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EXISTENCE OF NON-OSCILLATORY SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAYS AND M-ORDER LINEAR OPERATORS WITH VARIABLE COEFFICIENTS

JULIO G. DIX*
Department of Mathematics, Texas State University
601 University Drive, San Marcos, TX 78666, USA

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Abstract. This article concerns the existence of non-oscillatory solutions for non-linear, non-homogeneous differential equations. In these equations, the unknown function depends on distributed arguments that can be retarded or advanced and have variable delays. Using contraction mappings, we show the existence of solutions, and estimate their norms. Then our results are extended to dynamic equations on times scales, and to difference equations.

Keywords: Neutral equation; non-oscillatory solution; distributed delay.

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1 Introduction

This article concerns the existence of non-oscillatory solutions for the delay differential equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(r_{m-1}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(r_{m-2}(t)\cdots\frac{\mathrm{d}}{\mathrm{d}t}\Big(r_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)+p(t)\,x(\tau(t))\big)\Big)\cdots\Big)\Big)\\&+\int_a^b g(t,\xi,x(\delta(t,\xi)))\,\mathrm{d}\mu(\xi)=f(t)\,,\end{aligned}\tag{1.1}$$

*e-mail address: jd01@txstate.edu

where $m \geq 1$, $f \in C([0,+\infty),\mathbb{R})$, $g \in C([0,+\infty) \times [a,b] \times \mathbb{R},\mathbb{R})$, $p \in C([0,+\infty),\mathbb{R})$, r_i are positive functions having $m-i$ derivatives on $[0,+\infty)$. The delay arguments $\tau(t)$ and $\delta(t,\xi)$ are continuous functions satisfying conditions specified below, and the integral is defined in the Stieltjes sense where μ is non-decreasing.

Our goal is to study an equation that is more general than those studied in previous publications, consider various ranges for the coefficient $p(t)$, and extend these results to dynamic equations on times scales and to difference equations. With this in mind, we consider (1.1) because it has the following properties: higher order than the equations studied in [3, 9, 10, 12, 13]; a forcing term $f(t)$ not considered in [3, 4, 9, 13]; variable delays that include the fixed delays in [4, 9, 13] and the variable delays in [3, 10, 11, 12, 13]; a variable coefficient $p(t)$ that includes the fixed coefficients in [4, 9] and the variable coefficients in [3, 11, 12, 13]; and the coefficients $r_i(t)$ not necessarily equal to 1, as in the above references, except for [11, 12] where $r_1(t)$ is variable.

Oscillation results for delay differential equations can be found in [1, 5, 6, 7, 8], and their references. Non-oscillation results for equations related to (1.1), can be summarized as follows: Zhang [13] studied the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)\,x(t-\tau)\big]+Q_1(t)\,x(t-\delta_1)-Q_2(t)\,x(t-\delta_2)=0\,, \tag{1.2}$$

where τ,δ_1,δ_2 are non-negative constants, and Q_1,Q_2 are non-negative functions. Kulenovic [9] studied the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x(t)+p\,x(t-\tau)\big]+Q_1(t)\,x(t-\delta_1)-Q_2(t)\,x(t-\delta_2)=0\,, \tag{1.3}$$

where p is a constant not equal to ± 1. Candan [3] studied the first and second order ($k=1,2$) equations

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big[x(t)+p(t)\,x(t-\tau)\big]+\int_a^b q_1(t,\xi)\,x(t-\xi)\,\mathrm{d}\xi-\int_c^d q_2(t,\xi)\,x(t-\xi)\,\mathrm{d}\xi=0\,, \tag{1.4}$$

where τ is a non-negative constant. Note that equations (1.2)–(1.4) are linear and homogeneous. Li [10] studied the linear equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x(t)+p(t)\,x(\tau_0(t))\big]+q_1(t)\,x(\tau_1(t))-q_2(t)\,x(\tau_2(t))=e(t)\,, \tag{1.5}$$

on time scales. Chen [4] studied the nonlinear equation

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\big[x(t)+cx(t-\tau)\big]+g(t,x(t-\delta(t)))=0\,, \tag{1.6}$$

where c is a constant, $c\neq\pm1$. There, it is assumed that: For each t, $g(t,x)$ is non-decreasing, $xg(t,x)>0$ when $x\neq 0$; and for a fixed $x_0>0$, it is assumed that $\int_0^\infty s^{m-1}g(s,x_0)\,\mathrm{d}s<\infty$. Note that (H3) below is less restrictive than these assumptions. Rath [11] studied the non-linear non-homogeneous equation

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\big[r_1(t)\big[x(t)+p(t)\,x(\tau(t))\big]'\big]+q(t)G(x(h(t)))=f(t)\,. \tag{1.7}$$

In the above references, $p(t)$ is not allowed to oscillate around the values $p=0,-1,+1$. However, Rath [11] found a non-oscillatory solution for (1.7) when $p(t)=\pm1$ and τ is increasing. There it is assumed that $r_2 = \cdots = r_m = 1$ and $\int_0^\infty 1/r_1(t)\,\mathrm{d}t < \infty$. We do not assume these conditions for (1.1) and replace $q(t)\,G((x(h(t)))$ in (1.7) with a distributed delay.

Note that the Stieltjes integral allows the delay in (1.1) to include the delays in (1.2)–(1.7). Also note that an appropriate choice of the variable delays τ and δ, allows (1.1) to be an ordinary, an advanced, a retarded, or a neutral differential equation.

2 Results

By a solution, we mean a function $x \in C([t_0, +\infty), \mathbb{R})$ such that $x(t) + p(t)\,x(\tau(t))$ is m times continuous differentiable and satisfies (1.1) for $t \geq t_0$. We assume that an initial condition for (1.1) is available; i.e., a function ϕ defined on a sufficiently large interval $[-t^*, t_0]$, where $x(t) = \phi(t)$.

A solution is said to be oscillatory if it has zeros of arbitrarily large value; otherwise the solution is non-oscillatory.

In this article, we use the following assumptions.

(H1) The delay $\tau(t)$ is m times continuous differentiable and $\lim_{t\to\infty} \tau(t) = \infty$.

(H2) The delay δ is in $C([0,\infty) \times [a,b], \mathbb{R})$ and satisfies

$$\lim_{t\to\infty} \min_{a\leq\xi\leq b} \delta(t,\xi) = \infty.$$

(H3) The nonlinearity g satisfies $g(t,\xi,0) = 0$ and the Lipschitz condition

$$|g(t,\xi,x) - g(t,\xi,y)| \leq K(t,\xi)|x-y|,$$

for all $t \geq 0$, all $\xi \in [a,b]$, and all x, y in some interval $[-M_0, M_0]$. Furthermore, we assume that

$$\int_0^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \int_{s_{n-1}}^\infty \int_a^b K(s_n,\xi)\,\mathrm{d}\mu(\xi)\ \mathrm{d}s_n \cdots \mathrm{d}s_1 < \infty. \tag{2.1}$$

(H4) The right-hand side of (1.1) satisfies

$$\int_0^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \int_{s_{n-1}}^\infty |f(s_n)|\ \mathrm{d}s_n \cdots \mathrm{d}s_1 < \infty\,. \tag{2.2}$$

Note that assumptions (H3) and (H4) reduce the effect that g and f have on the solution of (1.1), as $t \to \infty$.

For simplicity of notation we define the operators

$$L_m[z](t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(r_{m-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}\Big(r_{m-2}(t)\cdots\frac{\mathrm{d}}{\mathrm{d}t}\Big(r_1(t)\frac{\mathrm{d}}{\mathrm{d}t}z(t)\Big)\cdots\Big)\Big), \tag{2.3}$$

$$\widehat{L}_m[z](t) = (-1)^m \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \int_{s_{n-1}}^\infty z(s_n)\ \mathrm{d}s_n \cdots \mathrm{d}s_1\,. \tag{2.4}$$

Then (1.1) can be written as

$$L_m[x(t) + p(t)\,x(\tau(t))] + \int_a^b g(t, \xi, x(\delta(t,\xi)))\ \mathrm{d}\mu(\xi) = f(t)\,.$$

Note that $L_m[\widehat{L_m}[z]] = z$.

In our first result $p(t)$ can oscillate about zero, but within certain bounds.

Theorem 2.1 *Assume* (H1)–(H4) *and that there exists a constant* p_1 *such that* $|p(t)| \le p_1 < 1/2$ *for all* $t \ge 0$*. Then for each constant* M *in* $(0, M_0(1-2p_1)/6]$*, there exist a time* t_0 *and a solution of* (1.1) *satisfying*

$$M \le x(t) \le \frac{6M}{1-2p_1} \quad \forall t \ge t_0\,.$$

Also there exists a solution of (1.1) *satisfying* $-\frac{6M}{1-2p_1} \le x(t) \le -M$ *for all* $t \ge t_0$.

Proof. Using (2.1), (2.2) and (2.3), we select $t_* \ge 0$ such that

$$\widehat{L}_m\big[\int_a^b K(\cdot,\xi)\ \mathrm{d}\mu(\xi)\big](t_*) < \frac{1-2p_1}{3}\,, \quad \widehat{L}_m[|f(\cdot)|](t_*) < \frac{1-2p_1}{3}\,. \tag{2.5}$$

Then select a time t_0 such that $t_* \le t_0$, $t_* \le \tau(t)$, and $t_* \le \delta(t,\xi))$ for all $t \ge t_0$ and all $\xi \in [a,b]$.

First we find a positive solution in the set of continuous functions

$$B_1 = \{x : M \le x(t) \le \frac{6M}{1-2p_1} \ \text{ for } \ t \ge t_0\}\,.$$

Note that this set is bounded, closed, convex, and complete under the supremum norm $\|x\| = \sup_{t\ge t_0}|x(t)|$. As standard technique, we transform (1.1) into an integral equation, and define an operator whose fixed points yield solutions of (1.1). Let

$$G_1[x](t) = \begin{cases} G_1[x](t_0) & \text{if } t < t_0, \\ \frac{M(7-2p_1)}{2(1-2p_1)} - p(t)\,x(\tau(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\ \mathrm{d}\mu(\xi) - f(\cdot)\big](t) & \text{if } t \ge t_0\,. \end{cases}$$

Certainly $G_1[x](t)$ being the integral of continuous functions is a continuous. For each x in B_1, using that $|p(t)| \le p_1 < 1/2$ and (2.5), we have

$$\begin{aligned} G_1[x](t) &\le \frac{M(7-2p_1)}{2(1-2p_1)} + p_1\frac{6M}{1-2p_1} + \frac{6M}{(1-2p_1)}\frac{(1-2p_1)}{3} + \frac{M}{2} = \frac{6M}{1-2p_1}\,, \\ G_1[x](t) &\ge \frac{M(7-2p_1)}{2(1-2p_1)} - p_1\frac{6M}{1-2p_1} - \frac{6M}{(1-2p_1)}\frac{(1-2p_1)}{3} - \frac{M}{2} = M\,. \end{aligned}$$

Therefore, G_1 maps B_1 into B_1. For any two functions x_1, x_2 in B_1, using that $0 \le p(t) \le p_1 < 1/2$ and (2.5), we have

$$|G_1[x_1](t) - G_1[x_2](t)| \le p_1\|x_1 - x_2\| + \frac{1-2p_1}{3}\|x_1 - x_2\| = \frac{p_1+1}{3}\|x_1 - x_2\|\,.$$

Since $p_1 < 1/2$, it follows that $\frac{p_1+1}{3} < 1$ and that G_1 is a contraction mapping in the complete set B_1. Then there exists a function x in B_1 such that $G_1[x] = x$. Applying the operator L_m to this equality, we show that x is a solution of (1.1).

Now we find a negative solution of (1.1) in the set

$$B_2 = \{x : -\frac{6M}{1-2p_1} \le x(t) \le -M \text{ for } t \ge t_0\}\,.$$

Let

$$G_2[x](t) = \begin{cases} G_2[x](t_0) & \text{if } t < t_0, \\ -\frac{M(7-2p_1)}{2(1-2p_1)} - p(t)\,x(\tau(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\;\mathrm{d}\mu(\xi) - f(\cdot)\big](t) & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_2, using that $0 \le p(t) \le p_1 < 1/2$ and (2.5), we have

$$-\frac{6M}{1-2p_1} \le G_2[x](t) \le -M\,.$$

Therefore, G_2 maps B_2 into B_2. For any two functions x_1, x_2 in B_2, using that $|p(t)| \le p_1 < 1/2$ and (2.5), we have

$$|G_2[x_1](t) - G_2[x_2](t)| \le \frac{p_1+1}{3}\|x_1 - x_2\|\,.$$

Since $p_1 < 1/2$, it follows that $\frac{p_1+1}{3} < 1$ and that G_2 has a fixed point in B_2, which is a solution of (1.1). This completes the proof. □

Next we consider wider ranges for $p(t)$, but $p(t)$ can not approach ± 1.

Theorem 2.2 *Assume* (H1)–(H4) *and that one of the following two conditions is satisfied:* $0 \le p(t) \le p_2 < 1$ *for all* $t \ge 0$*, or* $-1 < -p_2 \le p(t) \le 0$ *for all* $t \ge 0$*. Then for each constant* M *in* $(0, M_0(1-p_2)/6]$*, there exist a time* t_0 *and a solution of* (1.1) *satisfying*

$$M \le x(t) \le \frac{6M}{1-p_2} \quad \forall t \ge t_0\,.$$

Also there exists a solution of (1.1) *satisfying* $-\frac{6M}{1-p_2} \le x(t) \le -M$ *for all* $t \ge t_0$*.*

Proof. Using (2.1), we select $t_* \ge 0$ such that

$$\widehat{L}_m\big[\int_a^b K(\cdot,\xi)\;\mathrm{d}\mu(\xi)\big](t_*) < \frac{1-p_2}{3}\,, \quad \widehat{L}_m[|f(\cdot)|](t_*) < \frac{M}{2}\,. \tag{2.6}$$

Then $t_0 \ge t_*$ such that $t_* \le \tau(t)$, and $t_* \le \delta(t,\xi))$ for all $t \ge t_0$ and all $\xi \in [a,b]$.

First assume that $0 \le p(t) \le p_2 < 1$. We shall find a positive solution in the set

$$B_3 = \{x : M \le x(t) \le \frac{6M}{1-p_2} \text{ for } t \ge t_0\}\,.$$

Define the operator

$$G_3[x](t) = \begin{cases} G_3[x](t_0) & \text{if } t \le t_0, \\ \frac{M(7+5p_2)}{2(1-p_2)} - p(t)\,x(\tau(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](t) & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_3, using that $0 \le p(t) \le p_2 < 1$ and (2.6), we have

$$G_3[x](t) \le \frac{M(7+5p_2)}{2(1-p_2)} \;+\; 0 \;+\; \frac{6M}{(1-p_2)}\frac{(1-p_2)}{3} + \frac{M}{2} = \frac{6M}{1-p_2}\,,$$
$$G_3[x](t) \ge \frac{M(7+5p_2)}{2(1-p_2)} - p_2\,\frac{6M}{1-p_2} - \frac{6M}{(1-p_2)}\frac{(1-p_2)}{3} - \frac{M}{2} = M\,.$$

Therefore, G_3 maps B_3 into B_3. For any two functions x_1, x_2 in B_3, using that $0 \le p(t) \le p_2 < 1$ and (2.6), we have

$$|G_3[x_1](t) - G_3[x_2](t)| \le \frac{2p_2+1}{3}\|x_1 - x_2\|\,.$$

Since $p_2 < 1$, it follows that $\frac{2p_2+1}{3} < 1$ and that G_3 is a contraction mapping in B_3. Then there exists a function x in such that $G_3\,x = x$. Computing n derivatives, we show that x is a solution of (1.1).

Now assuming that $0 \le p(t) \le p_2 < 1$, we find a negative solution in the set

$$B_4 = \{x : -\frac{6M}{1-p_2} \le x(t) \le -M \text{ for } t \ge t_0\}\,.$$

Let

$$G_4[x](t) = \begin{cases} G_4[x](t_0) & \text{if } t \le t_0, \\ -\frac{M(7+5p_2)}{2(1-p_2)} - p(t)\,x(\tau(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](t) & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_4, using that $0 \le p(t) \le p_2 < 1$ and (2.6), we have

$$-\frac{M(2+p_2)}{1-p_2} \le G_4[x](t) \le -M\,.$$

Therefore, G_4 maps B_4 into B_4. For any two functions x_1, x_2 in B_4, using that $0 \le p(t) \le p_2 < 1$ and (2.5), we have

$$|G_4[x_1](t) - G_4[x_2](t)| \le \frac{2p_2+1}{3}\|x_1 - x_2\|\,.$$

Since $p_1 < 1$, it follows that $\frac{2p_2+1}{3} < 1$ and that G_4 has a fixed point in B_4, which is solution of (1.1).

Now assuming that $-1 < -p_2 \le p(t) \le 0$, we find a positive solution in the set

$$B_5 = \{x : M \le x(t) \le \frac{6M}{1-p_2} \text{ for } t \ge t_0\}\,.$$

Define the operator

$$G_5[x](t) = \begin{cases} G_5[x](t_0) & \text{if } t \le t_0, \\ \frac{7M}{2} - p(t)\,x(\tau(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](t) & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_5, using that $-1 < -p_2 \le p(t) \le 0$ and (2.6), we have

$$G_5[x](t) \le \frac{7M}{2} + p_2\,\frac{6M}{1-p_2} + \frac{6M}{(1-p_2)}\frac{(1-p_2)}{3} + \frac{M}{2} = \frac{6M}{1-p_2}\,,$$
$$G_5[x](t) \ge \frac{7M}{2} - p_2\,\frac{6M}{1-p_2} - \frac{6M}{(1-p_2)}\frac{(1-p_2)}{3} - \frac{M}{2} = M\,.$$

Therefore, G_5 maps B_5 into B_5. For any two functions x_1, x_2 in B_5, using that $-1 < -p_2 \le p(t) \le 0$ and (2.6), we have

$$|G_5[x_1](t) - G_5[x_2](t)| \le \frac{2p_2+1}{3}\|x_1 - x_2\|\,.$$

Since $p_2 < 1$, it follows that $\frac{2p_2+1}{3} < 1$ and that G_5 has a fixed point in B_5, which is a solution of (1.1).

Now assuming that $-1 < -p_2 \le p(t) \le 0$, we find a negative solution in the set

$$B_6 = \{x : -\frac{6M}{1-p_2} \le x(t) \le -M \text{ for } t \ge t_0\}\,.$$

Let

$$G_6[x](t) = \begin{cases} G_6[x](t_0) & \text{if } t \le t_0, \\ -\frac{7M}{2} - p(t)\,x(\tau(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\ \mathrm{d}\mu(\xi) - f(\cdot)\big](t) & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_6, using that $-1 < p_2 \le p(t) \le 0$ and (2.6), we have

$$-\frac{6M}{1-p_2} \le G_6[x](t) \le -M\,.$$

Therefore, G_6 maps B_6 into B_6. For any two functions x_1, x_2 in B_6, using that $-1 < p_2 \le p(t) \le 0$ and (2.6), we have

$$|G_6[x_1](t) - G_6[x_2](t)| \le \frac{2p_2+1}{3}\|x_1 - x_2\|\,.$$

Since $p_2 < 1$, it follows that $\frac{2p_2+1}{3} < 1$ and that G_6 has a fixed point in B_6, which is a solution of (1.1). This completes the proof. ☐

Theorem 2.3 *Assume* (H1)–(H4), *$\tau(t)$ is strictly increasing, and that one of the following two conditions is satisfied: $1 < p_3 \le p(t) \le p_4$ for all $t \ge 0$, or $-p_4 \le p(t) \le -p_3 < -1$ for all $t \ge 0$. Then for each constant M in $(0, \frac{M_0(p_3-1)}{3(p_4-1)}]$, there exist a time t_0 and a solution of* (1.1) *satisfying*

$$M \le x(t) \le \frac{3Mp_4}{p_3-1} \quad \forall t \ge t_0\,.$$

Also there exists a solution of (1.1) *satisfying $-\frac{3M(p_4-1)}{p_3-1} \le x(t) \le -M$ for all $t \ge t_0$.*

Proof. Using (2.1), we select $t_* \ge 0$ such that

$$\widehat{L}_m\big[\int_a^b K(\cdot,\xi)\ \mathrm{d}\mu(\xi)\big](t_*) < \frac{p_3-1}{3}\,, \quad \widehat{L}_m[|f(\cdot)|](t_*) < \frac{M}{2}\,. \tag{2.7}$$

Since $\tau(t)$ is strictly increasing and $\lim_{t\to\infty}\tau(t) = \infty$, the function τ is invertible and $\lim_{t\to\infty}\tau^{-1}(t) = \infty$, Then we select $t_0 \geq t_*$, such that $t_* \leq \tau^{-1}(t)$, and $t_* \leq \delta(t,\xi))$ for all $t \geq t_0$ and all $\xi \in [a,b]$.

First assuming that $1 < p_3 \leq p(t) \leq p_4$, we find a positive solution in the set

$$B_7 = \{x : M \leq x(t) \leq \frac{3Mp_4}{p_3 - 1} \text{ for } t \geq t_0\}\,.$$

Define the operator

$$G_7[x](t) = \begin{cases} G_7[x](t_0) & \text{if } t < t_0,\\ \frac{1}{p(\tau^{-1}(t))}\Big[M(2p_4 + \frac{3p_4}{p_3-1} + \frac{1}{2}) - x(\tau^{-1}(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](\tau^{-1}(t))\Big] & \text{if } t \geq t_0\,. \end{cases}$$

For each x in B_7, using that $1 < p_3 \leq p(t) \leq p_4$ and (2.7), we have

$$M \leq G_7[x](t) \leq \frac{3Mp_4}{p_3 - 1}\,.$$

Therefore, G_7 maps B_7 into B_7. For any two functions x_1, x_2 in B_7, using that $1 < p_3 \leq p(t) \leq p_4$ and (2.7), we have

$$|G_7[x_1](t) - G_7[x_2](t)| \leq (\frac{2}{3p_3} + \frac{1}{3})\|x_1 - x_2\|\,.$$

Since $p_3 > 1$, it follows that $\frac{2}{3p_3} + \frac{1}{3} < 1$ and that G_7 is a contraction mapping in B_7. Then G_7 has a fixed point which is a solution of (1.1).

Now assuming that $1 < p_3 \leq p(t) \leq p_4$, we find a negative solution in the set

$$B_8 = \{x : -\frac{3Mp_4}{p_3 - 1} \leq x(t) \leq -M \text{ for } t \geq t_0\}\,.$$

Define the operator

$$G_8[x](t) = \begin{cases} G_8[x](t_0) & \text{if } t \leq t_0,\\ \frac{1}{p(\tau^{-1}(t))}\Big[- M(2p_4 + \frac{3p_4}{p_3-1} + \frac{1}{2}) - x(\tau^{-1}(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](\tau^{-1}(t))\Big] & \text{if } t \geq t_0\,. \end{cases}$$

For each x in B_8, using that $1 < p_3 \leq p(t) \leq p_4$ and (2.7), we have

$$-\frac{3Mp_4}{p_3 - 1} \leq G_8[x](t) \leq -M\,.$$

Therefore, G_8 maps B_8 into B_8. For any two functions x_1, x_2 in B_8, using that $1 < p_3 \leq p(t) \leq p_4$ and (2.7), we have

$$|G_8[x_1](t) - G_8[x_2](t)| \leq (\frac{2}{3p_3} + \frac{1}{3})\|x_1 - x_2\|\,.$$

Since $p_3 > 1$, it follows that $\frac{2}{3p_3} + \frac{1}{3} < 1$ and that G_8 is a contraction mapping in B_8. Then G_8 has a fixed point which is a solution of (1.1).

Now assuming that $-p_4 \le p(t) \le -p_3 < -1$, we find a positive solution in the set

$$B_9 = \{x : M \le x(t) \le \frac{3Mp_4}{p_3 - 1} \text{ for } t \ge t_0\}\,.$$

Define the operator

$$G_9[x](t) = \begin{cases} G_9[x](t_0) & \text{if } t \le t_0,\\ \frac{1}{p(\tau^{-1}(t))}\Big[- M(2p_4 - \frac{1}{2}) - x(\tau^{-1}(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](\tau^{-1}(t))\Big] & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_7, using that $-p_4 \le p(t) \le -p_3 < -1$ and (2.7), we have

$$M \le G_9[x](t) \le \frac{3Mp_4}{p_3 - 1}\,.$$

Therefore, G_9 maps B_9 into B_9. For any two functions x_1, x_2 in B_9, using $-p_4 \le p(t) \le -p_3 < -1$ and (2.7), we have

$$|G_9[x_1](t) - G_9[x_2](t)| \le \big(\frac{2}{3p_3} + \frac{1}{3}\big)\|x_1 - x_2\|\,.$$

Since $p_3 > 1$, it follows that $\frac{2}{3p_3} + \frac{1}{3} < 1$ and that G_9 is a contraction mapping in B_9. Then G_9 has a fixed point which is a solution of (1.1).

Now assuming that $-p_4 \le p(t) \le -p_3 < -1$, we find a negative solution in the set

$$B_{10} = \{x : -\frac{3Mp_3}{3p_4 - 2p_3 - 1} \le x(t) \le -M \text{ for } t \ge t_0\}\,.$$

Define the operator

$$G_{10}[x](t) = \begin{cases} G_{10}[x](t_0) & \text{if } t \le t_0,\\ \frac{1}{p(\tau^{-1}(t))}\Big[M(p_3 - \frac{1}{2} + \frac{p_3(p_3-1)}{3p_4-2p_3-1)} - \frac{1}{2}) - x(\tau^{-1}(t)) & \\ -\widehat{L}_m\big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\,\mathrm{d}\mu(\xi) - f(\cdot)\big](\tau^{-1}(t))\Big] & \text{if } t \ge t_0\,. \end{cases}$$

For each x in B_{10}, using $-p_4 \le p(t) \le -p_3 < -1$ and (2.7), we have

$$-\frac{3Mp_3}{3p_4 - 2p_3 - 1} \le G_{10}[x](t) \le -M\,.$$

Therefore, G_{10} maps B_{10} into B_{10}. For any two functions x_1, x_2 in B_{10}, using $-p_4 \le p(t) \le -p_3 < -1$ and (2.7), we have

$$|G_{10}[x_1](t) - G_{10}[x_2](t)| \le \big(\frac{2}{3p_3} + \frac{1}{3}\big)\|x_1 - x_2\|\,.$$

Since $p_3 > 1$, it follows that $\frac{2}{3p_3} + \frac{1}{3} < 1$ and that G_{10} is a contraction mapping in B_{10}. Then G_{10} has a fixed point which is a solution of (1.1). This completes the proof. $\square$

Next, we allow $p(t)$ to reach ± 1. However, $p(t)$ must be constant. In this section, we follow the strategy in [11] where they assumed that $r_{m-2}(t) = \dots = r_2(t) = 1$ and $\int_0^\infty 1/r_1 < \infty$. We do not assume these conditions here, and correct two of their mistakes, as explained below. The main tool is the Schauder fixed point theorem that reads as follows: Let B be a closed, convex and non-empty subset of a Banach space X. Let $G : B \to B$ be a continuous mapping such that $G(B)$ is relatively compact in X. Then G has at least one fixed point in B.

Under the assumption that τ is increasing (strictly increasing), its inverse τ^{-1} exists and is also increasing. We define its iterated inverses as follows:

$$\tau^0 = t, \quad \tau^{-2}(t) = \tau^{-1}\big(\tau^{-1}(t)\big), \quad \tau^{-3}(t) = \tau^{-1}\Big(\tau^{-1}\big(\tau^{-1}(t)\big)\Big), \quad \dots .$$

First we claim that $\lim_{t\to\infty} \tau^{-1}(t) = \infty$. On the contrary suppose that there exists an upper bound, $\alpha > \tau^{-1}(t)$ for all $t \in \mathbb{R}$. Since τ is increasing, $t = \tau(\tau^{-1}(t)) < \tau(\alpha)$ for all $t \in \mathbb{R}$, which is a contradiction. Assuming that $\tau(t) < t$ for all t, since τ^{-1} is increasing, we have $t < \tau^{-1}(t) < \tau^{-2}(t) < \dots$. Next we claim that

$$\lim_{i\to\infty} \tau^{-i}(t) = \infty .$$

On the contrary suppose that the sequence $t_i := \tau^{-i}(t)$ is bounded above. Then $\{t_i\}$ being increasing, it converges to a finite number, $\lim_{i\to\infty} t_i = \alpha < \infty$. Since τ is increasing, from the inequality $t_i < \tau^{-1}(t_i) < \alpha$, we have $\tau(t_i) < t_i < \tau(\alpha)$. In the limit, $\alpha = \lim_{i\to\infty} t_i \le \tau(\alpha)$, which contradicts $\tau(t) < t$ for all t.

Theorem 2.4 *Assume* (H1)–(H4)*, $\tau(t)$ is strictly increasing and $\tau(t) < t$ for all $t \ge 0$. Then for each constant M in $(0, M_0/3]$, there exist a time t_0 and a solution of* (1.1) *with $p(t) = 1$ satisfying*

$$M \le x(t) \le 3M \quad \forall t \ge t_0 .$$

Also there exists a solution satisfying $-3M \le x(t) \le -M$ for all $t \ge t_0$.

Under the additional assumptions

$$\sum_{i=1}^\infty \int_{\tau^{-i}(0)}^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \int_{s_{n-1}}^\infty \int_a^b K(s_n,\xi)\, \mathrm{d}\mu(\xi)\, \mathrm{d}s_n \cdots\, \mathrm{d}s_1 < \infty, \tag{2.8}$$

$$\sum_{i=1}^\infty \int_{\tau^{-i}(0)}^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \int_{s_{n-1}}^\infty |f(s_n)|\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \infty, \tag{2.9}$$

there exists a solution of (1.1) *with $p(t) = -1$ satisfying $M \le x(t) \le 3M$ for all $t \ge t_0$. Also there exists a solution satisfying $-3M \le x(t) \le -M$ for all $t \ge t_0$.*

Proof. Since $\lim_{t\to\infty} \tau^{-1}(t) = \infty$, using (2.1) and (2.2), we select $t_* \ge 0$ such that

$$\sum_{i=1}^\infty \int_{\tau^{-(2i-1)}(t_*)}^{\tau^{-2i}(t_*)} \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \int_a^b K(s_n,\xi)\, \mathrm{d}\mu(\xi)\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{1}{6},$$

$$\sum_{i=1}^\infty \int_{\tau^{-(2i-1)}(t_*)}^{\tau^{-2i}(t_*)} \frac{1}{r_1(s_1)} \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty |f(s_n)|\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{M}{2},$$

Then we select $t_0 \ge t_*$, such that $t_* \le \tau(t)$, and $t_* \le \delta(t,\xi))$ for all $t \ge t_0$ and all $\xi \in [a,b]$.

First we assume that $p(t) = 1$, we look for positive solutions in the set

$$B_{11} = \{x : M \le x(t) \le 3M \text{ for } t \ge t_0\}\,.$$

Define the operator

$$G_{11}[x](t) = \begin{cases} G_{11}[x](t_0) & \text{if } t < t_0, \\ 2\,M - \sum_{i=1}^{\infty} \int_{\tau^{-(2i-1)}(t)}^{\tau^{-2i}(t)} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} & \\ \times \int_{s_{n-1}}^{\infty} \left[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\, \mathrm{d}\mu(\xi) - f(s_n)\right] \mathrm{d}s_n \cdots \mathrm{d}s_1 & \text{if } t \ge t_0\,. \end{cases}$$

The assumption $\tau(t) < t$ guarantees the continuity of $G_{11}[x](t)$. If there is an open interval (t_3, t_4) such that $\tau(t_3) = t_3$ and $\tau(t) \ne t$ on this interval, then $G_{11}[x]$ can not be continuous at t_3. This detail was overlooked in [11, p. 11]

Clearly for all $x \in B_{11}$, $2M - M \le G_{11}[x](t) \le 2M + M$. Next we show that $G_{11}(B_{11})$ is a collection of equi-continuous functions.

For each $\epsilon > 0$ and $t_1 \ge t_0$, we need to find a $\delta > 0$ (independent of $x \in B_{11}$) such that $|G_{11}[x](t) - G_{11}[x](t_1)| < \epsilon$ for all t for which $|t - t_1| < \delta$.

From (2.1) and (2.2), select a $t_2 \ge t_0$, such that

$$\begin{aligned} &\sum_{i=1}^{\infty} \int_{\tau^{-(2i-1)}(t_2)}^{\tau^{-2i}(t_2)} \frac{1}{r_1(s_1)} \cdots \int_a^b 3MK(s_n,\xi)\, \mathrm{d}\mu(\xi) + |f(s_n)|\, \mathrm{d}s_n \cdots \mathrm{d}s_1 \\ &\le \int_{\tau^{-1}(t_2)}^{\infty} \frac{1}{r_1(s_1)} \cdots \int_a^b 3\,M\,K(s_n,\xi)\, \mathrm{d}\mu(\xi) + |f(s_n|\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{\epsilon}{4}\,. \end{aligned}$$

Then $|G_{11}[x](t) - G_{11}[x](t_1)| < \epsilon/2$ for all $t_1, t \ge t_2$ and all $x \in B_{11}$.

Since $\lim_{i\to\infty} \tau^{-i}(t_1) = \infty$, there exists an integer N such that $t_2 \le \tau^{-2N}(t_1)$. Let

$$\widetilde{F} = \sup_{t_0 \le s_1 \le t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \int_a^b 3MK(s_n,\xi)\, \mathrm{d}\mu(\xi) + |f(s_n)|\, \mathrm{d}s_n \cdots \mathrm{d}s_2\,.$$

For t such that $t_1 \le t \le \tau^{-1}(t_1)$, we have $\tau^{-1}(t_1) \le \tau^{-1}(t) \le \tau^{-2}(t_1)$ and

$$|G_{11}[x](t) - G_{11}[x](t_1)| \le \sum_{i=1}^{N} \int_{\tau^{-(2i-1)}(t_1)}^{\tau^{-(2i-1)}(t)} \widetilde{F} + \frac{\epsilon}{2}\,.$$

Using the continuity of τ^{-i} there exists $\delta_i > 0$ such that $|t - t_1| < \delta_i$ implies

$$\int_{\tau^{-(2i-1)}(t_1)}^{\tau^{-(2i-1)}(t)} \widetilde{F} < \frac{\epsilon}{2N}\,.$$

For the case $t < t_1$, we just reverse the roles of t and t_1. By choosing $\delta = \min\{\delta_i : 1 \le i \le N\}$, we have the desired equi-continuity. Next by the Arzela-Ascoli theorem, the set $G_{11}(B_{11})$ is compact. By the Schauder theorem there is a fixed point x in B_{11}. For this fixed point we have

$$\begin{aligned} x(t) + x(\tau(t)) &= G_{11}[x](t) + G_{11}[x](\tau(t)) \\ &= 4M - \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \\ &\quad \times \int_{s_{n-1}}^{\infty} \Big[\int_a^b g(\cdot,\xi,x(\delta(\cdot,\xi)))\, \mathrm{d}\mu(\xi) - f(s_n)\Big]\, \mathrm{d}s_n \cdots \mathrm{d}s_1 \end{aligned}$$

Applying the operator L_m in both sides of this equation, we show that x is a solution to (1.1) with $p(t) = 1$.

To show the existence of a negative solution, we use the set

$$B_{12} = \{x : -3M \le x(t) \le -M \text{ for } t \ge t_0\},$$

and the operator

$$G_{12}[x](t) = \begin{cases} G_{12}[x](t_0) & \text{if } t < t_0, \\ -2M - \sum_{i=1}^{\infty} \int_{\tau^{-(2i-1)}(t)}^{\tau^{-2i}(t)} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} & \\ \times \int_{s_{n-1}}^{\infty} \left[\int_a^b g(\cdot, \xi, x(\delta(\cdot,\xi)))\, \mathrm{d}\mu(\xi) - f(s_n) \right] \mathrm{d}s_n \cdots \mathrm{d}s_1 & \text{if } t \ge t_0 . \end{cases}$$

Next we set $p(t) = -1$. Note that if $t \ge t_1$, then $\int_{\tau^{-i}(t)}^{\infty} \le \int_{\tau^{-i}(t_1)}^{\infty}$, as long as the integrand is non-negative. If there is time $t_2 > 0$ such that $\tau(t_2) = t_2$, then $\tau^{-i}(t_2) \le t_2$ for all i. In this case it is impossible to satisfy assumptions (2.8) and (2.9), and $G_{13}[x](t_2) = \infty$, as defined below. To avid this difficulty, we assume that $\tau(t) < t$ for all t. This detail was overlooked in [11, p. 11].

Since the series in (2.8) and (2.9) converge, there is an integer N such that

$$\sum_{i=N}^{\infty} \int_{\tau^{-i}(0)}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} \int_a^b K(s_n,\xi)\, \mathrm{d}\mu(\xi)\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{1}{6},$$

$$\sum_{i=N}^{\infty} \int_{\tau^{-i)}(0)}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} |f(s_n)|\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{M}{2},$$

Then we select $t_0 \ge t_*$, such that $t_* \le \tau(t)$, and $t_* \le \delta(t,\xi))$ for all $t \ge t_0$ and all $\xi \in [a,b]$. We will look for positive solutions in the set

$$B_{13} = \{x : M \le x(t) \le 3M \text{ for } t \ge t_0\}.$$

Define the operator

$$G_{13}[x](t) = \begin{cases} G_{13}[x](t_0) & \text{if } t < t_0, \\ 2M - \sum_{i=1}^{\infty} \int_{\tau^{-i}(t)}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} & \\ \times \int_{s_{n-1}}^{\infty} \left[\int_a^b g(\cdot, \xi, x(\delta(\cdot,\xi)))\, \mathrm{d}\mu(\xi) - f(s_n) \right] \mathrm{d}s_n \cdots \mathrm{d}s_1 & \text{if } t \ge t_0 . \end{cases}$$

Clearly for all $x \in B_{13}$, $2M - M \le G_{13}[x](t) \le 2M + M$. Next we show that $G_{13}(B_{13})$ is a collection of equi-continuous functions.

From (2.8) and (2.9), select an integer N such that

$$\sum_{i=N}^{\infty} \int_{\tau^{-i}(0)}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \int_a^b 3MK(s_n,\xi)\, \mathrm{d}\mu(\xi) + |f(s_n)|\, \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{\epsilon}{2}$$

Then for $t \ge t_1 \ge 0$,

$$|G_{13}[x](t) - G_{13}[x](t_1)| \le \sum_{i=1}^{N} \int_{\tau^{-i}(t_1)}^{\tau^{-i}(t)} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \mathrm{d}s_n \cdots \mathrm{d}s_1 + \frac{\epsilon}{2}$$

Using the continuity of τ^{-i} there exists $\delta_i > 0$ such that $|t - t_1| < \delta_i$ implies

$$\int_{\tau^{-i}(t_1)}^{\tau^{-i}(t)} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \mathrm{d}s_n \cdots \mathrm{d}s_1 < \frac{\epsilon}{2N}\,.$$

For the case $t < t_1$, we just reverse the roles of t and t_1. By choosing $\delta = \min\{\delta_i : 1 \le i \le N\}$, we have the equi-continuity of $G_{13}(B_{13})$. Next by the Arzela-Ascoli theorem, the set $G_{13}(B_{13})$ is compact. By the Schauder theorem there is a fixed point x in B_{13}. For this fixed point we have

$$\begin{aligned} x(t) - x(\tau(t)) &= G_{13}[x](t) - G_{13}[x](\tau(t)) \\ &= -\int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} \\ &\quad \times \int_{s_{n-1}}^{\infty} \Big[\int_a^b g(\cdot, \xi, x(\delta(\cdot,\xi)))\, \mathrm{d}\mu(\xi) - f(s_n) \Big]\ \mathrm{d}s_n \cdots \mathrm{d}s_1 \end{aligned}$$

Applying the operator L_m in both sides of this equation, we show that x is a solution to (1.1) with $p(t) = -1$.

To show the existence of a negative solution, we use the set

$$B_{14} = \{x : -3M \le x(t) \le -M \text{ for } t \ge t_0\}\,,$$

and the operator

$$G_{13}[x](t) = \begin{cases} G_{13}[x](t_0) & \text{if } t < t_0, \\ -2M - \sum_{i=1}^{\infty} \int_{\tau^{-i}(t)}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})} & \\ \times \int_{s_{n-1}}^{\infty} \left[\int_a^b g(\cdot, \xi, x(\delta(\cdot,\xi)))\, \mathrm{d}\mu(\xi) - f(s_n) \right]\ \mathrm{d}s_n \cdots \mathrm{d}s_1 & \text{if } t \ge t_0\,. \end{cases}$$

This completes the proof. □

3 Dynamic and difference equations

In this section we extend the previous results to equations on times scales, and to a discrete version of (1.1). First, we find non-oscillatory solutions to the dynamic equation

$$\begin{aligned} &\Big(r_{m-1}(t)\Big(\cdots\Big(r_1(t)\big(x(t) + p(t)\, x(\tau(t))\big)^{\Delta}\Big)^{\Delta} \cdots\Big)^{\Delta}\Big)^{\Delta} \\ &+ \int_a^b g(t, \xi, x(\delta(t,\xi)\, \Delta\xi = f(t)\,, \end{aligned} \tag{3.1}$$

on a time scale $\mathbb{T}$. This is, the variable t belongs to a non-empty closed subset $\mathbb{T}$ of real numbers. We assume that $\sup \mathbb{T} = \infty$, and that if $\mathbb{T}$ is right dense at a point t, then $\mathbb{T}$ is also right dense at the points $\tau(t)$ and $\delta(t,\xi)$ for all $\xi \in [a,b]$. Derivatives are understood as Hilger derivatives (also called Δ derivatives); see the book by Bohner [2] for information about time scales.

We assume that the following functions are rd-continuous on their domains. $f : \mathbb{T} \to \mathbb{R}$, $g : \mathbb{T} \times [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p : \mathbb{T} \to \mathbb{R}$, $r_i : \mathbb{T} \to \mathbb{R}$, $\tau : \mathbb{T} \to \mathbb{T}$ and $\delta : \mathbb{T} \times [a,b] \times \mathbb{R} \to \mathbb{R}$. Also we assume r_i is positive and has $m - i$ derivatives.

By a solution we mean a function x, from $\mathbb{T}$ to $\mathbb{R}$, that satisfies (3.1) for all t in $[t_0, \infty) \cap \mathbb{T}$. To state our results, assumptions (H1)–(H4) need some modifications:

(H1') The delay $\tau(t)$ is m times differentiable and $\lim_{t\to\infty}\tau(t)=\infty$.

(H2') The delay δ is rd-continuous, and satisfies

$$\lim_{t\to\infty}\min_{a\le\xi\le b}\delta(t,\xi)=\infty.$$

(H3') The nonlinearity g satisfies $g(t,\xi,0)=0$ and the Lipschitz condition

$$|g(t,\xi,x)-g(t,\xi,y)|\le K(t,\xi)|x-y|,$$

for all $t\ge 0$, all $\xi\in[a,b]$, and all x,y in some interval $[-M_0,M_0]$. Furthermore, we assume that

$$\int_0^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})}\int_{s_{n-1}}^\infty \int_a^b K(s_n,\xi)\,\Delta\xi\,\Delta s_n\cdots\Delta s_1<\infty. \tag{3.2}$$

(H4') The right-hand side of (3.1) satisfies

$$\int_0^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \cdots \frac{1}{r_{s_{n-1}}(s_{n-1})}\int_{s_{n-1}}^\infty |f(s_n)|\,\Delta s_n\cdots\Delta s_1<\infty\,. \tag{3.3}$$

The statements in Theorems 2.1 and 2.2 hold with obvious changes in notation. For Theorems 2.3 and 2.4, we need the additional assumption that if $\mathbb{T}$ is right dense at a point t, then $\mathbb{T}$ is right dense at $\tau^{-1}(t)$. Other than this, translating the previous results to time scales is a straight forward process.

Next, we find non-oscillatory sequences that satisfy a discrete version of (1.1). Functions of t are replaced by sequences with index n: $x(t)$ by x_n, $p(t)$ by p_n, and so on. Let Δ be the forward difference operator

$$\Delta x_n = x_{n+1}-x_n\,.$$

We consider the m-order difference equation

$$\Delta\Big(r_{m-1,n}\Delta\Big(\cdots\Delta\Big(r_{1,n}\Delta\big(x_n+p_n x_{\tau(n)}\big)\Big)\cdots\Big)\Big)+\sum_{\xi=a}^{b} g(n,\xi,x_{\delta(n,\xi)}=f_n\,. \tag{3.4}$$

where a,b,m,n,ξ are non-negative integers, f,g,p,x real-valued sequences. The delays are integer-valued functions: $\tau:\mathbb{Z}^+\to\mathbb{Z}$, $\delta:\mathbb{Z}^+\times\mathbb{Z}\to\mathbb{Z}$ that satisfy the assumptions stated below. The coefficients $r_{i,n}$ are positive sequences, and the operator Δ refers to the variable n. Note that there are no differentiability conditions, and that assumptions (H1)–(H4) need some modifications:

(H1") The delay τ satisfies $\lim_{n\to\infty}\tau(n)=\infty$.

(H2") The delay δ satisfies $\lim_{n\to\infty}\min_{a\le\xi\le b}\delta(n,\xi)=\infty$.

(H3") The non linearity g satisfies $g(n,\xi,0)=0$ and the Lipschitz condition

$$|g(n,\xi,x)-g(n,\xi,y)|\le K(n,\xi)|x-y|,$$

for all $n\ge 0$, all $\xi\in[a,b]$, and all x,y in some interval $[-M_0,M_0]$. Furthermore, we assume that

$$\sum_{s_1=0}^\infty \frac{1}{r_{1,s_1}}\sum_{s_2=s_1}^\infty \cdots \frac{1}{r_{m-1,s_{m-1}}}\sum_{s_m=s_{m-1}}^\infty \sum_{\xi=a}^{b} K(s_m,\xi)<\infty. \tag{3.5}$$

(H4”) The right-hand side of (3.4) satisfies

$$\sum_{s_1=0}^{\infty} \frac{1}{r_{1,s_1}} \sum_{s_2=s_1}^{\infty} \cdots \frac{1}{r_{m-1,s_{m-1}}} \sum_{s_m=s_{m-1}}^{\infty} |f(s_m)| < \infty \,. \tag{3.6}$$

Translating the results in Theorems 2.1–2.4 to sequences satisfying (3.4) is straight forward process.

Concluding remarks

We found non-oscillatory solutions for various ranges of the coefficient $p(t)$, in particular when it oscillates about zero. However, we are unable to obtain the same results when $p(t)$ oscillates about ± 1. This remains an open question.

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