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EXISTENCE AND UNIQUENESS OF A SOLUTION FOR A FRACTIONAL DIFFERENTIAL EQUATION BY ROTHE'S METHOD

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Abstract. In this paper, we establish the existence and uniqueness of a strong solution for a semilinear fractional differential equation by using Rothe's method. Thereafter, we show the continuous dependence of the solution on the initial data and the stability of the solution.

Keywords: Rothe's method; fractional differential equation; strong solution; a priori estimate.

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1 Introduction

Consider the following semilinear fractional differential equation in a Banach space X

$$D^{\alpha}u(t) + Au(t) = f(t, u(t)), \quad t \in (0, T], \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where D^{α} $(0 < \alpha < 1)$ denotes the standard Riemann–Liouville fractional derivative of order α, $-A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$ of contractions in X, $u_0 \in D(A)$, the domain of A, and the map f is continuous from $I \times D(A)$ into X. Here $I = [0, T]$.

Fractional differentiation and integration are the generalization of the ordinary differentiation and integration to arbitrary non-integral order. Fractional calculus is a powerful tool which plays

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an important role in the study of nonlinear oscillations of earthquakes and the modeling of multi-scale problems. Fractional differential equations have been recently applied to engineering, physics, signal processing and fractional dynamics problems etc. As indicated in [6], fractional differential equations involving Riemann-Liouville differential operator of fractional order $0 < \alpha < 1$, appear to be important in many physical problems. So they deserve an independent study parallel to the well known theory of ordinary differential equations.

Zhou [7] has established the existence and uniqueness of solutions for the following system of fractional differential equations by using Schauder's fixed point theorem

$$\begin{aligned} D^{\alpha}x(t) &= f(t, x(t)), \\ x(t_0) &= x_0, \end{aligned}$$

where D^{α} $(0 < \alpha < 1)$ denotes the fractional derivative in the sense of Caputo's definition.

Wang and Zhou [8] have proved the existence and uniqueness of a mild solution and established the optimal control in the α-norm for the following semilinear fractional evolution equations by using the fractional calculus, singular version Gronwall inequality and Leray–Schauder fixed point theorem

$$\begin{aligned} D^{q}x(t) &= -Ax(t) + f(t, x(t)), \quad t \in [0, T], \\ x(0) &= x_0, \end{aligned}$$

where D^q is the Caputo fractional derivative of order $0 < q < 1$, $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators and $f : J \times X_{\alpha} \to X$ is locally Lipschitz continuous in X_{α}. Here $X_{\alpha} = D(A^{\alpha})$ $(0 < \alpha < 1)$ is a Banach space with the norm $\|x\|_{\alpha} = \|A^{\alpha}x\|$ for $x \in X_{\alpha}$. For the approximation of fractional derivatives, we refer the readers to [4].

In this paper our aim is to use Rothe's method, also known as method of semidiscretization in time or the method of lines, to prove the existence and uniqueness of solution for a semilinear fractional differential equation and to study the continuous dependence of the solution on initial data and the stability of solution.

Since 1930, various classical types of initial boundary value problem have been investigated by many authors using the method of time discretization; see for instance [9] and [15] and references therein. The method of time discretization is a very efficient tool in the study of an approximate solution and its convergence to the solution of the problem. This method has been used by many authors to study the solutions of abstract cauchy problems with classical conditions.

For more applications of the Rothe's method to the parabolic boundary value problem, we refer the readers to [11, 12, 18–20], and the references therein.

In Rothe's method or the method of semidiscretization, we replace the time derivative by the corresponding difference quotients and we obtain a system of time independent equations. By using the m–accretivity of the operator these systems are guaranteed to have unique solutions. Using these approximate solutions we define the Rothe's sequence. After proving some a priori estimates for approximate solutions, we prove the convergence of the Rothe's sequence to the unique solution of the given problem.

The plan of the rest of the paper is as follows. In Section 2, we state all the assumptions and preliminaries. In Section 3, we state the main result. In the last section, we state and prove all the lemmas that are required to prove the main result and at the end of this section, we prove the main result.

2 Assumptions and Preliminaries

Definition 2.1 *The fractional order integral of a function $f(t)$ of order $\alpha > 0$ is defined by*

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau \,. $$

For convenience throughout the paper, we denote ${}_0D_t^{-\alpha}$ by $D^{-\alpha}$.

Definition 2.2 *The Riemann-Liouville derivative of order $0 < \alpha < 1$ of a function $f(t)$ is defined as*

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-s)^{\alpha}}\, \mathrm{d}\tau \,. $$

Definition 2.3 *The solution of* (1.1)–(1.2) *is stable if for every $\epsilon > 0$, there exists a $\delta > 0$ s.t.*

$$ \|u(t)\| < \epsilon \quad \textit{whenever} \quad \|u_0\| < \delta, \quad \forall t \in [0,T] \,. $$

Definition 2.4 *Let X be a Banach space and let X^* be its dual. For every $x \in X$ we define the duality map J as*

$$ J(x) = \{x^* : \, x^* \in X^* \quad \textit{and} \quad \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\} \,, $$

where $\langle x^*, x\rangle$ denotes the value of x^* at x.

Definition 2.5 *By a strong solution of the problem* (1.1)–(1.2)*, we mean an abstract function u such that the following conditions are satisfied:*

(i) $u \in C(I,X)$ *and* $u \in D(A)$*,*

(ii) D^{α} *exists and is continuous on I, where $0 < \alpha < 1$,*

(iii) u *satisfies* (1.1) *a.e. on I, with the initial condition $u(0) = u_0 \in D(A)$.*

Lemma 2.6 (Theorem 1.4.3, [5]) *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, then A is m–accretive, i.e.,*

$$ (Au, J(u)) \geq 0, \quad \textit{for} \quad u \in D(A) \,, $$

where J is the duality mapping and $R(I+\lambda A) = X$ for $\lambda > 0$, I is the identity operator on X and $R(\cdot)$ is the range of an operator.

We assume the following assumption:

(H) There exists a constant $k_1 > 0$ such that

$$ \|f(t,u) - f(s,v)\| \leq k_1[|t-s| + \|u-v\|], \quad \forall t,s \in [0,T],\ \forall u,v \in D(A) \,. $$

3 Main Result

Theorem 3.1 *Suppose that **H** is satisfied and A is m–accretive. Then problem* (1.1)–(1.2) *has a unique strong solution on I. Furthermore the solutions u_i $(i = 1, 2)$ corresponding to the initial data u_0^i $(i = 1, 2)$ satisfy the following estimates*

$$\|u_1(t) - u_2(t)\| \leq \|u_0^1 - u_0^2\| \exp\left(\frac{T^\alpha k_1}{2\Gamma(\alpha+1)}\right).$$

If, in addition, f satisfies the following condition

$$\|f(t, u(t))\| \leq k_2 \|u(t)\|\,,$$

then the solution of (1.1)–(1.2) *is stable.*

4 Discretization and a priori estimates

To apply the method of time discretization, we divide the interval $[0, T]$ into the subintervals of length $h_n = \frac{T}{n}$ and replace the equations (1.1)–(1.2) by the following approximate equations,

$$\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha} + A u_j^n = f(t_j^n, u_{j-1}^n), \quad j = 1, \cdots, n, \tag{4.1}$$

$$u_0^n = u_0. \tag{4.2}$$

Existence of unique $u_j^n \in D(A)$ satisfying (4.1)–(4.2) is a consequence of the m–accretivity of A. Let $f_0 = f(0, u_0)$.

Now we construct the Rothe's sequence $\{U^n(t)\}$ of functions from I into $D(A)$ defined by

$$U^n(t) = \begin{cases} u_0, \quad \text{for} \quad t \in [-\tau, 0], \\[2ex] u_{j-1}^n + \frac{t - t_{j-1}^n}{h_n^\alpha}(u_j^n - \alpha u_{j-1}^n) \quad \text{in} \quad I_j^n = (t_{j-1}^n, t_j^n], \\ \qquad\qquad j = 1, \cdots, n. \end{cases} \tag{4.3}$$

Now we prove the convergence of the sequence $\{U^n\}$ to the unique solution of the problem as $n \to \infty$. For this first we prove the some estimates for u_j^n and $\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}$ using **H**. Throughout the paper we denote the generic constant by C, which may have different value in the same discussion.

Lemma 4.1 *There exists a constant C (independent of n, j and h_n) such that*

$$\|u_j^n - u_0\| \leq C\,.$$

Proof. Taking $j = 1$ in (4.1) and using (4.2), we get

$$\frac{u_1^n - \alpha u_0}{h_n^\alpha} + A u_1^n = f(t_1^n, u_0).$$

Subtracting $\frac{1}{h_n^\alpha}u_0 + Au_0$, from both sides of above equation, we get

$$u_1^n - u_0 + h_n^\alpha(Au_1^n - Au_0) = h_n^\alpha[f(t_1^n, u_0) - Au_0] - (1-\alpha)u_0.$$

Applying $J(u_1^n - u_0)$ on both sides, we get

$$\begin{aligned}&\langle u_1^n - u_0, J(u_1^n - u_0)\rangle + h_n^\alpha\langle (Au_1^n - Au_0), J(u_1^n - u_0\rangle \\ &= \langle h_n^\alpha(f(t_1^n, u_0) - Au_0) - (1-\alpha)u_0, J(u_1^n - u_0)\rangle.\end{aligned}$$

By using m–accretivity of A and the definition of duality map J, we get

$$\begin{aligned}\|u_1^n - u_0\| &\leq h_n^\alpha[\|f(t_1^n, u_0)\| + \|Au_0\|] + (1-\alpha)\|u_0\| \\ &\leq h_n^\alpha[\|f(t_1^n, u_0) - f(0, u_0)\| + \|f(0, u_0)\| + \|Au_0\|] + (1-\alpha)\|u_0\|.\end{aligned}$$

Using **H**, we get

$$\begin{aligned}\|u_1^n - u_0\| &\leq h_n^\alpha[k_1|t_1^n| + \|f(0, u_0)\| + \|Au_0\|] + (1-\alpha)\|u_0\| \\ &\leq T^\alpha[k_1T + \|f_0\| + \|Au_0\|] + (1-\alpha)\|u_0\| \equiv C \text{ (say)}.\end{aligned}$$

We will prove this result by induction. For this we assume that

$$\|u_i^n - u_0\| \leq C, \quad \forall i < j\,.$$

Now we show that

$$\|u_j^n - u_0\| \leq C\,.$$

Subtracting $\frac{1}{h_n^\alpha}(u_0 - \alpha u_0) + Au_0$ from both sides of (4.1), we get

$$(u_j^n - u_0) + h_n^\alpha(Au_j^n - Au_0) = \alpha(u_{j-1}^n - u_0) + h_n^\alpha[f(t_j^n, u_{j-1}^n) - Au_0] - (1-\alpha)u_0.$$

Applying $J(u_j^n - u_0)$ on both sides and using using m–accretivity of A, we get

$$\begin{aligned}\langle u_j^n - u_0, J(u_j^n - u_0)\rangle \leq{}& \alpha\langle u_{j-1}^n - u_0, J(u_j^n - u_0)\rangle + h_n^\alpha\langle f(t_j^n, u_{j-1}^n), J(u_j^n - u_0)\rangle \\ &- h_n^\alpha\langle Au_0, J(u_j^n - u_0)\rangle - (1-\alpha)\langle u_0, J(u_j^n - u_0)\rangle.\end{aligned}$$

By using the definition of duality map J, we get

$$\|u_j^n - u_0\| \leq \alpha\|u_{j-1}^n - u_0\| + h_n^\alpha[\|f(t_j^n, u_{j-1}^n)\| + \|Au_0\|] + (1-\alpha)\|u_0\|.$$

By using **H** and induction hypothesis, we get

$$\begin{aligned}\|u_j^n - u_0\| &\leq \alpha C + h_n^\alpha[k_1(|t_j^n| + C) + \|f_0\| + \|Au_0\|] + (1-\alpha)\|u_0\| \\ \|u_j^n - u_0\| &\leq \alpha C + T^\alpha[k_1(T + C) + \|f_0\| + \|Au_0\|] + (1-\alpha)\|u_0\| \\ &\equiv C \text{ (say)}.\end{aligned}$$

This completes the proof of lemma. □

Lemma 4.2 *There exists a constant C (independent of n, j and h_n) such that*

$$\left\|\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}\right\| \leq C\,.$$

Proof. Taking $j = 1$ in (4.1), we get

$$\frac{u_1^n - \alpha u_0^n}{h_n^\alpha} + Au_1^n = f(t_1^n, u_0^n).$$

Subtracting $\alpha A u_0^n$ and applying $J(u_1^n - \alpha u_0^n)$ from both sides of the above equation, we get

$$\begin{aligned}&\frac{1}{h_n^\alpha}\langle u_1^n - \alpha u_0^n, J(u_1^n - \alpha u_0^n)\rangle + \langle Au_1^n - \alpha Au_0^n, J(u_1^n - \alpha u_0^n)\rangle \\ &= \langle f(t_1^n, u_0^n) - \alpha Au_0^n, J(u_1^n - \alpha u_0^n)\rangle.\end{aligned}$$

By using m–accretivity of A and the definition of duality map J, we get

$$\left\|\frac{u_1^n - \alpha u_0^n}{h_n^\alpha}\right\| \leq \|f(t_1^n, u_0^n)\| + \alpha\|Au_0^n\|.$$

Using **H**, we get:

$$\begin{aligned}\left\|\frac{u_1^n - \alpha u_0^n}{h_n^\alpha}\right\| &\leq k_1[|t_1^n| + \|f_0\|] + \alpha\|Au_0\| \\ &\leq k_1[T + \|f_0\|] + \alpha\|Au_0\| \equiv C \text{ (say)}.\end{aligned}$$

We will prove this result by induction. For this we assume that

$$\left\|\frac{u_i^n - \alpha u_{i-1}^n}{h_n^\alpha}\right\| \leq C, \quad \forall i < j.$$

Now we show that

$$\left\|\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}\right\| \leq C.$$

Subtracting α time of (4.1) written for $j-1$, from the same equation written for j, we get

$$\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha} - \alpha\frac{u_{j-1}^n - \alpha u_{j-2}^n}{h_n^\alpha} + Au_j^n - \alpha Au_{j-1}^n = f(t_j^n, u_{j-1}^n) - \alpha f(t_{j-1}^n, u_{j-2}^n).$$

Applying $J(u_j^n - \alpha u_{j-1}^n)$ to both sides and using the m–accretivity of A, we get

$$\left\|\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}\right\| \leq \alpha\left\|\frac{u_{j-1}^n - \alpha u_{j-2}^n}{h_n^\alpha}\right\| + \|f(t_j^n, u_{j-1}^n)\| + \alpha\|f(t_{j-1}^n, u_{j-2}^n)\|.$$

Using **H**, we get

$$\begin{aligned}\left\|\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}\right\| \leq \alpha\left\|\frac{u_{j-1}^n - \alpha u_{j-2}^n}{h_n^\alpha}\right\| + k_1(|t_j^n| + \|u_{j-1}^n - u_0\|) \\ +\alpha k_1(|t_{j-1}^n| + \|u_{j-2}^n - u_0\|) + (1+\alpha)\|f_0\|.\end{aligned}$$

Using Lemma 4.1 and the induction hypothesis, we get

$$\left\|\frac{u_j^n - \alpha u_{j-1}^n}{h_n^\alpha}\right\| \leq \alpha C + k_1(1+\alpha)(T + C) + (1+\alpha)\|f_0\| \equiv C \text{ (say)}.$$

This completes the proof of lemma. □

Now we define a sequence of step functions $\{X^n(t)\}$ as

$$X^n(t) \quad = \quad \begin{cases} u_0, & \text{at} \quad t = 0, \\ \\ u_j^n, & t \in (t_{j-1}^n, t_j^n]. \end{cases} \tag{4.4}$$

Remark 4.3 *From Lemma 4.1 and Lemma 4.2, it follows that the functions $U^n(t)$ are uniformly Lipschitz continuous on $(0, T]$ and $U^n(t) - X^n(t) \to 0$ in X as $n \to \infty$ on $(0, T]$. Furthermore sequences $\{U^n(t)\}$ and $\{X^n(t)\}$ are uniformly bounded in X.*

We denote

$$f^n(t) := f(t_j^n, u_{j-1}^n)\,.$$

Using (4.3) and (4.4) in (4.1), we get:

$$D^\alpha U^n(t) + AX^n(t) = f^n(t), \tag{4.5}$$

where D^α denotes the fractional derivative in $(0, T]$.

Integrating above between the limits 0 to t, we get

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AX^n(s)\, \mathrm{d}s = u_0 - U^n(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f^n(s)\, \mathrm{d}s. \tag{4.6}$$

Lemma 4.4 *There exist $u \in C([0, T], D(A))$ such that $U^n \to u$, as $n \to \infty$. Moreover u is Lipschitz continuous on $[0, T]$.*

Proof. Subtracting (4.5) written for m from the same equation written for n, and applying $J(X^n(t) - X^m(t))$ on both sides, also using m–accretivity of A, we get

$$\langle D^\alpha U^n(t) - D^\alpha U^m(t), J(X^n(t) - X^m(t))\rangle \leq \langle f^n(t) - f^m(t), J(X^n(t) - X^m(t))\rangle.$$

Using above equation and the fact

$$D^\alpha \|U^n(t) - U^m(t)\|^2 = \langle D^\alpha (U^n(t) - U^m(t)), J(U^n(t) - U^m(t))\rangle\,,$$

we conclude that

$$\begin{aligned} D^\alpha \|U^n(t) - U^m(t)\|^2 \quad \leq \quad & \langle f^n(t) - f^m(t), J(U^n(t) - U^m(t))\rangle \\ & + \langle D^\alpha U^n(t) - D^\alpha U^m(t), J(U^n(t) - U^m(t))\rangle \\ & - \langle D^\alpha U^n(t) - D^\alpha U^m(t), J(X^n(t) - X^m(t))\rangle \\ & + \langle f^n(t) - f^m(t), J(X^n(t) - X^m(t))\rangle \\ & - \langle f^n(t) - f^m(t), J(U^n(t) - U^m(t))\rangle\,. \end{aligned}$$

Now using the definition of duality map, we get

$$D^\alpha \|U^n(t) - U^m(t)\|^2 \leq \|f^n(t) - f^m(t)\| \|U^n(t) - U^m(t)\| + \epsilon_{nm}(t) \tag{4.7}$$

where

$$\epsilon_{nm}(t) = [\|D^\alpha (U^n(t) - U^m(t))\| + \|f^n(t) - f^m(t)\|]\, \epsilon'_{nm}(t)$$

and
$$\epsilon'_{nm}(t) = \|U^n(t) - X^n(t)\| + \|U^m(t) - X^m(t)\|\,.$$
According to Remark 4.3, $\epsilon_{nm}, \epsilon'_{nm} \to 0$ as $n, m \to \infty$.
Using **H**, we get

$$\begin{aligned}\|f^n(t) - f^m(t)\| &= \|f(t_j^n, u_{j-1}^n) - f(t_l^m, u_{l-1}^m)\| \\ &\leq k_1[|\ t_j^n - t_l^m\ | + \|u_{j-1}^n - u_{l-1}^m\|]\,.\end{aligned}$$

Using (4.3) and Lemma 4.2, we get

$$\|f^n(t) - f^m(t)\| \leq k_1[|t_j^n - t_l^m| + Ch_n + Ck_m] + k_1\|U^n(t) - U^m(t)\|\,.$$

Using above inequality in (4.7), we get

$$D^\alpha\|U^n(t) - U^m(t)\|^2 \leq \epsilon''_{nm}(t) + k_1\|U^n(t) - U^m(t)\|^2$$

where
$$\epsilon''_{nm}(t) = k_1[|t_j^n - t_l^m| + Ch_n + Ck_m]\|U^n(t) - U^m(t)\| + \epsilon_{nm}(t)\,.$$
It is clear that $\epsilon''_{nm} \to 0$ as $n, m \to \infty$.
Integrating between the limits 0 to t, and using Definition 2.1, we get

$$\|U^n(t) - U^m(t)\|^2 \leq g(t) + \frac{k_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U^n(s) - U^m(s)\|^2\,\mathrm{d}s\,,$$

where
$$g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\epsilon''_{nm}(t)\,\mathrm{d}s\,.$$

Applying Gronwall's inequality we conclude that there exist $u \in C([0,T], D(A))$ such that $U^n \to u$ as $n \to \infty$. As each U^n is uniformly Lipschitz continuous, u is Lipschitz continuous.

This completes the proof of lemma. □

Remark 4.5 *By Remark 4.3 and Lemma 4.4, we conclude that $X^n(t) \to u(t)$ as $n \to \infty$. Again according to Remark 4.3, sequence $\{AX^n(t)\}$ is uniformly bounded in $(0,T]$. So by Lemma 2.5 [3], we have $AX^n(t) \rightharpoonup Au(t)$. Here $\rightharpoonup$ denote weak convergence in X.*

Proof of Theorem 3.1. Now for every $x^* \in X^*$, $t \in (0,T]$, from (4.6) we have

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_0^t \langle (t-s)^{\alpha-1}AX^n(s), x^*\rangle\,\mathrm{d}s &= \langle u_0, x^*\rangle - \langle U^n(t), x^*\rangle \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \langle (t-s)^{\alpha-1}f^n(s), x^*\rangle\,\mathrm{d}s.\end{aligned}$$

By using bounded convergence theorem and Lemma 4.4, we get

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_0^t \langle (t-s)^{\alpha-1}Au(s), x^*\rangle\,\mathrm{d}s &= \langle u_0, x^*\rangle - \langle u(t), x^*\rangle \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \langle (t-s)^{\alpha-1}f(s,u(s)), x^*\rangle\,\mathrm{d}s.\end{aligned}$$

As $Au(t)$ is Bochner integrable on $(0, T]$, from the above equation, we obtain

$$D^\alpha u + Au(t) = f(t, u(t)), \quad \text{a.e. } t \in (0, T]. \tag{4.8}$$

Clearly $u \in C([0, T], X)$ and differentiable a.e. on $(0, T]$ with $u(t) \in D(A)$ a.e. on $(0, T]$ and $u(0) = u_0$ satisfying (4.8). Hence it will be a strong solution of the problem (1.1)–(1.2) on $[0, T]$.

Next we show the uniqueness of this strong solution. For this we assume that u_1 and u_2 are two strong solutions of the problem (1.1)–(1.2). Let $u = u_1 - u_2$, then from (4.8), we have

$$D^\alpha u(t) + Au_1(t) - Au_2(t) = f(t, u_1(t)) - f(t, u_2(t)), \quad t \in (0, T]\,.$$

Applying $J(u_1(t) - u_2(t))$ on both sides and using m–accretivity of A, we get

$$D^\alpha \langle u(t), J(u(t))\rangle \le \langle f(t, u_1(t)) - f(t, u_2(t)), J(u(t))\rangle\,.$$

By using **H**, we get

$$D^\alpha \|u(t)\|^2 \le k_2 \|u(t)\|^2\,.$$

Integrating between the limit 0 to t, we get

$$\|u(t)\|^2 \le \frac{k_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u(s)\|^2 \, \mathrm{d}s\,.$$

Applying Gronwall's inequality, we obtain

$$u(t) = 0, \quad t \in (0, T] \Rightarrow u_1(t) = u_2(t), \quad \forall t \in (0, T]\,,$$

which yields the uniqueness of the strong solution of (1.1)–(1.2).

Next we show the continuous dependence of the above solution on the initial data.

We are given that u_1 and u_2 are two strong solutions corresponding to the initial data u_0^1 and u_0^2 respectively.

From (4.8), we have

$$D^\alpha (u_1(t) - u_2(t)) + Au_1(t) - Au_1(t) = f(t, u_1(t)) - f(t, u_2(t))\,.$$

Applying $J(u_1(t) - u_2(t))$ and using m–accretivity of A, we get

$$D^\alpha \langle (u_1(t) - u_2(t)), J(u_1(t) - u_2(t))\rangle \le \langle f(t, u_1(t)) - f(t, u_2(t)), J(u_1(t) - u_2(t))\rangle\,.$$

Using the definition of duality map and **H**, we get

$$D^\alpha \|u_1(t) - u_2(t)\|^2 \le k_1 \|u_1(t) - u_2(t)\|^2\,.$$

Integrating and using Definition 2.1, we get

$$\|u_1(t) - u_2(t)\|^2 - \|u_0^1 - u_0^2\|^2 \le \frac{k_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_1(s) - u_2(s)\|^2 \, \mathrm{d}s\,.$$

Applying Grownwall's inequality, we obtain

$$\begin{aligned}
\|u_1(t) - u_2(t)\|^2 &\leq \|u_0^1 - u_0^2\|^2 \exp\left(\frac{k_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\right) \\
\|u_1(t) - u_2(t)\|^2 &\leq \|u_0^1 - u_0^2\|^2 \exp\left(\frac{T^\alpha k_1}{2\Gamma(\alpha+1)}\right).
\end{aligned}$$

In the last, we will prove the stability of the problem (1.1)–(1.2).

Applying $J(u(t))$ on the both sides of (4.8), we get

$$D^\alpha\langle u(t), J(u(t))\rangle + \langle Au(t), J(u(t))\rangle \leq \langle f(t,u(t)), J(u(t))\rangle .$$

By using the m–accretivity of A and the definition of duality map, we get:

$$\begin{aligned}
D^\alpha\|u(t)\|^2 &\leq \|f(t,u(t))\|\|u(t)\| \\
&\leq k_2\|u(t)\|^2.
\end{aligned}$$

Integrating and using Definition 2.1, we get

$$\|u(t)\|^2 \leq \|u(0)\|^2 + \frac{k_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)\|^2\,\mathrm{d}s.$$

Applying Grownwall's inequality, we obtain

$$\begin{aligned}
\|u(t)\|^2 &\leq \|u(0)\|^2 \exp\left(\frac{k_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\right) \\
&\leq \|u(0)\|^2 \exp\left(\frac{k_2 T^\alpha}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

This implies that

$$\|u(t)\| < \epsilon \quad \text{whenever} \quad \|u_0\| < \epsilon \exp\left(-\frac{k_2 T^\alpha}{2\Gamma(\alpha+1)}\right).$$

This completes the proof of the main result. □

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