann–Liouville fractional integral of u with order $q \in \mathbb{R}^+$ is given by*

$$J_t^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\,\mathrm{d}s,$$

where $u \in L^1((0,T);X)$.

Definition 2.2 *The Riemann–Liouville fractional derivative of u of order $q \in \mathbb{R}^+$ is given by*

$$D_t^q u(t) = D_t^m J_t^{m-q} u(t),$$

where $D_t^m = \frac{\mathrm{d}^m}{\mathrm{d}t^m}$, $u \in L^1((0,T);X)$, $J_t^{m-q}u \in W^{m,1}((0,T);X)$.

Definition 2.3 *The Caputo fractional derivative of u of order $q \in \mathbb{R}^+$ is given by*

$${}^cD_t^q u(t) = D_t^q\left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^k(0)\right),$$

where $u \in L^1((0,T);X) \cap C^{m-1}((0,T);X)$.

We also have

$$J_t^q({}^cD_t^q u(t)) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^k(0).$$

Definition 2.4 ([10, Chapter 1]) *Let $q > 0$. A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called a resolvent operator for the integral equation $u(t) = x + \frac{1}{\Gamma(q)}\int_0^t \frac{Au(s)}{(t-s)^{1-q}}\,\mathrm{d}s$, $t \geq 0$, $x \in X$, if the following conditions are satisfied*:

(a) *$S(t)$ is strongly continuous on $\mathbb{R}^+$ and $S(0) = I$;*

(b) $S(t)D(A) \subset D(A)$ *and* $AS(t)x = S(t)Ax$ *for all* $x \in D(A)$ *and* $t \geq 0$;

(c) $S(t)x$ *is a solution of*

$$u(t) = x + \frac{1}{\Gamma(q)} \int_0^t \frac{Au(s)}{(t-s)^{1-q}} \,\mathrm{d}s, \tag{2.3}$$

for all $x \in D(A)$, $t \geq 0$.

Definition 2.5 ([10, Chapter 1]) *A resolvent operator* $\{S(t)\}_{t\geq 0}$ *for* (2.3) *is called differentiable if* $S(\cdot)x \in W^{1,1}_{loc}(\mathbb{R}^+, X)$ *for all* $x \in D(A)$ *and there exists* $\varphi_A \in L^1_{loc}(\mathbb{R}^+)$ *such that* $\|S'(t)x\| \leq \varphi_A(t)\|x\|_{[D(A)]}$ *for all* $x \in D(A)$.

Definition 2.6 ([10, Chapter 2]) *A resolvent operator* $\{S(t)\}_t \geq 0$ *for* (2.3) *is called analytic if the operator function* $S(\cdot)\colon (0,\infty) \to L(X)$ *admits an analytic extension to a sector* $\Sigma_{0,\,\theta_0} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \theta_0\}$ *for some* $0 < \theta_0 \leq \frac{\pi}{2}$.

Definition 2.7 *A function* $u \in C([-\tau, T]; X)$ *is said to be a mild solution of the history valued neutral problem* (1.1)–(1.2) *if* $u(t) = \phi(t) + h(u)(t)$ *for* $t \in [-\tau, 0]$, $\int_0^t \frac{u(s)}{(t-s)^{1-q}}\,\mathrm{d}s \in D(A)$ *for* $t \in [0,T]$ *and*

$$\begin{aligned} u(t) = \phi(0) + h(u)(0) + g(0, (\phi + h(u))(0), \phi + h(u)) - g(t, u(t), u_t) \\ + \frac{A}{\Gamma(q)} \int_0^t \frac{u(s)}{(t-s)^{1-q}}\,\mathrm{d}s + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), u_s)}{(t-s)^{1-q}}\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

The above equation (2.4) can also be written as the integral equation

$$u(t) = k(t) + \frac{1}{\Gamma(q)} \int_0^t \frac{Au(s)}{(t-s)^{1-q}}\,\mathrm{d}s, \tag{2.5}$$

where

$$\begin{aligned} k(t) = \phi(0) + h(u)(0) + g(0, (\phi + h(u))(0), \phi + h(u)) - g(t, u(t), u_t) \\ + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), u_s)}{(t-s)^{1-q}}\,\mathrm{d}s, \quad \text{for } t \in [0,T], \end{aligned}$$

is a continuous function due to continuity of f and g on $[0,T]$.

The next results assemble different properties of the mild solution of (2.5).

Lemma 2.1 ([10]) *Assume* $S(t)$ *is the resolvent operator for* (2.5). *Then*

(i) *if* $u \in C([0,T]; X)$ *is a mild solution of* (2.5), *then* $S * k$ *is continuously differentiable on* $[0,T]$ *and*

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t S(t-s)k(s)\,\mathrm{d}s, \quad \forall\, t \in [0,T],$$

(ii) *if* $\{S(t)\}_{t\geq 0}$ *is analytic,* $k \in C^{\alpha}([0,T];X)$ *for some* $\alpha \in (0,1)$ *and* $k(0) = 0$*, then the function given by*

$$u(t) = S(t)k(t) + \int_0^t S'(t-s)[k(s) - k(t)]\,\mathrm{d}s, \quad \forall\, t \in [0,T],$$

is a mild solution of (2.5) *and* $u \in C^{\alpha}([0,T];X)$*.*

(iii) *if* $\{S(t)\}_{t\geq 0}$ *is analytic and* $k \in C^{\alpha}([0,T];X)$*, then the function defined by*

$$u(t) = S(t)(k(t) - k(0)) + \int_0^t S'(t-s)[k(s) - k(t)]\,\mathrm{d}s + S(t)k(0), \quad \forall\, t \in [0,T],$$

is a mild solution of (2.5)*;*

(iv) *if* $\{S(t)\}_{t\geq 0}$ *is differentiable and* $k \in C([0,T];[D(A)])$*, then the function* $u\colon [0,T] \to X$ *given by*

$$u(t) = \int_0^t S'(t-s)k(s)\,\mathrm{d}s + k(t), \qquad t \in [0,T],$$

is a mild solution of (2.5)*.*

3 Existence of mild solution

In this section, we discuss the existence of a mild solution for the problem (1.1)–(1.2). Throughout the work, we assume that $f, g\colon [0,T] \times X \times C([-\tau,0];X) \to [D(A)]$ are continuous functions and the resolvent operator $\{S(t)\}_{t\geq 0}$ is a differentiable operator.

By considering Lemma 2.1 (iv), we have

$$u(t) = \begin{cases} \phi(t) + h(u)(t), & t \in [-\tau,0], \\ Gu(t) + Fu(t) + \displaystyle\int_0^t S'(t-s)[Gu(s) + Fu(s)]\,\mathrm{d}s, & t \in [0,T], \end{cases} \tag{3.1}$$

where

$$Gu(t) = \phi(0) + h(u)(0) + g(0, (\phi + h(u))(0), \phi + h(u)) - g(t, u(t), u_t), \tag{3.2}$$

$$Fu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s), u_s)\,\mathrm{d}s. \tag{3.3}$$

Since A is the infinitesimal generator of the differentiable resolvent operator $\{S(t)\}_{t\geq 0}$ and there exists a function φ_A in $L^1_{loc}([0,\infty),\mathbb{R}^+)$ such that $\|S'(t)x\| \leq \varphi_A \|x\|_{[D(A)]}$, for all $t > 0$. Without loss of generality, we assume that $\|S(t)\| \leq M$, for $t \in [0,T]$.

Here we list the following assumptions:

(A1) The function $f\colon [0,T] \times X \times C([-\tau,0];X) \to [D(A)]$ is a Lipschitz continuous function such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\|_{[D(A)]} \leq L_f\big(\|x_1 - x_2\| + \|y_1 - y_2\|_{[-\tau,0]}\big)$$

for all $x_i \in X$, $y_i \in C([-\tau,0];X)$ $(i = 1,2)$ and $t \in [0,T]$ and L_f is a positive constant and $B = \sup_{t\in[0,T]} \|f(t,0,0)\|$.

(A2) The function $g\colon [0,T]\times X\times C([-\tau,0];X)\to[D(A)]$ is a continuous function such that

$$\|g(t,x,y)\|_{[D(A)]}\le c_1\big(\|x\|+\|y\|_{[-\tau,0]}\big)+c_2,$$
$$\|g(t,x_1,y_1)-g(t,x_2,y_2)\|_{[D(A)]}\le L_g\big(\|x_1-x_2\|+\|y_1-y_2\|_{[-\tau,0]}\big)$$

for all $t\in[0,T]$ and $x, x_1, x_2\in X$, $y, y_1, y_2\in C([-\tau,0];X)$ and c_1, c_2 and L_g are positive constants.

(A3) The function $h\colon C([-\tau,\ 0];X)\to C([-\tau,0];[D(A)])$ is a Lipschitz continuous function, *i.e.*, there exists a positive constant L_h such that

$$\|h(x)-h(y)\|_{[D(A)]}\le L_h(\|x-y\|),$$

and let h be also uniformly bounded, *i.e.*, there is a positive constant N such that

$$\|h(x)\|_{[D(A)]}\le N$$

for all $x, y\in C([-\tau,\ 0];X)$.

From the equations (3.2)–(3.3) and assumptions (A1)–(A3), it is clear that $G, F\in C([0,T];[D(A)])$ in (3.1). Our first result for the existence of a solution of the nonlocal problem (1.1)–(1.2) is based on the Banach fixed point theorem.

Theorem 3.1 *Let $f,g\in C([0,T]\times X\times C([-\tau,0];X);[D(A)])$ and $\phi\in C([-\tau,0];[D(A)])$. Suppose that the assumptions* (A1)–(A3) *are satisfied and*

$$L_B=(L_h+2L_hL_g+2L_g)\times(1+\|\varphi_A\|_{L^1})+\left[2L_f\frac{T^q}{\Gamma(1+q)}(1+||\varphi_A||_{L^1})\right]<1. \tag{3.4}$$

Then the problem (1.1)–(1.2) *has a unique mild solution on* $[-\tau,\ T]$.

Proof. By Lemma 2.1 (iv), we have that the mild solution $u\colon[-\tau,\ T]\to X$ of problem (1.1) is given by the integral equation (3.1). Define a map $Q\colon C([-\tau,\ T],X)\to C([-\tau,\ T],X)$ such that

$$Qu(t)=\begin{cases}\phi(t)+h(u)t, & t\in[-\tau,0],\\ Gu(t)+Fu(t)+\displaystyle\int_0^t S'(t-s)[G(u)(s)+F(u)(s)]\,\mathrm{d}s, & t\in[0,T].\end{cases}$$

Now, we prove that Q has a fixed point which is a unique mild solution of the equation (1.1).

Let $u,v\in C([-\tau,T],X)$. For $t\in[-\tau,\ 0]$, we have

$$\begin{aligned}\|Qu(t)-Qv(t)\|&\le\|hu(t)-v(t)\|_{[D(A)]}\\&\le L_h\|u(t)-v(t)\|,\end{aligned} \tag{3.5}$$

and for $t\in[0,T]$, we get

$$\begin{aligned}&\|Qu(t)-Qv(t)\|\\&\quad\le\|Gu(t)-Gv(t)\|+\|Fu(t)-Fv(t)\|\\&\qquad+\int_0^t\|S'(t-s)[Gu(s)-Gv(s)+Fu(s)-Fv(s)]\|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le \|h(u)(0)-h(v)(0)\|_{[D(A)]} + \|g(0,(\phi+h(u))(0),\phi+h(u)) \\
&\qquad\qquad - g(0,(\phi+h(v))(0),\phi+h(v)\|_{[D(A)]} + \|g(t,u(t),u_t)-g(t,v(t),v_t)\|_{[D(A)]} \\
&\quad + \int_0^t \varphi_A(t-s)\Big[\|h(u)(0)-h(v)(0)\|_{[D(A)]} \\
&\qquad + \|g(0,(\phi+hu)(0),\phi+h(u)) - g(0,(\phi+h(v))(0),\phi+h(v))\|_{[D(A)]} \\
&\qquad + \|g(s,u(s),u_s)-g(s,v(s),v_s)\|_{[D(A)]}\Big]\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[\|f(s,u(s),u_s)-f(s,v(s),v_s)\|_{[D(A)]}\Big]\,\mathrm{d}s \\
&\quad + \int_0^t \varphi_A(t-s)\left[\frac{1}{\Gamma(q)}\int_0^s (s-\xi)^{q-1}\|f(\xi,u(\xi),u_\xi)-f(\xi,v(\xi),v_\xi)\|_{D(A)}\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\le L_h\|u-v\| + L_g\big(L_h\|u-v\| + L_h\|u-v\|_{[-\tau,t]}\big) + L_g\big(\|u(t)-v(t)\| + \|u_t-v_t\|_{[-\tau,0]}\big) \\
&\quad + \int_0^t \varphi_A(t-s)\Big[L_h\|u-v\| + L_g\big(L_h\|u(s)-v(s)\| \\
&\qquad + L_h\|u_s-v_s\|_{[-\tau,0]}\big) + L_g\big(\|u(s)-v(s)\| + \|u_s-v_s\|_{[-\tau,0]}\big)\Big]\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}L_f\big(\|u(s)-v(s)\| + \|u_s-v_s\|_{[-\tau,0]}\big)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t \varphi_A(t-s)\left[\int_0^s (s-\xi)^{q-1}L_f\big(\|u(\xi)-v(\xi)\| + \|u_\xi-v_\xi\|_{[-\tau,0]}\big)\,\mathrm{d}\xi\right]\mathrm{d}s.
\end{aligned}
$$

Since $\|u_t\|_{[-\tau,0]} = \sup_{\theta\in[-\tau,0]}\|u_t(\theta)\|$ and $\sup_{t\in[0,T]}\|u_t\|_{[-\tau,0]} \le \sup_{t\in[-\tau,T]}\|u(t)\|$. Therefore, we get

$$
\begin{aligned}
&\|Qu(t)-Qv(t)\| \\
&\qquad \le \big[(L_h+2L_hL_g+2L_g) + (L_h+2L_hL_g+2L_g)\|\varphi_A\|_{L^1}\big] \times \|\, u-v\|_{[-\tau,T]} \\
&\qquad\quad + \left[2L_f\frac{T^q}{\Gamma(1+q)} + 2L_f\frac{T^q}{\Gamma(1+q)}\|\varphi_A\|_{L^1}\right]\|u-v\|_{[-\tau,T]}, \quad \forall\, t\in[0,T], \\
&\qquad = L_B\|u-v\|_{[-\tau,T]},
\end{aligned}
$$

where $L_B = (L_h+2L_hL_g+2L_g)\times(1+\|\varphi_A\|_{L^1}) + [2L_f\frac{T^q}{\Gamma(1+q)}(1+||\varphi_A||_{L^1})]$. Taking the supremum over $[-\tau,T]$, we get

$$
\|Q(u)-Q(v)\|_{[-\tau,T]} \le L_B||u-v||_{[-\tau,T]}.
$$

Since $L_B<1$. Therefore, it implies that $\|Q(u)-Q(v)\| < ||u-v||_{[-\tau,T]}$, *i.e.*, Q is a contraction mapping on $C([-r,T];X)$ and there exists a unique fixed point of the map Q which is just a unique mild solution for the system (1.1)–(1.2) by Banach fixed point theorem. □

Our second existence result is based on the Krasnoselskii's fixed point theorem.

Lemma 3.2 *Let X be a Banach space and B be a non-empty closed bounded and convex subset of X. Let Q_1 and Q_2 be two operators such that*

(1) $Q_1x+Q_2y\in B$, *for every pair* $x,y\in B$;

(2) *Q_1 is a contraction mapping;*

(3) *Q_2 is completely continuous.*

Then, there exists a fixed point of map $Q = Q_1 + Q_2$.

Theorem 3.3 *Let $\phi \in C([-\tau, 0]; [D(A)])$, $h \in C(C([-\tau, 0]; X), C([-\tau, 0]; [D(A)]))$, $g \in C([0, T] \times X \times C([-\tau, 0]; X); [D(A)])$ and $f \in C([0, T] \times X \times C([-\tau, 0]; X); [D(A)])$ be completely continuous such that f, g, h satisfy the hypotheses* (A1)–(A3)*, respectively. If*

$$\left(2c_1 + 2L_f \frac{T^q}{q\Gamma(q)}\right) \times (1 + \|\varphi_A\|_{L^1}) < 1 \tag{3.6}$$

and

$$L = (L_h + 2L_h L_g + 2L_g)(1 + \|\varphi_A\|_{L^1}) < 1, \tag{3.7}$$

then there exists a mild solution of (1.1)–(1.2) *on* $[-\tau, T]$.

Proof. To prove the result we convert the problem of the existence of a solution of (1.1)–(1.2) into a fixed point problem.

Consider the map $Q\colon C([-\tau, T]; X) \to C([-\tau, T]; X)$ defined by

$$Qu(t) = \begin{cases} \phi(t) + (hu)(t), & t \in [-\tau, 0], \\ Gu(t) + Fu(t) + \displaystyle\int_0^t S'(t-s)[G(u)(s) + F(u)(s)]\,\mathrm{d}s, & t \in [0, T]. \end{cases} \tag{3.8}$$

Further, define

$$B_k(C([-\tau, T]; X)) = \big\{u \in C([-\tau, T]; X); \|u(t)\| \le k, \forall\, t \in [-\tau, T]\big\} = B_k,$$

which is a closed and convex ball with center at the origin and radius k. Now, we introduce the decomposition of the map Q into Q_1 and Q_2 on $B_k(C([-\tau, T]; X))$ given by

$$Q_1u(t) = \begin{cases} \phi(t) + h(u)t, & t \in [-\tau, 0], \\ Gu(t) + \displaystyle\int_0^t S'(t-s)[G(u)(s)]\,\mathrm{d}s, & t \in [0, T], \end{cases}$$

$$Q_2u(t) = \begin{cases} 0, & t \in [-\tau, 0], \\ Fu(t) + \displaystyle\int_0^t S'(t-s)[F(u)(s)]\,\mathrm{d}s, & t \in [0, T]. \end{cases}$$

Since F and G are both continuous. By the assumptions (A1)–(A3) we have that for any $u \in C([0, T]; X)$,

$$\begin{aligned} &\left\|\int_0^t S'(t-s)[Gu(s) + Fu(s)]\,\mathrm{d}s\right\| \\ &\qquad \le \int_0^t \|S'(t-s)[Gu(s) + Fu(s)]\|\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \varphi_A(t-s)\|Gu(s)\|_{[D(A)]}\,\mathrm{d}s \\
&\qquad\qquad + \int_0^t \varphi_A(t-s)\left[\frac{1}{\Gamma(q)}\int_0^s \frac{\|f(\xi,u(\xi),u_\xi)\|_{[D(A)]}}{(s-\xi)^{1-q}}\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\le \left(\sup_{s\in[-\tau,T]}\|Gu(s)\|_{[D(A)]} + \sup_{s\in[-\tau,T]}\|f(s,u(s),u_s)\|_{[D(A)]}\frac{T^q}{\Gamma(q+1)}\right)\times\|\varphi_A\|_{L^1},
\end{aligned}
$$

it follows that the function $s\mapsto S'(t-s)[Gu(s)+Fu(s)]$ is integrable on $(0,t)$ for all $t\in[0,T]$. Therefore mapping Q is well-defined with the values in $C([-\tau,T];X)$.

It is easily observed that a fixed point of the map Q is a mild solution of the problem (1.1)–(1.2). We give the proof in several steps.

Step 1. Next, we prove that $Q(B_k)\subset B_k$. Suppose, on the contrary that for each $k\in\mathbb{N}$ there exists $u^k\in B_k$ and some $t^k\in[-\tau,T]$ such that $k<\|Qu^k(t^k)\|$. If $t^k\in[-\tau,0]$, then we get

$$
\begin{aligned}
k &< \|Qu^k(t^k)\| \le \|\phi(t^k)\|_{[D(A)]} + \|h(u^k)(t^k)\|_{[D(A)]} \\
&\le \|\phi\|_{[-\tau,0]} + N,
\end{aligned}
\tag{3.9}
$$

and if $t^k\in[0,T]$, we get

$$
k<\|Q(u^k)(t^k)\|\le\|Q_1(u^k)(t^k)\|+\|Q_2(u^k)(t^k)\|. \tag{3.10}
$$

Now,

$$
\begin{aligned}
&\|Q_1(u^k)(t^k)\| \\
&\quad\le \|Gu^k(t^k)\| + \int_0^{t^k}\|S'(t^k-s)Gu^k(s)\|\,\mathrm{d}s \\
&\quad\le \|Gu^k(t^k)\| + \int_0^{t^k}\varphi_A(t^k-s)\|Gu^k(s)\|_{[D(A)]}\,\mathrm{d}s \\
&\quad\le \|\phi(0)\| + \|h(u^k)(0)\| + \|g(0,(\phi+h(u^k))(0),\phi+h(u^k))\| \\
&\qquad\quad + \|g(t^k,u^k(t^k),u^k_{t^k})\| + \int_0^{t^k}\varphi_A(t^k-s)\big[\|\phi(0)\| + \|h(u^k)(0)\|_{[D(A)]} \\
&\qquad\quad + \|g(0,(\phi+h(u^k))(0),\phi+h(u^k))\|_{[D(A)]} + \|g(s,u^k(s),u^k_s)\|_{[D(A)]}\big]\,\mathrm{d}s \\
&\quad\le \|\phi(0)\| + N + [2c_1(\|\phi\|_{[-\tau,0]}+N)+c_2] + [2c_1k+c_2] \\
&\qquad\quad + \int_0^{t^k}\varphi_A(t^k-s)\big[\|\phi(0)\| + N + [2c_1(\|\phi\|_{[-\tau,0]}+N)+c_2] + [2c_1k+c_2]\big]\,\mathrm{d}s \\
&\quad\le \big(\|\phi(0)\| + N + [2c_1(\|\phi\|_{[-\tau,0]}+N)+c_2] + [2c_1k+c_2]\big)\times(1+\|\varphi_A\|_{L^1})
\end{aligned}
\tag{3.11}
$$

and

$$
\|Q_2u^k(t^k)\| \le \|Fu^k(t^k)\| + \int_0^{t^k}\|S'(t^k-s)Fu^k(s)\|\,\mathrm{d}s. \tag{3.12}
$$

Since we have

$$\begin{aligned}\left\|\frac{1}{\Gamma(q)}\int_0^{t^k}(t^k-s)^{q-1}f(s,u^k(s),u_s^k)\,\mathrm{d}s\right\| \\ &\le \frac{1}{\Gamma(q)}\int_0^{t^k}\frac{(\|f(s,u^k(s),u_s^k)-f(s,0,0)\|+\|f(s,0,0)\|)}{(t^k-s)^{1-q}}\,\mathrm{d}s \\ &\le \frac{1}{\Gamma(q)}\int_0^{t^k}(t^k-s)^{q-1}\big[L_f(\|u^k(s)\|+\|u_s^k\|_{[-\tau,0]})+B\big]\,\mathrm{d}s \\ &\le (2L_fk+B)\frac{T^q}{\Gamma(q+1)},\end{aligned} \tag{3.13}$$

therefore from (3.12) and (3.13), we get

$$\|Q_2u^k(t^k)\| \le (2L_fk+B)\frac{T^q}{\Gamma(q+1)}(1+\|\varphi_A\|_{L^1}). \tag{3.14}$$

Then equations (3.10), (3.11) and (3.14) imply that

$$\begin{aligned}k<\|Qu^k(t^k)\| \le \Big\{&\|\phi(0)\|+N+[2c_1(\|\phi\|_{[-\tau,0]}+N)+c_2]+[2c_1k+c_2] \\ &+(2L_fk+B)\frac{T^q}{\Gamma(q+1)}\Big\}\times(1+\|\varphi_A\|_{L^1})=L_k \text{ (say)}.\end{aligned} \tag{3.15}$$

From the inequality (3.9) and (3.15), we get

$$k<\max\{\|\phi\|_{[-\tau,0]}+N, L_k\}. \tag{3.16}$$

We divide both sides of the equation (3.16) by k and taking limit $k\to\infty$, we obtain

$$\left(2c_1+2L_f\frac{T^q}{\Gamma(q+1)}\right)\times(1+\|\varphi_A\|_{L^1})\ge 1,$$

which contradicts (3.6). Therefore, we have that there exists a positive integer $k\in\mathbb{N}$ such that $Q(B_k)\subset B_k$.

To prove the result that Q has a fixed point, firstly we show that Q_1 is a contraction and secondly that Q_2 is completely continuous.

Step 2. Q_1 is a contraction. For $u,v\in B_k$, we have

$$\begin{aligned}\|Q_1u(t)-Q_1v(t)\| &\le \|h(u)(t)-h(v)(t)\| \\ &\le L_h\|u(t)-v(t)\|, \quad \text{if} \quad t\in[-\tau,0],\end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
&\|Q_1u(t)-Q_1v(t)\| \\
&\quad\le \|h(u)(0)-h(v)(0)\| + \|g(0,(\phi+hu)0,\phi+h(u)) \\
&\qquad\qquad - g(0,(\phi+hv)0,\phi+h(v))\| + \|g(t,u(t),u_t)-g(t,v(t),v_t)\| \\
&\qquad + \int_0^t \big\|S'(t-s)\big[h(u)(0)-h(v)(0)+g(0,(\phi+hu)(0),\phi+h(u)) \\
&\qquad\qquad - g(0,(\phi+h(v))(0),\phi+h(v))+g(s,u(s),u_s)-g(s,v(s),v_s)\big]\big\|\,\mathrm{d}s \\
&\quad\le (L_h+2L_hL_g+2L_g)\|u-v\|_{[-\tau,T]} \qquad\qquad (3.18)\\
&\qquad + \int_0^t \varphi_A(t-s)\big[\|h(u)(0)-h(v)(0)\|_{[D(A)]} \\
&\qquad\qquad + \|g(0,(\phi+hu)(0),\phi+h(u))-g(0,(\phi+h(v))0,\phi+h(v))\|_{[D(A)]} \\
&\qquad\qquad + \|g(s,u(s),u_s)-g(s,v(s),v_s)\|_{[D(A)]}\big]\,\mathrm{d}s \\
&\quad\le (L_h+2L_hL_g+2L_g)(1+\|\varphi_A\|_{L^1})\times\|u-v\|_{[-\tau,T]}, \quad \forall\, t\in[0,T], \\
&\quad= L\|u-v\|_{[-\tau,T]}, \quad \text{if } t\in[0,T],
\end{aligned}
$$

where $L=(L_h+2L_hL_g+2L_g)(1+\|\varphi_A\|_{L^1})<1$. Taking supremum over $[-\tau,T]$ and from (3.17) and (3.18), we get

$$\|Q_1u-Q_1v\|_{[-\tau,T]}\le L\|u-v\|_{[-\tau,T]},$$

whence it follows that Q_1 is a contraction on B_k.

Step 3. Q_2 is completely continuous in X. Since the function

$$s\mapsto \int_0^t S'(t-s)\int_0^s (s-\xi)^{q-1}f(\xi,u(\xi),u_\xi)\,\mathrm{d}\xi\,\mathrm{d}s$$

is integrable on $[0,T]$ from the assumption (A1). Now we show that Q_2 is uniformly bounded. For $t\in[0,T]$, we have

$$
\begin{aligned}
\|Q_2u(t)\| &\le \|Fu(t)\| + \int_0^t \|S'(t-s)Fu(s)\|\,\mathrm{d}s \\
&\le (2L_fk+B)\frac{T^q}{\Gamma(q+1)}(1+\|\varphi_A\|_{L^1}),
\end{aligned}
$$

and for $t\in[-\tau,0]$, we have $Q_2u(t)=0$. It implies that Q_2 is uniformly bounded on $[-\tau,T]$.

Now, let $\{u_n\}$ be a sequence in B_k such that $u_n\to u$ in B_k. By the continuity of f, we have

$$f(s,u_n(s),(u_n)_s)\to f(s,u(s),u_s), \qquad \text{as} \qquad n\to\infty.$$

For each $t\in[0,T]$, we have

$$
\begin{aligned}
&\|Q_2u_n(t)-Q_2u(t)\| \\
&\quad\le \|Fu_n(t)-Fu(t)\| + \int_0^t \|S'(t-s)[Fu_n(s)-Fu(s)]\|\,\mathrm{d}s \\
&\quad\le \frac{1}{\Gamma(q)}\int_0^t \frac{\|f(s,u_n(s),(u_n)_s)-f(s,u(s),u_s)\|}{(t-s)^{1-q}}\,\mathrm{d}s \\
&\qquad + \int_0^t \varphi_A(t-s)\frac{1}{\Gamma(q)}\left[\int_0^s \frac{\|f(\xi,u_n(\xi),(u_n)_\xi)-f(\xi,u(\xi),u_\xi)\|_{[D(A)]}}{(s-\xi)^{1-q}}\,\mathrm{d}\xi\right]\mathrm{d}s,
\end{aligned}
$$

which tends to zero when $n \to \infty$ and for $t \in [-\tau, 0]$, we have $Q_2 u(t) = 0$. Thus, Q_2 is continuous on $[-\tau, T]$.

Next, we show that the set $\{Q_2 u(t) : u \in B_k\}$ is relatively compact in X for all $t \in [-\tau, T]$. For $t \in [-\tau, 0]$, the set $\{Q_2 u(t) : u \in B_k\}$ is compact. For $t \in [0, T]$, let ε be a real number satisfying $0 < \varepsilon < t$. For $u \in B_k$, define the operator Q_2^ε by

$$Q_2^\varepsilon u(t) = \frac{1}{\Gamma(q)} \int_0^{t-\varepsilon} \frac{f(s, u(s), u_s)}{(t-s)^{1-q}} \,\mathrm{d}s + \int_0^{t-\varepsilon} S'(t-s) \frac{1}{\Gamma(q)} \int_0^s \frac{f(\xi, u(\xi), u_\xi)}{(s-\xi)^{1-q}} \,\mathrm{d}\xi \,\mathrm{d}s.$$

By the assumption (A1), f is completely continuous, the set $X_\varepsilon = \{Q_2^\varepsilon u(t);\ u \in B_k\}$ is precompact in X, for every $\varepsilon > 0$, $\varepsilon \in (0, t)$. Moreover, for every $u \in B_k$, we have

$$\begin{aligned}\|Q_2 u(t) - Q_2^\varepsilon u(t)\| \le{}& \frac{1}{\Gamma(q)} \int_{t-\varepsilon}^t \frac{\|f(s, u(s), u_s)\|}{(t-s)^{1-q}} \,\mathrm{d}s \\ &+ \int_{t-\varepsilon}^t \left\| S'(t-s) \frac{1}{\Gamma(q)} \int_0^s \frac{f(\xi, u(\xi), u_\xi)}{(s-\xi)^{1-q}} \,\mathrm{d}\xi \right\| \mathrm{d}s.\end{aligned}$$

It shows that precompact sets X_ε are arbitrary close to the set $\{Q_2 u(t) : u \in B_k\}$. Hence, the set $\{Q_2 u(t) : u \in B_k\}$ is precompact in X.

Step 4. Next, we show that $Q_2(B_k)$ is equicontinuous. For $t \in [-\tau, 0]$, we have $Q_2 u(t) = 0$ and for $t \in [0, T]$ and $0 < t < t + h \le T$, $h > 0$, we get that

$$\begin{aligned}&\|Q_2 u(t+h) - Q_2 u(t)\| \\ &\quad\le \frac{1}{\Gamma(q)} \left\| \int_0^{t+h} \frac{f(s, u(s), u_s)}{(t+h-s)^{1-q}} \,\mathrm{d}s - \int_0^t \frac{f(s, u(s), u_s)}{(t-s)^{1-q}} \,\mathrm{d}s \right\| \\ &\qquad + \frac{1}{\Gamma(q)} \left\| \int_0^{t+h} S'(t+h-s) F u(s) \,\mathrm{d}s - \int_0^t S'(t-s) F u(s) \,\mathrm{d}s \right\| \\ &\quad\le \frac{1}{\Gamma(q)} \int_0^t \left[\frac{1}{(t+h-s)^{1-q}} - \frac{1}{(t-s)^{1-q}} \right] \|f(s, u(s), u_s)\| \,\mathrm{d}s \\ &\qquad + \frac{1}{\Gamma(q)} \int_t^{t+h} \frac{1}{(t+h-s)^{1-q}} \|f(s, u(s), u_s)\| \,\mathrm{d}s \\ &\qquad + \int_0^h \left\| S'(t+h-s) \frac{1}{\Gamma(q)} \int_0^s \frac{f(\xi, u(\xi), u_\xi)}{(s-\xi)^{1-q}} \,\mathrm{d}\xi \right\| \mathrm{d}s \\ &\qquad + \int_0^t \varphi_A(t-s) \frac{1}{\Gamma(q)} \left\{ \left\| \int_0^{s+h} \frac{f(\xi, u(\xi), u_\xi)}{(s+h-\xi)^{1-q}} \,\mathrm{d}\xi \right.\right. \\ &\qquad\qquad \left.\left. - \int_0^s \frac{f(\xi, u(\xi), u_\xi)}{(s-\xi)^{1-q}} \,\mathrm{d}\xi \right\|_{[D(A)]} \right\} \mathrm{d}s,\end{aligned}$$

which tends to zero as $h \to 0$, and therefore the set $\{Q_2(u) :\ u \in B_k\}$ is equicontinuous. Thus, we proved that $Q_2(B_k)$ is relatively compact for $t \in [-\tau, T]$. Thus, Q_2 is completely continuous by Arzelà–Ascoil theorem. Hence, by the Kranoselskii fixed point theorem, there exists a fixed point $u \in B_k$ such that $Qu(t) = (Q_1 + Q_2)u(t) = u(t)$ which is a mild solution of the problem (1.1) with nonlocal conditions (1.2). $\square$

4 Application

Let $X = L^2(0,1)$, $0 < \alpha < 1$ and $\tau > 0$. Now consider the following fractional order partial differential equation of the form

$$\begin{aligned}
&\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\biggl(w(t,x) + \biggl[f_1(t,x) + \int_0^1 h_1\bigl(w(t,x), \partial_x w(t,x)\bigr)\,\mathrm{d}x \\
&\qquad\qquad + \int_{-\tau}^0 k_1(-\theta) h_2\bigl(w(t+\theta,x), \partial_x w(t+\theta,x)\bigr)\,\mathrm{d}\theta\biggr]\biggr) - \partial_x^2 w(t,x) \\
&= f_2(t,x) + \int_0^1 h_3\bigl(w(t,x), \partial_x w(t,x)\bigr)\,\mathrm{d}x \\
&\qquad\qquad + \int_{-\tau}^0 k_2(-\theta) h_4\bigl(w(t+\theta,x), \partial_x w(t+\theta,x)\bigr)\,\mathrm{d}\theta, \quad x \in (0,1), t > 0, \\
&\qquad w(t,0) = w(t,1) = 0, \quad t \in [0,T], \quad 0 < T < \infty, \\
&\qquad w(t,x) = \chi(t,x) + h(w(t,x))(t,x), \quad t \in [-\tau, 0], \quad x \in (0,1),
\end{aligned} \tag{4.1}$$

where h_1, h_2, h_3, h_4, f_1 and f_2 are real-valued smooth functions, k_1 and k_2 are square integrable functions and χ is a continuous function on $[-\tau, 0]$ and $h\colon C([-\tau,0];X) \to C([-\tau,0];X)$ is Lipschitz continuous function.

Further we define an operator A as

$$Au = u'' \quad \text{with} \quad u \in D(A) = \{u \in H_0^1(0,1) \cap H^2(0,1)\};$$

here, the operator A is the infinitesimal generator of an analytic semi-group $S(t)$. Now, equation (4.1) can be reformulated as

$$\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}[u(t) + g(t, u(t), u_t)] &= Au(t) + f(t, u(t), u_t), \quad t \in [0,T], \\
u(t) &= \chi(t) + h(u)(t), \quad t \in [-\tau, 0],
\end{aligned} \tag{4.2}$$

where $u(t)(x) = w(t,x)$, $u_t(\theta)(x) = w(t+\theta, x)$, $t \in [0,T]$, $\theta \in [-\tau, 0]$, $x \in (0,1)$ and $u(t)(x) = \chi(t,x) + h(u)(t)(x)$, $t \in [-\tau, 0]$, $x \in (0,1)$.

To represent the system (4.1)–(4.2) in the abstract form (1.1)–(1.2), we define the function $g\colon [0,T] \times X \times C([-\tau,0];X) \to [D(A)]$ as

$$\begin{aligned}
g(t,\psi,\vartheta)(x) &= f_1(t,x) \\
&\quad + \int_0^1 h_1\bigl(\psi(x), \psi'(x)\bigr)\,\mathrm{d}x + \int_{-\tau}^0 k_1(-\theta) h_2\bigl(\vartheta(\theta)(x), \partial_x \vartheta(\theta)(x)\bigr)\,\mathrm{d}\theta,
\end{aligned}$$

where $f_1\colon [0,T] \times (0,1) \to \mathbb{R}$ is a function such that

(a1) $f_1(0,\cdot) \in L^2(0,1)$,

(a2) $|f_1(t,x) - f_1(s,x)| \le k_3(x)|t-s|^\theta, \quad \forall\, t,s \in [0,T]$, a.e. $x \in (0,1)$,

where $k_3 \in L^2(0,1)$ and h_1, h_2 are Lipschitz continuous non-decreasing functions such that

$$\|h_1(x_1,y_1) - h_1(x_2,y_2)\| \le L_1\bigl[\|x_1 - y_1\| + \|x_2 - y_2\|\bigr],$$

$$\|h_2(x_1,y_1)-h_2(x_2,y_2)\| \le L_2\big[\|x_1-y_1\|_{[-\tau,0]}+\|x_2-y_2\|_{[-\tau,0]}\big],$$

and $\|h_1(0,0)\| = H_1$, $\|h_2(0,0)\| = H_2$. Therefore g satisfies the following conditions

$$\begin{aligned} &\|g(t,u_1,v_1)-g(t,u_2,v_2)\| \le L_g\big[\|u_1-u_2\|+\|v_1-v_2\|_{[-\tau,0]}\big],\\ &\|\ g(t,u,v)\| \le F_1+2L_1(\|x\|)+H_1+2L_2\|k\|_{L^1}(\|v\|_{[-\tau,0]})+H_2, \end{aligned} \tag{G}$$

where $L_g=\max\{2L_1,2L_2\|k_1\|_{L^1}\}$. It implies that g satisfy the assumption (A2).

Similarly, we can define the function $f\colon [0,T]\times X\times C([-\tau,0];X)\to[D(A)]$ such that

$$f(t,\psi,\vartheta)(x)=f_2(t,x)+\int_0^1 h_3\big(\psi(x),\psi^{'}(x)\big)\,\mathrm{d}x \\ +\int_{-\tau}^0 k_2(-\theta)h_4\big(\vartheta(\theta)(x),\partial_x\vartheta(\theta)(x)\big)\,\mathrm{d}\theta,$$

and $f_2\colon [0,T]\times(0,1)\to\mathbb{R}$ satisfies the following conditions

(b1) $|f_2(t,x)-f_2(s,x)|\le k_4(x)|t-s|^{\theta_1}, \quad \forall\, t,s\in[0,T]$, a.e. $x\in(0,1)$,

(b2) $f_2(0,\cdot)\in L^2(0,1)$,

where $k_4\in L^2(0,1)$ and h_3, h_4 are Lipschitz continuous non-decreasing functions such that

$$\begin{aligned} &\|h_3(x_1,y_1)-h_3(x_2,y_2)\| \le L_3\big[\|x_1-y_1\|+\|x_2-y_2\|\big],\\ &\|h_4(x_1,y_1)-h_5(x_2,y_2)\| \le L_4\big[\|x_1-y_1\|_{[-\tau,0]}+\|x_2-y_2\|_{[-\tau,0]}\big]. \end{aligned}$$

Thus f satisfies the assumption (A1), *i.e.*,

$$\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \le L_f\big[\|u_1-u_2\|+\|v_1-v_2\|_{[-\tau,0]}\big], \tag{F}$$

where $L_f=\max\{2L_3,2L_4\|k_2\|_{L^1}\}$ and $B_1=\|f(t,0,0)\|$.

Also $h\colon C([-\tau,0];X)\to C([-\tau,0];[D(A)])$ satisfies the following condition

$$\|h(u)-h(v)\|\le L_h\|u-v\|, \qquad \|h(u)\|\le N_1. \tag{H}$$

Therefore, all the assumptions of Theorem 3.1 and Theorem 3.3 hold which guarantees the existence of a solution of the equation (4.2).

Theorem 4.1 *Suppose inequalities* (G), (F) *and* (H) *hold and A generates resolvent operator $\{S(t)\}_{t\ge0}$ which is differentiable. If*

$$\left[L_h+2(1+L_h)(L_g)+2L_f\frac{T^q}{\Gamma(q+1)}\right](1+\|\varphi_A\|_{L^1})<1,$$

then the problem (4.2) *has a unique solution.*

Theorem 4.2 *Let* (G), (F) *and* (H) *hold. If*

$$\left[2L_g+2L_f\frac{T^q}{\Gamma(q+1)}\right]<1$$

and

$$[L_h+2(1+L_h)L_g+2L_g](1+\|\varphi\|_{L^1})<1,$$

where $L_g=\max\{2L_1,2L_2\|k_1\|_{L^1}\}$, $L_f=\max\{2L_3,2L_4\|k_2\|_{L^1}\}$, then the problem (4.2) *has a mild solution.*

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