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APPROXIMATION OF A SOLUTION OF A SEMILINEAR DIFFERENTIAL EQUATION WITH A DEVIATED ARGUMENT

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Abstract. In this paper we shall study the approximation of a solution to a class of semilinear evolution equation with a deviated argument in a separable Hilbert space. With the help of analytic semigroup theory and Banach fixed theorem we shall establish the existence and uniqueness of solutions of the problem considered. Next we consider the Faedo-Galerkin approximation of the solution and prove some convergence results.

Keywords: Semilinear evolution equation with a deviated argument, Banach fixed point theorem, Faedo-Galerkin approximation and analytic semigroup.

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1 Introduction

In the present study we are interested in the Faedo-Galerkin approximation of solutions to the following class of semilinear evolution equation with a deviated argument in a separable Hilbert space $(H, \|.\|, (.,.))$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[u(t)+g(t,u(t))]+Au(t) = f(t,u(t),u[h(u(t),t)]), \\ u(0) = u_0, \quad 0<t\leq T<\infty. \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a close, densely defined, positive definite, self adjoint linear operator and satisfies assumption (H1) stated later. Functions f, g and h are suitably defined satisfying certain conditions to be stated later.

The initial results related to the differential equations with the deviated arguments can be found in some research papers of the last decade but still a complete theory seems to be missing. For the initial works on the existence, uniqueness and stability of various types of solutions of different kinds of differential equations, we refer to [5]–[16] and the references cited in these papers.

The existence and uniqueness of a solution to the following semilinear evolution equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(u(t)+g(t,u(t))) &= Au(t)+f(t,u(t)), \quad t>0, \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

has been studied by Hernández [4] under the assumptions that A is the infinitesimal generator of an analytic semigroup of bounded linear operators defined on a Banach space B and f and g are appropriate continuous functions on $[0,T]\times W$ into B where W is an open subset of B.

Hernandez and Henriquez [11, 12] established some results concerning the existence, uniqueness and qualitative properties of the solution operator of the following general partial neutral functional differential equation with the unbounded delay:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(u(t)-g(t,u_t)) &= Au(t)+f(t,u_t), \ \ t\geq 0, \\ u_0 &= \varphi\in C_0, \end{aligned} \tag{1.3}$$

where A generates an analytic semigroup on a Banach space B, g and f are continuous functions from $[0,\infty)\times C_0$ into B and for each $u : (-\infty,b]\to B,\ b>0$ and $t\in[0,b]$, u_t represents, as usual, the mapping defined from $(-\infty,0]$ into B by

$$u_t(\theta)=u(t+\theta) \ \text{ for } \ \theta\in(-\infty,0].$$

Also the approximation of a solution to a nonlinear Sobolev type evolution equation has been studied by Bahuguna and Shukla [1] in a separable Hilbert space $(H, \|.\|, (.,.))$, where the linear operator A satisfies the assumption $(H1)$ stated below so that A generates an analytic semigroup. The functions f and g are some appropriate continuous functions of their arguments in H. The Faedo-Galerkin approximations of a solution to the particular case of (1.1) where $g,h=0$ and $f(t,u)=M(u)$ has been considered by Milleta [3]. The more general case has been dealt with by D. Bahuguna, S.K. Srivastava and S. Singh [2].

2 Preliminaries and Assumptions

In this section, we shall provide the assumptions, notations, notions and related results needed for the subsequent sections. We assume that the operator A satisfies the following.

(H1) A is a closed, positive definite, self-adjoint, linear operator from the domain $D(A) \subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, i.e., $Au_i = \lambda_i u_i$ and $(u_i, u_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

These assumptions on A guarantee that $-A$ generates an analytic semigroup, denoted by $S(t)$, $t \geq 0$.

We mention some notions and preliminaries essential for our purpose. It is well known that there exist constants $\tilde{M} \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq \tilde{M}e^{\omega t}, \quad t \geq 0.$$

Since $-A$ generates the analytic semigroup $S(t)$, $t \geq 0$, we may add cI to $-A$ for some constant c, if necessary, and in what follows we may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \leq M$ and $0 \in \rho(A)$. In this case it is possible to define the fractional power A^α for $0 \leq \alpha \leq 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$ (cf. Pazy [17], pp. 69-75 and p. 195). Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|,$$

defines a norm on $D(A^\alpha)$. Henceforth we represent by H_α the space $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_\alpha)^*$, the dual space of H_α is a Banach space endowed with the norm $\|x\|_{-\alpha} = \|A^{-\alpha}x\|$. In the view of the facts mentioned above we have the following result for an analytic semigroup $S(t)$, $t \geq 0$ (cf. Pazy [17] pp. 195-196).

Lemma 2.1 *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following properties.*

(i) *H_α is a Banach space for $0 \leq \alpha \leq 1$.*

(ii) *For $0 < \delta \leq \alpha < 1$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous.*

(iii) *A^α commutes with $S(t)$ and there exists a constant $C_\alpha > 0$ depending on $0 \leq \alpha \leq 1$ such that*

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}, \quad t > 0.$$

For more details on the fractional powers of closed linear operators we refer to Pazy [17].

It can be seen easily that $\mathcal{C}_t^\alpha = C([0,t]; H_\alpha)$, for all $t \in [0,T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\alpha} := \sup_{0 \leq r \leq t} \|\psi(r)\|_\alpha, \ \psi \in C_t^\alpha.$$

We set, $\mathcal{C}_T^{\alpha-1} = C([0,T]; H_{\alpha-1}) = \{y \in \mathcal{C}_T^{\alpha} : \|y(t) - y(s)\|_{\alpha-1} \le L|t-s|, \forall t, s \in [0,T]\}$, where L is a suitable positive constant to be specified later and $0 \le \alpha < 1$.

We assume the following conditions:

(H2) Let $U_1 \subset \mathrm{Dom}(f)$ is an open subset of $\mathbb{R}_+ \times H_\alpha \times H_{\alpha-1}$ and for each $(t,u,v) \in U_1$ there is a neighborhood $V_1 \subset U_1$ of (t,u,v). The nonlinear map $f : \mathbb{R}_+ \times H_\alpha \times H_{\alpha-1} \to H$ satisfies the following condition,

$$\|f(t,x,\psi) - f(s,y,\tilde{\psi})\| \le L_f[|t-s|^{\theta_1} + \|x-y\|_\alpha + \|\psi - \tilde{\psi}\|_{\alpha-1}],$$

where $0 < \theta_1 \le 1$, $0 \le \alpha < 1$, $L_f > 0$ is a constant, $(t,x,\psi) \in V_1$, and $(s,y,\tilde{\psi}) \in V_1$.

(H3) Let $U_2 \subset$ Dom(h) is an open subset of $H_\alpha \times \mathbb{R}_+$ and for each $(x,t) \in U_2$ there is a neighborhood $V_2 \subset U_2$ of (x,t). The map $h : H_\alpha \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following condition,

$$|h(x,t) - h(y,s)| \le L_h[\|x-y\|_\alpha + |t-s|^{\theta_2}],$$

where $0 < \theta_2 \le 1$, $0 \le \alpha < 1$, $L_h > 0$ is a constant, $(x,t),\ (y,s) \in V_2$ and $h(.,0) = 0$.

(H4) Let $U_3 \subset$ Dom(g) is an open subset of $[0,T] \times H_{\alpha-1}$ and for each $(t,x) \in U_3$ there is a neighborhood $V_3 \subset U_3$ of (x,t). There exist positive constants $0 < \alpha < \beta < 1$, such that the function $A^\beta g$ is continuous for $(t,u) \in [0,T_0] \times H_{\alpha-1}$ such that

$$\begin{aligned} \|A^\beta g(t,x) - A^\beta g(s,y)\| &\le L_g\{|t-s| + \|x-y\|_{\alpha-1}\}, \text{ and} \\ 4L_g\|A^{\alpha-\beta-1}\| &< \; = \eta < 1 \end{aligned}$$

where $L_g, \eta > 0$ is positive constants and $(x,t),\ (y,s) \in V_3$.

3 Approximate solutions and convergence

The existence of a solution to (1.1) is closely related to the following integral equation (3.1).

Definition 3.1 *A continuous function $u : [0,T] \to H$ is said to be a mild solution of equation (1.1) if u is the solution of the following integral equation*

$$\begin{aligned} u(t) = S(t)[u(0) + g(0,u_0)] - g(t,u(t)) + \int_0^t AS(t-s)g(s,u(s))\,\mathrm{d}s \\ + \int_0^t S(t-s)f(s,u(s),u[h(u(s),s)])\,\mathrm{d}s, \\ t \in [0,T] \end{aligned} \tag{3.1}$$

and satisfies the initial condition $u(0) = u_0$.

Definition 3.2 *By a solution of the problem (1.1), we mean a function $u : [0,T] \to H$ satisfying the following four conditions:*

(i) $u(.) + g(., u(.)) \in C_T^{\alpha-1} \cap C^1((0,T), H) \cap C([0,T], H)$,

(ii) $u(t) \in D(A)$, *and* $f(t, u(t), u[h(u(t),t)]) \in U_1$,

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}[u(t) + g(t, u(t))] + A[u(t)] = f(t, u(t), u[h(u(t),t)])$ *for all* $t \in (0,T]$,

(iv) $u(0) = u_0$.

Let $H_n \subseteq H$ denote the finite subspace spanned by in $\{u_0, u_1, \cdots, u_n\}$ and let $P^n : H \longrightarrow H_n$ be the corresponding projection operator for $n = 0, 1, 2, \cdots$. We can prove that assumptions (H2)–(H3), $0 \leq \alpha < 1$ and $u \in C_{T_0}^{\alpha}$ imply that $f(s, u(s), u[h(u(s), s)])$ is continuous on $[0, T_0]$. Therefore, we can show that there exists a positive constant N such that

$$\|f(s, u(s), u[h(u(s), s)])\| \leq N = L_f[{T_0}^{\theta_1} + R(1 + LL_h) + LL_h T_0^{\theta_2}] + N_0,$$

where $N_0 = \|f(0, u_0, u_0)\|$. Similarly with the help of the assumption $(H4)$, we can show easily that $\|A^{\beta} g(t, u(t))\| \leq L_g[T_0 + R] + \|g(0, u_0)\|_{\alpha} = N_1$.

We define

$$g_n : \mathbb{R}_+ \times H \longrightarrow H \; ; \; g_n(t, u(t)) = g(t, P^n u(t)) \; .$$

and

$$f_n : \mathbb{R}_+ \times H \times H \longrightarrow H \; ; \; f_n(t, u(t), u[h(u(s), s)]) = f(t, P^n u(t), P^n u[(h(u(t), t))])$$

We set

$$\mathcal{W} = \{u \in \mathcal{C}_{T_0}^{\alpha} \cap \mathcal{C}_{T_0}^{\alpha-1} : u(0) = u_0, \;\; \|u - u_0\|_{T_0, \alpha} \leq R\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of $\mathcal{C}_{T_0}^{\alpha-1}$.

Theorem 3.3 *Let us assume that the assumptions (H1)–(H4) are satisfied and* $u_0 \in D(A^{\alpha})$ *for* $0 \leq \alpha < 1$. *Then there exists a unique* $u_n \in \mathcal{C}_{T_0}^{\alpha-1} \cap \mathcal{C}_{T_0}^{\alpha}$ *such that* $F_n u_n = u_n$ *for each* $n = 0, 1, 2, \ldots$, *i.e.,* u_n *satisfies the approximate integral equation*

$$\begin{aligned} u_n(t) \;\; &= \;\; S(t)[u(0) + g_n(0, u_0)] - g_n(t, u(t)) + \int_0^t AS(t-s))g_n(s, u(s))\,\mathrm{d}s \\ &\quad + \int_0^t S(t-s) f_n(s, u(s), u[h(u(s), s)])\,\mathrm{d}s, t \in [0, T_0] \; . \end{aligned} \tag{3.2}$$

Proof. For a fixed $R > 0$, we choose $0 < T_0 = T_0(\alpha, \beta, u_0) \leq T$ such that

$$C_{\alpha+1-\beta} L_g \frac{T_0^{\beta-\alpha}}{\beta - \alpha} + C_{\alpha} L_f [2 + LL_h] T_0^{1-\alpha} \leq 1 - \eta \tag{3.3}$$

where $\eta = 4L_g \|A^{\alpha-\beta-1}\| < 1$ and $T_0 < \min(d_1, d_2)$ with

$$d_1 = \left(\frac{R}{4}(\beta - \alpha)(C_{1+\alpha-\beta} L_g)^{-1}\right)^{\frac{1}{\beta-\alpha}}, \tag{3.4}$$

$$d_2 = \left(\frac{R}{4}(1 - \alpha)(C_{\alpha}[2 + LL_h] L_f)^{-1}\right)^{\frac{1}{1-\alpha}} \tag{3.5}$$

and satisfying the following

$$\|(S(t)-I)A^{\alpha}[u_0+g_n(0,u_0)]\|+\|A^{\alpha-\beta}\|L_g[T_0+R]\leq \frac{R}{2}, \tag{3.6}$$

for all $t\in[0,T_0]$.

$$C_{\alpha+1-\beta}N_1\frac{T_0^{\beta-\alpha}}{\beta-\alpha}+C_{\alpha}N\frac{T_0^{1-\alpha}}{1-\alpha}\leq \frac{R}{2}. \tag{3.7}$$

For more details of choosing such a T_0, we refer Theorem 2.2 of [10].

We define a map $\mathcal{F}_n:\mathcal{W}\to\mathcal{W}$ given by

$$\begin{aligned}(\mathcal{F}_n u)(t)&=S(t)[u(0)+g_n(0,u_0)]-g_n(t,u(t))+\int_0^t AS(t-s)g_n(s,u(s))\,\mathrm{d}s\\&+\int_0^t S(t-s)f_n(s,u(s),u[h(u(s),s)])\,\mathrm{d}s,\, t\in[0,T_0].\end{aligned} \tag{3.8}$$

In order to prove this theorem first we need to show that $\mathcal{F}_n u\in\mathcal{C}_{T_0}^{\alpha-1}$ for any $u\in\mathcal{C}_{T_0}^{\alpha-1}$. Clearly, $\mathcal{F}_n:\mathcal{C}_{T_0}^{\alpha}\to\mathcal{C}_{T_0}^{\alpha}$.

If $u\in\mathcal{C}_{T_0}^{\alpha-1},\quad T_0>t_2>t_1>0,$ and $0\leq\alpha<1$, then we get

$$\begin{aligned}&\|(\mathcal{F}_n u)(t_2)-(\mathcal{F}_n u)(t_1)\|_{\alpha-1}\\&\quad\leq\|(S(t_2)-S(t_1))(u_0+g_n(0,u_0))\|_{\alpha-1}\\&\quad+\|A^{\alpha-1-\beta}\|\|A^{\beta}g_n(t_2,u(t_2))-A^{\beta}g_n(t_1,u(t_1))\|\\&\quad+\int_0^{t_1} S(t_2-s)-S(t_1-s))A^{\alpha-1}\|\ \|g_n(s,u(s))\|\,\mathrm{d}s\\&\quad+\int_{t_1}^{t_2} S(t_2-s)A^{\alpha-1}\|\|g_n(s,u(s))\|\,\mathrm{d}s\\&\quad+\int_0^{t_1} S(t_2-s)-S(t_1-s))A^{\alpha-1}\|\ \|f_n(s,u(s),u[h(u(s),s)])\|\,\mathrm{d}s\\&\quad+\int_{t_1}^{t_2} S(t_2-s)A^{\alpha-1}\|\|f_n(s,u(s),u[h(u(s),s)])\|\,\mathrm{d}s.\end{aligned} \tag{3.9}$$

For the first part of right hand side of (3.9), we have,

$$\begin{aligned}&\|(S(t_2)-S(t_1))(u_0+g_n(0,u_0))\|_{\alpha-1}\\&\quad\leq\int_{t_1}^{t_2}\|A^{\alpha-1}S'(s)(u_0+g_n(0,u_0))\|\,\mathrm{d}s\\&\quad=\int_{t_1}^{t_2}\|A^{\alpha}S(s)(u_0+g_n(0,u_0))\|\,\mathrm{d}s\\&\quad\leq\int_{t_1}^{t_2}\|S(s)\|\ \|[\|u_0\|_{\alpha}+\|A^{\alpha-\beta}\|\ \|g_n(0,u_0))\|_{\beta}\,\mathrm{d}s]\\&\quad\leq C_1(t_2-t_1)\end{aligned} \tag{3.10}$$

where $C_1 = [\|u_0\|_\alpha + \|A^{\alpha-\beta}\|\|g_n(0,u_0)\|_\beta]M$.

For the second part of right hand side of (3.9), we can see that

$$\begin{aligned}
&\|A^{\alpha-\beta-1}\|\|A^\beta g_n(t_2,u(t_2)) - A^\beta g_n(t_1,u(t_1))\| \\
&\quad \le \|A^{\alpha-\beta-1}\|L_g[(t_2-t_1) + \|u(t_2)-u(t_1)\|_{\alpha-1}] \\
&\quad \le \|A^{\alpha-\beta-1}\|[L_g(1+L)](t_2-t_1) \\
&\quad \le C_2(t_2-t_1). \qquad (3.11)
\end{aligned}$$

where $C_2 = \|A^{\alpha-\beta-1}\|[L_g(1+L)]$. To handle the third and fifth part of the right hand side of (3.9), observe that,

$$\begin{aligned}
\|S(t_2-s)-S(t_1-s)\|_{\alpha-1} &\le \int_0^{t_2-t_1} \|A^{\alpha-1}S'(l)S(t_1-s)\|\,\mathrm{d}l \\
&\le \int_0^{t_2-t_1} \|S(l)A^\alpha S(t_1-s)\|\,\mathrm{d}l \\
&\le MC_\alpha(t_2-t_1)(t_1-s)^{-\alpha}. \qquad (3.12)
\end{aligned}$$

Now we use the inequality (3.12) to get the bound for fifth part as given below,

$$\begin{aligned}
&\int_0^{t_1} \|(S(t_2-s)-S(t_1-s))A^{\alpha-1}\|\;\|f_n(s,u(s),u[h(u(s),s)])\|\,\mathrm{d}s \\
&\quad \le C_3(t_2-t_1), \qquad (3.13)
\end{aligned}$$

where $C_3 = NMC_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}$. Similarly for third part we have,

$$\begin{aligned}
&\int_0^{t_1} \|(S(t_2-s)-S(t_1-s))A^{\alpha-\beta}\|\;\|A^\beta g_n(s,u(s))])\|\,\mathrm{d}s \\
&\le C_4(t_2-t_1) \qquad (3.14)
\end{aligned}$$

where $C_4 = N_1MC_\alpha T_0^{1+\alpha-\beta}$. For the bound of the sixth part, we have,

$$\begin{aligned}
&\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-1}\|\|f_n(s,u(s),u[h(u(s),s)])\|\,\mathrm{d}s \\
&\quad \le C_5(t_2-t_1), \qquad (3.15)
\end{aligned}$$

where $C_5 = \|A^{\alpha-1}\|MN$. Finally for the fourth part we have the following

$$\begin{aligned}
&\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-\beta}\|\|A^\beta g_n(s,u(s))\|\,\mathrm{d}s \\
&\quad \le C_6(t_2-t_1), \qquad (3.16)
\end{aligned}$$

where $C_6 = \|A^{\alpha-\beta}\|MN_1$.

We use the inequalities (3.10), (3.11), (3.14)–(3.16) in inequality (3.9) to get the following inequality,

$$\|(\mathcal{F}_n u)(t_2) - (\mathcal{F}_n u)(t_1)\|_{\alpha-1} \le L|t_2-t_1|, \qquad (3.17)$$

where, $L = \max\{C_i,\ i = 1, 2, \cdots 6\}$. Hence, $\mathcal{F}_n : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ follows.

Our next task is to show that $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$. Now, for $t \in (0, T_0]$ and $u \in \mathcal{W}$, we have

$$\begin{aligned}
\|(\mathcal{F}_n u)(t) - u_0\|_\alpha \leq\; & \|(S(t) - I)A^\alpha[u_0 + g_n(0, u_0)]\| \\
& + \|A^{\alpha-\beta}\| \|A^\beta g_n(t, u(t))) - A^\beta g_n(0, u(0))\| \\
& + \int_0^t \|S(t-s)A^{1+\alpha-\beta}\| \|A^\beta g_n(s, u(s)))])\| \,\mathrm{d}s \\
& + \int_0^t \|S(t-s)A^\alpha\| \|f_n(s, u(s), u[h(u(s), s)])\| \,\mathrm{d}s \\
& \leq \|(S(t) - I)A^\alpha[u_0 + g_n(0, u_0)]\| + \|A^{\alpha-\beta}\| L_g[T_0 + R] \\
& + C_{1+\alpha-\beta} N_1 \frac{T_0^{\beta-\alpha}}{\beta - \alpha} + C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

Hence, from inequalities (3.6) and (3.7), we get

$$\|\mathcal{F}_n u - u_0\|_{T_0,\alpha} \leq R.$$

Therefore, $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$.

Now, if $t \in (0, T_0]$ and $u, v \in \mathcal{W}$, then

$$\begin{aligned}
& \|(\mathcal{F}_n u)(t) - (\mathcal{F}_n v)(t)\|_\alpha \\
& \leq \|A^{\alpha-\beta}\| \|A^\beta g_n(t, u(t)) - A^\beta g_n(t, v(t))\| \\
& + \int_0^t \|S(t-s)A^{1+\alpha-\beta}\| \|A^\beta g_n(s, u(s))) - A^\beta g_n(s, v(s))\| \,\mathrm{d}s. \\
& + \int_0^t \|S(t-s)A^\alpha\| \|f_n(s, u(s), u[h(u(s), s)]) \\
& \qquad - f_n(s, v(s), v[h(v(s), s)])\| \,\mathrm{d}s.
\end{aligned} \tag{3.18}$$

We have the following inequalities,

$$\|A^\beta g_n(t, u(t))) - A^\beta g_n(t, v(t))\| \leq L_g \|A^{-1}\| \|u - v\|_{T_0,\alpha}, \tag{3.19}$$

$$\begin{aligned}
& \|f_n(s, u(s), u[h(u(s), s)]) - f_n(s, v(s), v[h(v(s), s)])\| \\
& \quad \leq L_f[2 + LL_h] \|u - v\|_{T_0,\alpha}.
\end{aligned} \tag{3.20}$$

We use the inequalities (3.19) and (3.20) in the inequality (3.18), we get

$$\begin{aligned}
\|(\mathcal{F}_n u)(t) - (\mathcal{F}_n v)(t)\|_\alpha \;\; \leq \;\; & [(L_g \|A^{\alpha-\beta-1}\| + C_{1+\alpha-\beta} L_g \frac{T_0^{(\beta-\alpha)}}{\beta-\alpha}) \\
& + C_\alpha L_f [2 + LL_h] \frac{T_0^{(1-\alpha)}}{1-\alpha}]] \|u - v\|_{T_0,\alpha}.
\end{aligned} \tag{3.21}$$

Hence from inequality (3.3), we get the following inequality given below

$$\|\mathcal{F}_n u - \mathcal{F}_n v\|_{T_0,\alpha} < \|u - v\|_{T_0,\alpha}.$$

Therefore, the map $\mathcal{F}_n$ has a unique fixed point $u_n \in \mathcal{W}$ which is given by,

$$u_n(t) = S(t)[u_0 + g_n(0,u_0)] - g_n(t,u(t)) + \int_0^t AS(t-s)g_n(s,u(s)))\,\mathrm{d}s, \\ + \int_0^t S(t-s)f_n(s,u(s),u[h(u(s),s)])\,\mathrm{d}s, \quad t \in [0,T_0]. \tag{3.22}$$

Hence, the mild solution u_n of equation (1.1) is given by the equation (3.22) and belong to $\mathcal{W}$, hence, the theorem is proved. □

Lemma 3.4 *Let (H1)–(H3) hold. If $u_0 \in D(A^\alpha)$, then $u_n(t) \in D(A^\vartheta)$ for all $t \in (0,T_0]$ where $0 \le \vartheta \le \beta < 1$. Furthermore, if $u_0 \in D(A)$, then $u_n(t) \in D(A^\vartheta)$ for all $t \in [0,T_0]$, where $0 \le \vartheta \le \beta < 1$.*

Proof. From Theorem 3.3, we have the existence of a unique $u_n \in C^\alpha_{T_0} \cap C^{\alpha-1}_{T_0}$ satisfying (3.22). Part (a) of Theorem 2.6.13 in Pazy [17] implies that for $t > 0$ and $0 \le \vartheta < 1$, $S(t) : H \to D(A^\vartheta)$ and for $0 \le \vartheta \le \beta < 1$, $D(A^\beta) \subseteq D(A^\vartheta)$. (H2)–(H4) implies that the map $t \mapsto A^\beta g(t,u_n(t)$ is Hölder continuous on $[0,T_0]$ with the exponent $\rho = \min\{\gamma,\vartheta\}$ since the Hölder continuity of u_n can be easily established using the similar arguments from (3.9) to (3.16). It follows that (cf. Theorem 4.3.2 in [17])

$$\int_0^t S(t-s)A^\beta g_n(s,u_n(s))\,\mathrm{d}s \in D(A).$$

Also from Theorem 1.2.4 in Pazy [17], we have $S(t)x \in D(A)$ if $x \in D(A)$. The required result follows from these facts and the fact that $D(A) \subseteq D(A^\vartheta)$ for $0 \le \vartheta \le 1$. □

Lemma 3.5 *Let $(H1)$ and $(H2)$ hold. If $u_0 \in D(A^\alpha)$ and $t_0 \in (0,T_0]$ then*

$$\|u_n(t)\|_\vartheta \le U_{t_0}, \quad \alpha < \vartheta < \beta, \quad t \in [t_0,T_0], \quad n = 1,2,\cdots,$$

for some constant U_{t_0}, dependent of t_0 and

$$\|u_n(t)\|_\vartheta \le U_0, \quad 0 < \vartheta \le \alpha, \quad t \in [0,T_0], \quad n = 1,2,\cdots,$$

for some constant U_0. Moreover, if $u_0 \in D(A)$, then there exists a constant U_0, such that

$$\|u_n(t)\|_\vartheta \le U_0, \quad 0 < \vartheta < \beta, \quad t \in [0,T_0], \quad n = 1,2,\cdots.$$

Proof. First, we assume that $u_0 \in D(A^\alpha)$. Applying A^ϑ on both the sides of (3.22) and using (iii) of Lemma (2.1), for $t \in [t_0,T_0]$ and $\alpha < \vartheta < \beta$, we have

$$\begin{aligned}\|u_n(t)\|_\vartheta \le & \|A^\vartheta S(t)(u_0 + g_n(0,u_0)\| + \|A^{\vartheta-\beta}\|\,\|A^\beta g_n(t,u_n(t))\| \\ & + \int_0^t \|A^{1+\vartheta-\beta}S(t-s)\|\,\|A^\beta g_n(s,u_n(s))\|\,\mathrm{d}s \\ & + \int_0^t \|S(t-s)A^\vartheta\|\,\|f_n(s,u_n(s),u_n[h(u_n(s),s)])\|\,\mathrm{d}s \\ \le & C_\vartheta t_0^{-\vartheta}(\|u_0\| + \|g_n(0,u_0\|) + \|A^{\vartheta-\beta}\|N_1 \\ & + \ C_{1+\vartheta-\beta}N_1\frac{T_0{}^{(\beta-\vartheta)}}{\beta-\vartheta} + C_\vartheta N\frac{T_0{}^{(1-\vartheta)}}{1-\vartheta} \le U_{t_0}.\end{aligned}$$

Again, for $t \in [0, T_0]$ and $0 < \vartheta \le \alpha$, $u_0 \in D(A^\vartheta)$ and

$$\begin{aligned}\|u_n(t)\|_\vartheta \le & M(\|A^\vartheta u_0\| + \|g_n(0, \tilde{u_0}\|_\vartheta) + \|A^{\vartheta-\beta}\|N_1 \\ & + C_{1+\vartheta-\beta}N_1 \frac{T_0{}^{(\beta-\vartheta)}}{\beta-\vartheta} + C_\vartheta N \frac{T_0{}^{(1-\vartheta)}}{1-\vartheta} \le U_0.\end{aligned}$$

Furthermore, if $u_0 \in D(A)$ then $u_0 \in D(A^\vartheta)$ for $0 < \vartheta \le \beta$ and we can easily get the required estimate. This completes the proof of the proposition. □

4 Convergence of Solutions

In this section we establish the convergence of the solution $u_n \in X_\alpha(T_0)$ of the approximate integral equation (3.22) to a unique solution u of (3.1).

Theorem 4.1 *Let (H1)–(H4) hold. If $u_0 \in D(A^\alpha)$, then for any $t_0 \in (0, T_0]$,*

$$\lim_{m\to\infty} \sup_{\{n\ge m,\ t_0\le t\le T_0\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof. Let $0 < \alpha < \vartheta < \beta$. For $n \ge m$, we have

$$\begin{aligned}&\|f_n(t, u_n(t), u_n[h(u_n(t), t)]) - f_m(t, u_m(t), u_m[h(u_m(t), t)])\| \\ &\quad \le \|f_n(t, u_n(t), u_n[h(u_n(t), t)]) - f_n(t, u_m(t), u_m[h(u_m(t), t)])\| \\ &\qquad + \|f_n(t, u_m(t), u_m[h(u_m(t), t)]) - f_m(t, u_m(t), u_m[h(u_m(t), t)])\| \\ &\quad \le L_f(1 + LL_h)[\|u_n(t) - u_m(t)\|_\alpha + \|(P^n - P^m)u_m(t)\|_\alpha].\end{aligned}$$

Also,

$$\|(P^n - P^m)u_m(t)\|_\alpha \le \|A^{\alpha-\vartheta}(P^n - P^m)A^\vartheta u_m(t)\| \le \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|.$$

Thus, we have

$$\begin{aligned}&\|f_n(t, u_n(t), u_n[h(u_n(t), t)]) - f_m(t, u_m(t), u_m[h(u_m(t), t)])\| \\ &\quad \le L_f(1 + LL_h)[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|].\end{aligned}$$

Similarly

$$\begin{aligned}&\|A^\beta g_n(t, u_n(t)) - A^\beta g_m(t, u_m(t))\| \\ &\le \|A^\beta g_n(t, u_n(t)) - A^\beta g_n(t, u_m(t))\| + \|A^\beta g_n(t, u_m(t)) - A^\beta g_m(t, u_m(t))\| \\ &\le L_g\|A^{-1}\|[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|].\end{aligned}$$

Now, for $0 < t_0' < t_0$, we may write

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_\alpha \\
&\le \|S(t)A^\alpha(g_n(0,u_0) - g_m(0,u_0))\| \\
&+ \|A^{\alpha-\beta}\| \, \|A^\beta g_n(t,u_n(t)) - A^\beta g_m(t,u_m(t))\| \\
&+ \Big(\int_0^{t_0'} + \int_{t_0'}^t\Big)\|A^{1+\alpha-\beta}S(t-s)\|\|A^\beta g_n(s,u_n(s)) - A^\beta g_m(s,u_m(s))\| \,\mathrm{d}s \\
&+ \Big(\int_0^{t_0'} + \in_{t_0'}^t\Big)\|A^\alpha S(t-s)\|\|f_n(s,u_n(s),u_n[h(u_n(s),s)]) \\
&- f_m(s,u_m(s),u_m(h(u_m(s),s)))\| \,\mathrm{d}s.
\end{aligned}$$

We estimate the first term as

$$\begin{aligned}
&\|S(t)A^\alpha(g_n(0,u_0) - g_m(0,u_0))\| \\
&\qquad \le M\|A^{\alpha-\beta}\| \, \|A^\beta g(0,P^n u_0) - A^\beta g(0,P^m u_0)\| \\
&\qquad \le M\|A^{\alpha-\beta-1}\|L_g\|(P^n - P^m)A^\alpha u_0\|.
\end{aligned}$$

The first and the third integrals are estimated as

$$\begin{aligned}
\int_0^{t_0'} &\|A^{1+\alpha-\beta}S(t-s)\| \, \|A^\beta g_n(s,u_n(s)) - A^\beta g_m(s,u_m(s))\| \,\mathrm{d}s \\
&\le 2C_{1+\alpha-\beta}N_1(t_0 - t_0')^{-(1+\alpha-\beta)}t_0', \\
\int_0^{t_0'} &\|A^\alpha S(t-s)\| \, \|f_n(s,u_n(s),u_n[h(u_n(s),s)]) - f_m(s,u_m(s),u_m[h(u_m(s),s)])\| \,\mathrm{d}s \\
&\le 2C_\alpha N(t_0 - t_0')^{-\alpha}t_0'.
\end{aligned}$$

For the second and the fourth integrals, we have

$$\begin{aligned}
&\int_{t_0'}^t \|A^{1+\alpha-\beta}S(t-s)\| \, \|A^\beta g_n(s,u_n(s)) - A^\beta g_m(s,u_m(s))\| \,\mathrm{d}s \\
&\le C_{1+\alpha-\beta}L_g\|A^{-1}\| \int_{t_0'}^t [\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(s)\|] \,\mathrm{d}s \\
&\le C_{1+\alpha-\beta}L_g\|A^{-1}\|\Big(\frac{U_{t_0'}T_0{}^{(\beta-\alpha)}}{\lambda_m^{\vartheta-\alpha}(\beta-\alpha)} + \int_{t_0'}^t (t-s)^{(\beta-\alpha)-1}\|u_n(s) - u_m(s)\|_\alpha \,\mathrm{d}s\Big),
\end{aligned}$$

$$\begin{aligned}
&\int_{t_0'}^t \|A^\alpha S(t-s)\|\|f_n(s,u_n(s),u_n[h(u_n(s),s)]) - f_m(s,u_m(s),u_m[h(u_m(s),s)])\| \,\mathrm{d}s \\
&\le C_\alpha L_f(1 + LL_h) \int_{t_0'}^t [\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(s)\|] \,\mathrm{d}s \\
&\le C_\alpha L_f(1 + LL_h)\Big(\frac{U_{t_0'}T_0{}^{(1-\alpha)}}{\lambda_m^{\vartheta-\alpha}(1-\alpha)} + \int_{t_0'}^t (t-s)^{(1-\alpha)-1}\|u_n(s) - u_m(s)\|_\alpha \,\mathrm{d}s\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}\|u_n(t)-u_m(t)\|_\alpha \leq & M\|A^{\alpha-\beta-1}\|L_g\|(P^n-P^m)A^\alpha u_0\| \\ & + \|A^{\alpha-\beta-1}\|L_g\Big(\|u_n(t)-u_m(t)\|_\alpha + \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\Big) \\ & + 2\Big(\frac{C_{1+\alpha-\beta}N_1}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha N}{(t_0-t_0')^\alpha}\Big)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\ & + \int_{t_0'}^t \Big(\frac{C_\alpha L_f(1+LL_h)}{(t-s)^{\eta(\alpha-1)+1}} + \frac{C_{1+\alpha-\beta}L_g\|A^{-1}\|}{(t-s)^{\eta(\alpha-\beta)+1}}\Big)\|u_n(s)-u_m(s)\|_\alpha\,\mathrm{d}s,\end{aligned}$$

where

$$C_{\alpha,\beta} = C_\alpha L_f(1+LL_h)\frac{T_0{}^{(1-\alpha)}}{1-\alpha} + C_{1+\alpha-\beta}L_g\|A^{-1}\|\frac{T_0{}^{(\beta-\alpha)}}{\beta-\alpha}.$$

Since $\|A^{\alpha-\beta-1}\|L_g < 1$, we have

$$\begin{aligned}\|u_n(t)-u_m(t)\|_\alpha \leq & \frac{1}{(1-\|A^{\alpha-\beta-1}\|L_g)}\Big\{M\|(P^n-P^m)A^\alpha u_0\| + \|A^{\alpha-\beta-1}\|L_g\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\ & + 2\Big(\frac{C_{1+\alpha-\beta}N_1}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha N}{(t_0-t_0')^\alpha}\Big)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\Big\} \\ & + \int_{t_0'}^t \Big(\frac{C_\alpha L_f(1+LL_h)}{(t-s)^{\eta(\alpha-1)+1}} + \frac{C_{1+\alpha-\beta}L_g\|A^{-1}\|}{(t-s)^{\eta(\alpha-\beta)+1}}\Big)\|u_n(s)-u_m(s)\|_\alpha\,\mathrm{d}s.\end{aligned}$$

Lemma 5.6.7 in [17] implies that there exists a constant C such that

$$\begin{aligned}&\|u_n(t)-u_m(t)\|_\alpha \\ &\leq \frac{1}{(1-\|A^{\alpha-\beta-1}\|L_g)}\Big\{M\|(P^n-P^m)A^\alpha u_0\| + (\|A^{\alpha-\beta-1}\|L_g + C_{\alpha,\beta})\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\ &+ 2\Big(\frac{C_{1+\alpha-\beta}N_1}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha N}{(t_0-t_0')^\alpha}\Big)t_0'\Big\}C.\end{aligned}$$

Taking supremum over $[t_0, T_0]$ and letting $m\to\infty$, we obtain

$$\begin{aligned}&\lim_{m\to\infty}\sup_{\{n\geq m, t\in[t_0,T_0]\}}\|u_n(t)-u_m(t)\|_\alpha \\ &\leq \frac{2}{(1-\|A^{\alpha-\beta-1}\|L_g)}\Big(\frac{C_{1+\alpha-\beta}N_1}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha N}{(t_0-t_0')^\alpha}\Big)t_0' C.\end{aligned}$$

As t_0' is arbitrary, the right hand side may be made as small as desired by taking t_0' sufficiently small. This completes the proof of the proposition. □

Corollary 4.1 *If* $u_0 \in D(A)$ *then*

$$\lim_{m\to\infty}\sup_{\{n\geq m,\ 0\leq t\leq T_0\}}\|u_n(t)-u_m(t)\|_\alpha = 0.$$

Proof. Propositions 3.4 and 3.5 imply that in the proof of Proposition 4.1 we may take $t_0 = 0$.

With the help of theorems 3.3 and 4.1 For the convergence of the solution $u_n(t)$ of the approximate integral equation (3.22) we have the following result. □

Theorem 4.2 *Let (H1)–(H4) hold and let* $u_0 \in D(A^\alpha)$*. Then there exists a unique function* $u_n \in \mathcal{W}$

$$\begin{aligned} u_n(t) &= S(t)[u(0) + g_n(0, u_0)] - g_n(t, u_n(t)) \\ &+ \int_0^t AS(t-s)g_n(s, u_n(s))\,\mathrm{d}s \\ &+ \int_0^t S(t-s)f_n(s, u_n(s), u_n[h_n(u_n(s), s)])\,\mathrm{d}s,\ t \in [0, T_0] \end{aligned}$$

and $u \in \mathcal{W}$

$$\begin{aligned} u(t) &= S(t)[u(0) + g(0, u_0)] - g(t, u(t)) \\ &+ \int_0^t AS(t - sg(s, u(s))\,\mathrm{d}s + \int_0^t S(t-s)f(s, u(s), u[h(u(s), s)])\,\mathrm{d}s,\ t \in [0, T_0] \end{aligned}$$

such that $u_n \to u$ *as* $n \to \infty$ *in* $\mathcal{W}$ *and* u *satisfies (3.1) on* $[0, T_0]$.

5 Faedo-Galerkin Approximations

For any $0 < t < T_0$, we have a unique $u \in \mathcal{W}$ satisfying the integral equation

$$\begin{aligned} u(t) = S(t)[u(0) + g(0, u_0)] - g(t, u(t)) + \int_0^t AS(t-s)g(s, u(s))\,\mathrm{d}s \\ + \int_0^t S(t-s)f(s, u(s), u[h(u(s), s)])\,\mathrm{d}s,\ t \in [0, T_0]. \end{aligned} \tag{5.1}$$

Also, we have a unique solution $u_n \in X_\alpha(T_0)$ of the approximate integral equation

$$\begin{aligned} u_n(t) &= S(t)[u(0) + g_n(0, u_0)] - g_n(t, u_n(t)) + \int_0^t AS(t-s)g_n(s, u_n(s))\,\mathrm{d}s \\ &+ \int_0^t S(t-s)f_n(s, u_n(s), u_n[h_n(u_n(s), s)])\,\mathrm{d}s,\ t \in [0, T_0]. \end{aligned} \tag{5.2}$$

If we project (5.2) onto H_n, we get the Faedo-Galerkin approximation $\hat{u}_n(t) = P^n u_n(t)$ satisfying

$$\begin{aligned} \hat{u}_n(t) &= S(t)[u(0) + g_n(0, u_0)] - g_n(t, \hat{u}_n(t)) + \int_0^t AS(t-s)g_n(s, \hat{u}_n(s))\,\mathrm{d}s \\ &+ \int_0^t S(t-s)f_n(s, \hat{u}_n(s), \hat{u}_n(h_n(\hat{u}_n(s), s)))\,\mathrm{d}s,\ t \in [0, T_0]. \end{aligned} \tag{5.3}$$

The solution u of (5.1) and $\hat{u}_n$ of (5.3), have the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)u_i, \quad \alpha_i(t) = (u(t), u_i), \quad i = 0, 1, \ldots; \tag{5.4}$$

$$\hat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)u_i, \quad \alpha_i^n(t) = (\hat{u}_n(t), u_i), \quad i = 0, 1, \ldots; \tag{5.5}$$

As a consequence of Theorems 3.3 and 4.1, we have the following result.

Theorem 5.1 *Let (H1)–(H4) hold and let $u_0 \in D(A^\alpha)$. Then there exists a unique function $\hat{u}_n \in \mathcal{W}$*

$$\begin{aligned}\hat{u}_n(t) &= S(t)[u(0) + g_n(0, u_0)] - g_n(t, \hat{u}_n(t)) + \int_0^t AS(t-s)g_n(s, \hat{u}_n(s))\,\mathrm{d}s \\ &\quad + \int_0^t S(t-s)f_n(s, \hat{u}_n(s), \hat{u}_n[h_n(\hat{u}_n(s), s)])\,\mathrm{d}s,\ t \in [0, T_0]\end{aligned}$$

and $u \in \mathcal{W}$

$$\begin{aligned}u(t) &= S(t)[u(0) + g(0, u_0)] - g_n(t, \hat{u}(t)) \\ &\quad + \int_0^t AS(t-s)g(s, u(s))\,\mathrm{d}s \\ &\quad + \int_0^t S(t-s)f(s, u(s), u[h(u(s), s)])\,\mathrm{d}s,\ t \in [0, T_0]\end{aligned}$$

such that $\hat{u}_n \to u$ as $n \to \infty$ in $\mathcal{W}$ and u satisfies (3.1) on $[0, T_0]$.

Now, we shall show the convergence of $\alpha_i^n(t) \to \alpha_i(t)$. It can easily be checked that

$$A^\alpha[u(t) - \hat{u}_n(t)] = A^\alpha\Big[\sum_{i=0}^{\infty}(\alpha_i(t) - \alpha_i^n(t))u_i\Big] = \sum_{i=0}^{\infty}\lambda_i^\alpha(\alpha_i(t) - \alpha_i^n(t))u_i.$$

Thus, we have

$$\|A^\alpha[u(t) - \hat{u}_n(t)]\|^2 \geq \sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2.$$

We have the following convergence theorem.

Theorem 5.1 *Let $(H1)$ and $(H2)$ hold. Then we have the following.*

(a) If $u_0 \in D(A^\alpha)$, then for any $0 < t_0 \leq T_0$,

$$\lim_{n\to\infty}\sup_{t_0\leq t\leq T_0}\Big[\sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2\Big] = 0.$$

(b) If $u_0 \in D(A)$, then

$$\lim_{n\to\infty}\sup_{0\leq t\leq T_0}\Big[\sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2\Big] = 0.$$

The assertion of this theorem follows from the facts mentioned above and the following result.

Proposition 5.2 *Let $(H1)$ and $(H2)$ hold and let T_0 be any number such that $0 < T_0 < t_{\max}$, then we have the following.*

(a) If $u_0 \in D(A^\alpha)$, then for any $0 < t_0 \leq T_0$,

$$\lim_{n\to\infty}\sup_{\{n\geq m, t_0\leq t\leq T_0\}}\|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

(b) If $u_0 \in D(A)$, then

$$\lim_{n\to\infty} \sup_{\{n\geq m, 0\leq t\leq T_0\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

Proof. For $n \geq m$, we have

$$\begin{aligned}\|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| &= \|A^\alpha[P^n u_n(t) - P^m u_m(t)]\| \\ &\leq \|P^n[u_n(t) - u_m(t)]\|_\alpha + \|(P^n - P^m)u_m\|_\alpha \\ &\leq \|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m\|.\end{aligned}$$

If $u_0 \in D(A^\alpha)$ then the result in (a) follows from Proposition 4.1. If $u_0 \in D(A)$, (b) follows from Corollary 4.1. □

6 Examples

Let $X = L^2(0,1)$. We consider the following partial differential equations with a deviated argument,

$$\begin{cases} \partial_t[w(t,x) + \partial_x f_1(t, w(t,x))] - \partial_x^2[w(t,x)] \\ \quad = f_2(x, w(t,x)) + f_3(t, x, w(t,x)), \quad x \in (0,1),\ t > 0, \\ \quad w(t,0) = w(t,1) = 0,\ t \in [0,T],\ 0 < T < \infty, \\ \quad w(0,x) = u_0,\ x \in (0,1), \end{cases} \tag{6.1}$$

where

$$f_2(x, w(t,x)) = \int_0^x K(x,s) w(s, h(t)(a_1|w(s,t)| + b_1|w_s(s,t)|))\, \mathrm{d}s.$$

The function $f_3 : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u and uniformly in x. Further we assume that $a_1, b_1 \geq 0, (a_1, b_1) \neq (0,0)$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $h(0) = 0$ and $K : [0,1] \times [0,1] \to \mathbb{R}$.

We define an operator A as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = \{u \in H_0^1(0,1) \cap H^2(0,1)\ \ u'' \in X\}. \tag{6.2}$$

Here clearly the operator A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2}) = H_0^1(0,1)$ is the Banach space endowed with the norm,

$$\|x\|_{1/2} := \|A^{1/2}x\|,\ \ x \in D(A^{1/2})$$

and we denote this space by $X_{1/2}$. Also, for $t \in [0,T]$, we denote

$$C_t^{1/2} = C([0,t]; D(A^{1/2})),$$

endowed with the sup norm

$$\|\psi\|_{t,1/2} := \sup_{0\leq r\leq t} \|\psi(r)\|_\alpha, \quad \psi \in \mathcal{C}_t^{1/2}.$$

We observe some properties of the operators A and $A^{1/2}$ defined by (6.2). For $u \in D(A)$ and $\lambda \in \mathbb{R}$, with $Au = -u'' = \lambda u$, we have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\left\langle -u'', u \right\rangle = |u'|^2_{L^2} = \lambda |u|^2_{L^2}$$

so $\lambda > 0$. A solution u of $Au = \lambda u$ is of the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the corresponding solution is given by

$$u_n(x) = D\sin(\sqrt{\lambda_n}x).$$

We have $\langle u_n, u_m\rangle = 0$ for $n \neq m$ and $\langle u_n, u_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n u_n(x), \quad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\,\alpha_n\, u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$. $X_{-\frac{1}{2}} = H^1(0,1)$ is a Sobolev space of negative index with the equivalent norm $\|.\|_{-\frac{1}{2}} = \sum_{n=1}^{\infty} |\langle ., u_n\rangle|^2$. For more details on the Sobolev space of negative index, we refer to Gal [10].

The equation (6.1) can be reformulated as the following abstract equation in $X = L^2(0,1)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[u(t) + g(t, u(t))] + Au(t) &= f(t, u(t), u[h(u(t), t)]) \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{6.3}$$

where $u(t) = w(t,.)$ that is $u(t)(x) = w(t,x)$, $x \in (0,1)$. The function $g : \mathbb{R}_+ \times X_{1/2} \to X$, such that $g(t,u(t))(x) = \partial_x f_1(t, w(t,x))$ and the operator A is same as in equation (6.2).

The function $f : \mathbb{R}_+ \times X_{1/2} \times X_{-1/2} \to X$, is given by

$$f(t, \psi, \xi)(x) = f_2(x, \xi) + f_3(t, x, \psi), \tag{6.4}$$

where $f_2 : [0,1] \times X \to H^1_0(0,1)$ is given by

$$f_2(t, \xi) = \int_0^x K(x,y)\xi(y)\,\mathrm{d}y, \tag{6.5}$$

and $f_3 : \mathbb{R} \times [0,1] \times H^2(0,1) \to H^1_0(0,1)$ satisfies the following

$$\|f_3(t, x, \psi)\| \leq Q(x,t)(1 + \|\psi\|_{H^2(0,1)}) \tag{6.6}$$

with $Q(.,t) \in X$ and Q is continuous in its second argument. We can easily verified that the function f is satisfied the assumptions (H1)–(H4). For more details see [10].

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