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# STABILITY IN NONLINEAR NEUTRAL VOLTERRA DIFFERENCE EQUATIONS WITH VARIABLE DELAYS

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**Abstract.** In this paper we use the contraction mapping theorem to obtain asymptotic stability results of the zero solution of a nonlinear neutral Volterra difference equation with variable delays. Some conditions which allow the coefficient sequences to change sign and do not ask the boundedness of delays are given. An asymptotic stability theorem with a sufficient condition is proved. The obtained results improve and generalize those due to Raffoul (2006) [11], Yankson (2009) [15] and Yankson (2006) [16].

**Keywords:** Fixed point, Stability, Neutral Volterra difference equations, Variable delays.

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## 1 Introduction

Certainly, the Lyapunov direct method has been, for more than 100 years, the main tool for the study of stability properties of ordinary, functional, partial differential and difference equations. Nevertheless, the application of this method to problems of stability in differential and difference equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms ( [2, 3, 5–8, 13]). Recently, Burton, Furumochi, Zhang, Raffoul, Islam, Yankson and others have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1–3, 9, 11, 12, 15–17]). The fixed point theory does not only solve the

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problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [2]).

In this paper we consider the nonlinear neutral Volterra difference equation with variable delays

$$\triangle x(n) = -a(n)\,x(n-\tau_1(n)) + c(n)\,\triangle x(n-\tau_2(n)) + \sum_{s=n-\tau_2(n)}^{n-1} k(n,s)\,q(x(s)), \qquad (1.1)$$

with the initial condition

$$x(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z},$$

where $\psi$ is bounded sequence and for each $n_0 \in \mathbb{Z}^+$,

$$m_j(n_0) = \inf\{n - \tau_j(n),\ n \geq n_0\},\ m(n_0) = \min\{m_j(n_0),\ j = 1,2\}.$$

Here $\triangle$ denotes the forward difference operator $\triangle x(t) = x(n+1) - x(n)$ for any sequence $\{x(n),\ n \in \mathbb{Z}^+\}$. Throughout this paper we assume that $a, c : \mathbb{Z}^+ \to \mathbb{R}$, $k : \mathbb{Z}^+ \times ([m_2(n_0), \infty) \cap \mathbb{Z}) \to \mathbb{R}$, $q : \mathbb{R} \to \mathbb{R}$ and $\tau_1, \tau_2 : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $n - \tau_1(n) \to \infty$ and $n - \tau_2(n) \to \infty$ as $n \to \infty$. The function $q(x)$ is locally Lipschitz in $x$. That is, there is positive constant $L$ so that if $|x|, |y| \leq L_1$ for some positive constant $L_1$ then

$$|q(x) - q(y)| \leq L\,\|x - y\| \text{ and } q(0) = 0. \qquad (1.2)$$

Equation (1.1) and its special cases have been investigated by many authors. For example, Raffoul in [11] and Yankson in [15] have studied the equation

$$\triangle x(n) = -a(n)\,x(n - \tau_1(n)), \qquad (1.3)$$

and proved the following.

**Theorem A** (Raffoul [11]). *Suppose that $\tau_1(n) = r$ and $a(n+r) \neq 1$ and there exists a constant $\alpha < 1$ such that*

$$\sum_{s=n-r}^{n-1} |a(s+r)| + \sum_{s=0}^{n-1} \left( |a(s+r)| \left| \prod_{k=s+1}^{n-1} [1 - a(k+r)] \right| \sum_{u=s-r}^{s-1} |a(u+r)| \right) \leq \alpha, \qquad (1.4)$$

*for all $n \in \mathbb{Z}^+$ and $\prod_{s=0}^{n-1} [1 - a(s+r)] \to 0$ as $n \to \infty$. Then, for every small initial sequence $\psi : [-r, 0] \cap \mathbb{Z} \to \mathbb{R}$,* the *solution $x(n) = x(n, 0, \psi)$ of (1.3) is bounded and tends to zero as $n \to \infty$.*

**Theorem B** (Yankson [15]). *Suppose that the inverse sequence $g$ of $n - \tau_1(n)$ exists, $1 - a(g(n)) \neq 0$ and there exists a constant $\alpha \in (0,1)$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ such that*

$$\sum_{s=n-\tau_1(n)}^{n-1} |a(g(s))| + \sum_{s=n_0}^{n-1} \left( |a(g(s))| \left| \prod_{k=s+1}^{n-1} [1 - a(g(s))] \right| \sum_{u=s-\tau_1(s)}^{s-1} |a(g(s))| \right) \leq \alpha. \qquad (1.5)$$

*Then the zero solution of (1.3) is asymptotically stable if $\prod_{s=n_0}^{n-1} [1 - a(g(s))] \to 0$ as $n \to \infty$.*

Obviously, Theorem B improves and generalizes Theorem A. On other hand, Yankson in [16] considered the following nonlinear neutral Volterra difference equation

$$x(n+1) = a_1(n)\,x(n) + c(n)\,\triangle x(n-\tau_2(n)) + \sum_{s=n-\tau_2(n)}^{n-1} k(n,s)\,q(x(s)), \tag{1.6}$$

where $a_1 : \mathbb{Z}^+ \to \mathbb{R}$ and $0 \leq \tau_2(n) \leq \tau_0$ for some constant $\tau_0$, and obtained the following theorem.

**Theorem C** (Yankson [16]). *Let $a_1(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$. Suppose that (1.2) holds and there exists a constant $\alpha \in (0,1)$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ such that*

$$|c(n-1)| + \sum_{s=n_0}^{n-1} \left\{ |\phi_2(s)| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \right\} \left| \prod_{u=s+1}^{n-1} a_1(u) \right| \leq \alpha, \tag{1.7}$$

*where $\phi_2(s) = c(s) - c(s-1)\,a_1(s)$. Then the zero solution of (1.6) is asymptotically stable if $\prod_{s=n_0}^{n-1} a_1(s) \to 0$ as $n \to \infty$.*

**Remark 1.1** *The Theorem C is still true if the delay $\tau_2$ is unbounded.*

Our purpose here is to give, by using the contraction mapping principle, asymptotic stability results of a nonlinear neutral Volterra difference equation with variable delays (1.1). For details on contraction mapping principle we refer the reader to [14] and for more on the calculus of difference equations, we refer the reader to [4] and [10]. The results presented in this paper improve and generalize the main results in [11, 15, 16].

## 2 Main results

Let $D(n_0)$ denote the set of bounded sequences $\psi : [m(n_0), n_0] \cap \mathbb{Z} \to \mathbb{R}$ with the maximum norm $\|.\|$. Also, let $(\mathbb{B}, \|.\|)$ be the Banach space of bounded sequences $x : [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ with the maximum norm. For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, a solution of (1.1) through $(n_0, \psi)$ is a sequence $[m(n_0), n_0 + \alpha] \cap \mathbb{Z} \to \mathbb{R}$ for some positive constant $\alpha > 0$ such that $x$ satisfies (1.1) on $[n_0, n_0 + \alpha) \cap \mathbb{Z}$ and $x = \psi$ on $[m(n_0), n_0] \cap \mathbb{Z}$. We denote such a solution by $x(n) = x(n, n_0, \psi)$. For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, there exists a unique solution $x(n) = x(n, n_0, \psi)$ of (1.1) defined on $[m(n_0), \infty) \cap \mathbb{Z}$. For a fixed $n_0$, we define $\|\psi\| = \max\{|\psi(n)| : n \in [m(n_0), n_0] \cap \mathbb{Z}\}$.

Let $h_j : [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ be an arbitrary sequence. Rewrite (1.1) as

$$\begin{aligned} \triangle x(n) = {} & -\sum_{j=1}^{2} h_j(n)\,x(n) + \triangle_n \sum_{j=1}^{2} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) + \sum_{j=1}^{2} h_j(n-\tau_j(n))\,x(n-\tau_j(n)) \\ & - a(n)\,x(n-\tau_1(n)) + c(n)\,\triangle x(n-\tau_2(n)) + \sum_{s=n-\tau_2(n)}^{n-1} k(n,s)\,q(x(s)), \end{aligned} \tag{2.1}$$

where $\triangle_n$ represents that the difference is with respect to $n$. If we let $H(n) = 1 - \sum_{j=1}^{2} h_j(n)$ then (2.1) is equivalent to

$$\begin{aligned} x(n+1) = H(n)\, x(n) + \triangle_n \sum_{j=1}^{2} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\, x(s) + \sum_{j=1}^{2} h_j(n-\tau_j(n))\, x(n-\tau_j(n)) \\ - a(n)\, x(n-\tau_1(n)) + c(n) \triangle x(n-\tau_2(n)) + \sum_{s=n-\tau_2(n)}^{n-1} k(n,s)\, q(x(s)). \qquad (2.2) \end{aligned}$$

**Lemma 2.1** *Suppose that $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$. Then $x$ is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(n) = &\left\{ x(n_0) - c(n_0-1)\, x(n_0-\tau_2(n_0)) - \sum_{j=1}^{2} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s) x(s) \right\} \prod_{u=n_0}^{n-1} H(u) \\ &+ c(n-1)\, x(n-\tau_2(n)) + \sum_{j=1}^{2} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\, x(s) \\ &+ \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \left\{ [h_1(s-\tau_1(s)) - a(s)]\, x(s-\tau_1(s)) \right. \\ &\left. + [h_2(s-\tau_2(s)) - \phi(s)]\, x(s-\tau_2(s)) + \sum_{u=s-\tau_2(s)}^{s-1} k(s,u)\, q(x(u)) \right\} \\ &- \sum_{j=1}^{2} \sum_{s=n_0}^{n-1} \{1 - H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\, x(u), \qquad (2.3) \end{aligned}$$

*where*

$$\phi(n) = c(n) - c(n-1)\, H(n). \qquad (2.4)$$

*Proof.* Let $x$ be a solution of (1.1). By multiplying both sides of (2.2) by $\prod_{u=n_0}^{n} H^{-1}(u)$ and by summing from $n_0$ to $n-1$ we obtain

$$\begin{aligned} \sum_{s=n_0}^{n-1} \triangle \left[ \prod_{u=n_0}^{s-1} H^{-1}(u)\, x(s) \right] = &\sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u) \triangle_s \sum_{j=1}^{2} \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\, x(u) \\ &+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u) \sum_{j=1}^{2} \{h_j(s-\tau_j(s))\}\, x(s-\tau_j(s)) \\ &+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u) \left\{ -a(s) x(s-\tau_1(s)) + \sum_{u=s-\tau_2(s)}^{s-1} k(s,u)\, q(x(u)) \right\} \\ &+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u)\, c(s) \triangle x(s-\tau_2(s)). \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&\prod_{u=n_0}^{n-1} H^{-1}(u)\, x(n) - \prod_{u=n_0}^{n_0-1} H^{-1}(u)\, x(n_0) \\
&\quad = \sum_{j=1}^{2} \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u) \triangle_s \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\, x(u) \\
&\qquad + \sum_{j=1}^{2} \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u) \left\{ h_j(s-\tau_j(s)) \right\} x(s-\tau_j(s)) \\
&\qquad + \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u) \left\{ -a(s)x(s-\tau_1(s)) + \sum_{u=s-\tau_2(s)}^{s-1} k(s,u)\, q(x(u)) \right\} \\
&\qquad + \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} H^{-1}(u)\, c(s) \triangle x(s-\tau_2(s)).
\end{aligned}$$

By dividing both sides of the above expression by $\prod_{u=n_0}^{n-1} H^{-1}(u)$ we get

$$\begin{aligned}
x(n) &= x(n_0) \prod_{u=n_0}^{n-1} H(u) \\
&\quad + \sum_{j=1}^{2} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \triangle_s \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\, x(u) \\
&\quad + \sum_{j=1}^{2} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \left\{ h_j(s-\tau_j(s)) \right\} x(s-\tau_j(s)) \\
&\quad + \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \left\{ -a(s)x(s-\tau_1(s)) + \sum_{u=s-\tau_2(s)}^{s-1} k(s,u)\, q(x(u)) \right\} \\
&\quad + \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u)\, c(s) \triangle x(s-\tau_2(s)). \qquad (2.5)
\end{aligned}$$

By performing a summation by parts, we have

$$\begin{aligned}
&\sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \triangle_s \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\, x(u) \\
&= \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) - \prod_{u=n_0}^{n-1} H(u) \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)x(s) \\
&\quad - \sum_{s=n_0}^{n-1} \left\{ 1 - H(s) \right\} \prod_{u=s+1}^{n-1} H(u) \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\, x(u), \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} H(u)\, c(s)\triangle x(s-\tau_2(s)) \\
&\quad = -c(n_0-1)\,x(n_0-\tau_2(n_0))\prod_{u=n_0}^{n-1} H(u) + c(n-1)\,x(n-\tau_2(n)) \\
&\qquad -\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} H(u)\,\phi(s)x(s-\tau_2(s)),
\end{aligned} \tag{2.7}$$

where $\phi$ is given by (2.4). Finally, substituting (2.6) and (2.7) into (2.5) completes the proof. □

**Definition 2.2** *The zero solution of (1.1) is Lyapunov stable if for any $\epsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ implies $|x(n, n_0, \psi)| \leq \epsilon$ for $n \in [n_0, \infty) \cap \mathbb{Z}$.*

**Theorem 2.3** *Let $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$. Suppose that (1.2) holds, and there exists a positive constant $M$ and a constant $\alpha \in (0,1)$ such that for $n \in [n_0, \infty) \cap \mathbb{Z}$*

$$\left|\prod_{u=n_0}^{n-1} H(u)\right| \leq M, \tag{2.8}$$

*and*

$$\begin{aligned}
&|c(n-1)| + \sum_{j=1}^{2}\sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\
&+ \sum_{s=n_0}^{n-1}\left|\prod_{u=s+1}^{n-1} H(u)\right| \left\{ |h_1(s-\tau_1(s)) - a(s)| + |h_2(s-\tau_2(s)) - \phi(s)| + L\sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \right\} \\
&+ \sum_{j=1}^{2}\sum_{s=n_0}^{n-1} |1-H(s)| \left|\prod_{u=s+1}^{n-1} H(u)\right| \sum_{u=s-\tau_j(s)}^{s-1} |h_j(u)| \leq \alpha\ .
\end{aligned} \tag{2.9}$$

*Then the zero solution of (1.1) is stable.*

*Proof.* Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$(M + \alpha M)\,\delta + \alpha\epsilon \leq \epsilon.$$

Let $\psi \in D(n_0)$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$. Define

$$\mathbb{S} = \{\varphi \in \mathbb{B} : \varphi(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z},\ \|\varphi\| \leq \epsilon\}.$$

This $(\mathbb{S}, \|.\|)$ is a complete metric space where $\|.\|$ is the maximum norm.

Use (2.3) to define the operator $P:\mathbb{S}\to\mathbb{S}$ by $(P\varphi)(n)=\psi(n)$ for $n\in[m(n_0),n_0]\cap\mathbb{Z}$ and

$$\begin{aligned}(P\varphi)(n) = &\left\{\psi(n_0) - c(n_0-1)\psi(n_0-\tau_j(n_0)) - \sum_{j=1}^{2}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)\psi(s)\right\}\prod_{u=n_0}^{n-1} H(u) \\ &+ c(n-1)\varphi(n-\tau_2(n)) + \sum_{j=1}^{2}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s) \\ &+ \sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} H(u)\left\{[h_1(s-\tau_1(s))-a(s)]\varphi(s-\tau_1(s))\right. \\ &+ \left.[h_2(s-\tau_2(s))-\phi(s)]\varphi(s-\tau_2(s)) + \sum_{u=s-\tau_2(s)}^{s-1} k(s,u)q(\varphi(u))\right\} \\ &- \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\{1-H(s)\}\prod_{u=s+1}^{n-1} H(u)\sum_{u=s-\tau_j(s)}^{s-1} h_j(u)\varphi(u), \end{aligned} \tag{2.10}$$

for $n\in[n_0,\infty)\cap\mathbb{Z}$. Clearly, $P\varphi$ is bounded. We first show that $P$ maps from $\mathbb{S}$ to $\mathbb{S}$. We have

$$\begin{aligned}&|(P\varphi)(n)| \\ &\le M\delta + \alpha M\delta + \left\{|c(n-1)| + \sum_{j=1}^{2}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|\right. \\ &+ \sum_{s=n_0}^{n-1}\left|\prod_{u=s+1}^{n-1} H(u)\right|\left\{|h_1(s-\tau_1(s))-a(s)| + |h_2(s-\tau_2(s))-\phi(s)| + L\sum_{u=s-\tau_2(s)}^{s-1}|k(s,u)|\right\} \\ &+ \left.\sum_{j=1}^{2}\sum_{s=n_0}^{n-1}|1-H(s)|\left|\prod_{u=s+1}^{n-1} H(u)\right|\sum_{u=s-\tau_j(s)}^{s-1}|h_j(u)|\right\}\|\varphi\| \\ &\le (M+\alpha M)\delta + \alpha\epsilon \\ &\le \epsilon,\end{aligned}$$

by (1.2), (2.8) and (2.9). Thus $P$ maps $\mathbb{S}$ into itself. We next show that $P$ is a contraction. Let $\varphi_1,\varphi_2\in\mathbb{S}$, then

$$\begin{aligned}&|(P\varphi_1)(n) - (P\varphi_2)(n)| \\ &\le \left\{|c(n-1)| + \sum_{j=1}^{2}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|\right. \\ &+ \sum_{s=n_0}^{n-1}\left|\prod_{u=s+1}^{n-1} H(u)\right|\left\{|h_1(s-\tau_1(s))-a(s)| + |h_2(s-\tau_2(s))-\phi(s)| + L\sum_{u=s-\tau_2(s)}^{s-1}|k(s,u)|\right\} \\ &+ \left.\sum_{j=1}^{2}\sum_{s=n_0}^{n-1}|1-H(s)|\left|\prod_{u=s+1}^{n-1} H(u)\right|\sum_{u=s-\tau_j(s)}^{s-1}|h_j(u)|\right\}\|\varphi_1-\varphi_2\| \\ &\le \alpha\|\varphi_1-\varphi_2\|,\end{aligned}$$

by (1.2) and (2.9). This shows that $P$ is a contraction with contraction constant $\alpha$. Thus, by the contraction mapping principle ( [14], p. 2), $P$ has a unique fixed point $x$ in $\mathbb{S}$ which is a solution of (1.1) with $x = \psi$ on $[m(n_0), n_0] \cap \mathbb{Z}$ and $|x(n)| = |x(n, n_0, \psi)| \leq \epsilon$ for $n \in [n_0, \infty) \cap \mathbb{Z}$. This proves that the zero solution of (1.1) is stable. □

**Definition 2.4** *The zero solution of (1.1) is asymptotically stable if it is Lyapunov stable and if for any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ implies $x(n, n_0, \psi) \to 0$ as $n \to \infty$.*

**Theorem 2.5** *Assume that the hypotheses of Theorem 2.3 hold. Also assume that*

$$\prod_{u=n_0}^{n-1} H(u) \to 0 \text{ as } n \to \infty. \tag{2.11}$$

*Then the zero solution of (1.1) is asymptotically stable.*

*Proof.* *We have already proved that the zero solution of (1.1) is stable. Let $\psi \in D(n_0)$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ and define*

$$\mathbb{S}^* = \{\varphi \in \mathbb{B} : \varphi(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z},\ \|\varphi\| \leq \epsilon \text{ and } \varphi(n) \to 0 \text{ as } n \to \infty\}.$$

*Define $P : \mathbb{S}^* \to \mathbb{S}^*$ by (2.10). From the proof of Theorem 2.3, the map $P$ is a contraction with the contraction constant $\alpha$ and for every $\varphi \in \mathbb{S}^*$, $\|P\varphi\| \leq \epsilon$.*

*We next show that $(P\varphi)(n) \to 0$ as $n \to \infty$. There are five terms on the right hand side in (2.10). Denote them, respectively, by $I_k$, $k = 1, 2, ..., 5$. It is obvious that the first term $I_1$ tends to zero as $t \to \infty$, by condition (2.11). Also, due to the facts that $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ for $j = 1, 2$ as $n \to \infty$, the second term $I_2$ tends to zero, as $n \to \infty$. Left to show that each one of the remaining terms in (2.10), go to zero at infinity.*

*Let $\varphi \in \mathbb{S}^*$ be fixed. For the given $\epsilon_1 > 0$, we choose $N_0 > n_0$ large enough such that $n - \tau_j(n) \geq N_0$, $j = 1, 2$ implies $|\varphi(s)| < \epsilon_1$ if $s \geq n - \tau_j(n)$. Therefore, the third term $I_3$ in (2.10) satisfies*

$$\begin{aligned} |I_3| &= \left| \sum_{j=1}^{2} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s) \right| \\ &\leq \sum_{j=1}^{2} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \, |\varphi(s)| \\ &\leq \epsilon_1 \sum_{j=1}^{2} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \leq \alpha\epsilon_1 < \epsilon_1. \end{aligned}$$

*Thus, $I_3 \to 0$ as $n \to \infty$. Now for a given $\epsilon_1 > 0$, there exists a $N_1 > n_0$ such that $s \geq N_1$ implies*

*$|\varphi(s-\tau_j(s))| < \epsilon_1$ for $j=1,2$. Thus, for $n \geq N_1$, the term $I_4$ in (2.10) satisfies*

$$\begin{aligned}
|I_4| &= \Bigg| \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \Bigg\{ [h_1(s-\tau_1(s)) - a(s)]\,\varphi(s-\tau_1(s)) \\
&\quad + [h_2(s-\tau_2(s)) - \phi(s)]\,\varphi(s-\tau_2(s)) + \sum_{u=s-\tau_2(s)}^{s-1} k(s,u)\,q(\varphi(u)) \Bigg\} \Bigg| \\
&\leq \sum_{s=n_0}^{N_1-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| \Bigg\{ |h_1(s-\tau_1(s)) - a(s)|\,|\varphi(s-\tau_1(s))| \\
&\quad + |h_2(s-\tau_2(s)) - \phi(s)|\,|\varphi(s-\tau_2(s))| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)|\,|\varphi(u)| \Bigg\} \\
&\quad + \sum_{s=N_1}^{n-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| \Bigg\{ |h_1(s-\tau_1(s)) - a(s)|\,|\varphi(s-\tau_1(s))| \\
&\quad + |h_2(s-\tau_2(s)) - \phi(s)|\,|\varphi(s-\tau_2(s))| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)|\,|\varphi(u)| \Bigg\} \\
&\leq \sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \sum_{s=n_0}^{N_1-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| \Bigg\{ |h_1(s-\tau_1(s)) - a(s)| \\
&\quad + |h_2(s-\tau_2(s)) - \phi(s)| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \Bigg\} \\
&\quad + \epsilon_1 \sum_{s=N_1}^{n-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| \Bigg\{ |h_1(s-\tau_1(s)) - a(s)| \\
&\quad + |h_2(s-\tau_2(s)) - \phi(s)| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \Bigg\}.
\end{aligned}$$

*By (2.11), we can find $N_2 > N_1$ such that $n \geq N_2$ implies*

$$\begin{aligned}
&\sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \sum_{s=n_0}^{N_1-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| \Bigg\{ |h_1(s-\tau_1(s)) - a(s)| \\
&\quad + |h_2(s-\tau_2(s)) - \phi(s)| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \Bigg\} \\
&= \sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \left| \prod_{u=N_2}^{n-1} H(u) \right| \sum_{s=n_0}^{N_1-1} \left| \prod_{u=s+1}^{N_2-1} H(u) \right| \Bigg\{ |h_1(s-\tau_1(s)) - a(s)| \\
&\quad + |h_2(s-\tau_2(s)) - \phi(s)| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \Bigg\} < \epsilon_1 .
\end{aligned}$$

*Now, apply (2.9) to have $|I_4| < \epsilon_1 + \alpha\epsilon_1 < 2\epsilon_1$. Thus, $I_4 \to 0$ as $n \to \infty$. Similarly, by using (2.9),* *then, if $n \geq N_2$ then term $I_5$ in (2.10) satisfies*

$$\begin{aligned}|I_5| &= \left| \sum_{j=1}^{2} \sum_{s=n_0}^{n-1} \{1 - H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{u=s-\tau_j(s)}^{s-1} h_j(u) \varphi(u) \right| \\ &\leq \sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \left| \prod_{u=N_2}^{n-1} H(u) \right| \sum_{j=1}^{2} \sum_{s=n_0}^{N_1-1} |1 - H(s)| \left| \prod_{u=s+1}^{N_2-1} H(u) \right| \sum_{u=s-\tau_j(s)}^{s-1} |h_j(u)| \\ &+ \epsilon_1 \sum_{j=1}^{2} \sum_{s=N_1}^{n-1} |1 - H(s)| \left| \prod_{u=s+1}^{n-1} H(u) \right| \sum_{u=s-\tau_j(s)}^{s-1} |h_j(u)| \\ &< \epsilon_1 + \alpha\epsilon_1 < 2\epsilon_1.\end{aligned}$$

*Thus, $I_5 \to 0$ as $n \to \infty$. In conclusion $(P\varphi)(n) \to 0$ as $n \to \infty$, as required. Hence $P$ maps $\mathbb{S}^*$ into $\mathbb{S}^*$.*

*By the contraction mapping principle, $P$ has a unique fixed point $x \in \mathbb{S}^*$ which solves (1.1). Therefore, the zero solution of (1.1) is asymptotically stable.* □

Letting $\tau_1 = 0$, $a = 1 - a_1$, we have

**Corollary 2.6** *Let $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$. Suppose that (1.2) holds and there exists a constant $\alpha \in (0, 1)$ such that for $n \in [n_0, \infty) \cap \mathbb{Z}$*

$$\begin{aligned}&|c(n-1)| + \sum_{s=n-\tau_2(n)}^{n-1} |h_2(s)| \\ &+ \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| \left( |h_1(s) - 1 + a_1(s)| + |h_2(s - \tau_2(s)) - \phi(s)| + L \sum_{u=s-\tau_2(s)}^{s-1} |k(s,u)| \right) \\ &+ \sum_{s=n_0}^{n-1} |1 - H(s)| \left| \prod_{u=s+1}^{n-1} H(u) \right| \sum_{u=s-\tau_2(s)}^{s-1} |h_2(u)| \ \leq \alpha \ ,\end{aligned} \tag{2.12}$$

*where $\phi(n) = c(n) - c(n-1) H(n)$. Then the zero solution of (1.6) is asymptotically stable if*

$$\prod_{u=n_0}^{n-1} H(u) \to 0 \text{ as } n \to \infty.$$

**Remark 2.7** *When $h_1(s) = 1 - a_1(s)$ and $h_2(s) = 0$, Corollary 2.6 reduces to Theorem C.*

For the special case $c(n) = 0$ and $q(x) = 0$, we can get

**Corollary 2.8** *Suppose that $1 - h_1(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$, and there exists a constant*

$\alpha \in (0,1)$ *such that for* $n \in [n_0, \infty) \cap \mathbb{Z}$

$$\sum_{s=n-\tau_1(n)}^{n-1} |h_1(s)| + \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} [1 - h_1(n)] \right| |h_1(s - \tau_1(s)) - a(s)| + \sum_{s=n_0}^{n-1} |h_1(s)| \left| \prod_{u=s+1}^{n-1} [1 - h_1(n)] \right| \sum_{u=s-\tau_1(s)}^{s-1} |h_1(u)| \le \alpha\,. \tag{2.13}$$

*Then the zero solution of (1.3) is asymptotically stable if*

$$\prod_{u=n_0}^{n-1} [1 - h_1(n)] \to 0 \text{ as } n \to \infty.$$

**Remark 2.9** *When* $h_1(s) = a(g(s))$*, Corollary 2.8 reduces to Theorem B.*

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