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ASYMPTOTIC STABILITY IN NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

SAÏD ABBAS*
Laboratoire de Mathématiques, Université de Saïda;
B.P. 138, 20000, Saïda, Algérie

MOUFFAK BENCHOHRA†
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès;
B.P. 89, 22000, Sidi Bel-Abbès, Algérie

GASTON M. N'GUÉRÉKATA‡
Department of Mathematics, Morgan State University
1700 E. Cold Spring Lane, Baltimore M.D. 21252, USA

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Abstract. Our aim in this work is to study the existence and the local stability of solutions for a system of nonlinear delay partial differential equations of fractional order. We use the Schauder fixed point theorem for the existence of solutions, and we prove that all solutions are locally asymptotically stable.

Keywords: Delay differential equation; left-sided mixed Riemann-Liouville integral of fractional order; Caputo fractional-order derivative; stability; solution; fixed point.

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*e-mail address: abbasmsaid@yahoo.fr
†e-mail address: benchohra@univ-sba.dz
‡e-mail address: Gaston.N'Guerekata@morgan.edu

1 Introduction

The subject of fractional calculus is as old as the differential calculus and it has been developed up to nowadays (see Kilbas *et al.* [12], Hilfer [11]). Fractional differential and integral equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering and so on. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [4], Baleanu *et al.* [8], Hilfer [11], Kilbas *et al.* [12], Lakshmikantham *et al.* [13], Podlubny [14], and Tarasov [19], and the papers by Abbas *et al.* [1, 2, 3, 5], Agarwal *et al.* [6], Cuevas *et al.* [9], Qian *et al.* [15, 16, 17], Tenreiro Machado *et al.* [20], Vityuk and Golushkov [21].

In [16], Qian studied the global attractivity of solutions of the following nonlinear delay differential equation

$$x'(t) = p(t)x(t)[f(x(t)) - g(x(t-\tau))];\ t \geq 0, \tag{1.1}$$

where $\tau \in (0,\infty)$ and $p, f, g : [0,\infty) \to [0,\infty)$ are given continuous functions. Notice that equation (1.1) has applications in population dynamics, dynamics of price, production, and consumption of a particular commodity.

Motivated by [16], in this paper we establish sufficient conditions for the existence and the stability of solutions of the following system of nonlinear delay differential equations of fractional order of the form

$$^{c}D^{r}_{\theta}u(t,x) = p(t,x)\,u(t,x)\Big[f(t,x,u(t,x)) - g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m))\Big],$$
$$\text{for } (t,x) \in J := \mathbb{R}_+ \times [0,b], \tag{1.2}$$

$$u(t,x) = \Phi(t,x), \qquad \text{for } (t,x) \in \tilde{J} := [-T,\infty) \times [-\xi,b] \setminus (0,\infty) \times (0,b], \tag{1.3}$$

$$\begin{cases} u(t,0) = \varphi(t); & t \in [0,\infty), \\ u(0,x) = \psi(x); & x \in [0,b]. \end{cases} \tag{1.4}$$

where $b > 0,\ \theta = (0,0),\ \mathbb{R}_+ = [0,\infty),\ \tau_i, \xi_i \geq 0;\ i = 1\dots,m,\ T = \max\limits_{i=1\dots,m}\{\tau_i\},\ \xi = \max\limits_{i=1\dots,m}\{\xi_i\},\ ^{c}D^{r}_{\theta}$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1],\ p : J \to \mathbb{R},\ f : J \times \mathbb{R} \to \mathbb{R},\ g : J \times \mathbb{R}^m \to \mathbb{R}$ are given continuous functions, $\varphi : \mathbb{R}_+ \to \mathbb{R},\ \psi : [0,b] \to \mathbb{R}$ are absolutely continuous functions with $\lim_{t\to\infty}\varphi(t) = 0$, and $\psi(x) = \varphi(0)$ for each $x \in [0,b]$, and $\Phi : \tilde{J} \to \mathbb{R}^n$ is continuous with $\varphi(t) = \Phi(t,0)$ for each $t \in \mathbb{R}_+$, and $\psi(x) = \Phi(0,x)$ for each $x \in [0,b]$.

This paper initiates the concept of local attractivity of solutions of problem (1.2)-(1.4).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,a] \times [0,b]);\ a,b > 0$, we denote the space of Lebesgue-integrable functions $u : [0,a] \times [0,b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(t,x)|\,\mathrm{d}x\,\mathrm{d}t.$$

By $BC := BC([-T,\infty)\times[-\xi,b])$ we denote the Banach space of all bounded and continuous functions from $[-T,\infty)\times[-\xi,b]$ into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x)\in[-T,\infty)\times[-\xi,b]} |u(t,x)|.$$

For $u_0 \in BC$ and $\eta \in (0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1 *[21] Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1([0,a]\times[0,b])$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}u(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x) = \int_0^t\int_0^x u(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau;$$

for almost all $(t,x)\in[0,a]\times[0,b]$, where $\sigma=(1,1)$. For instance, $I_\theta^r u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1([0,a]\times[0,b])$. Note also that when $u\in C([0,a]\times[0,b])$, then $(I_\theta^r u)\in C([0,a]\times[0,b])$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t\in[0,a],\ x\in[0,b].$$

Example 2.2 *Let $\lambda,\omega\in(-1,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then*

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2},\ \textit{for almost all}\ (t,x)\in[0,a]\times[0,b].$$

By $1-r$ we mean $(1-r_1,1-r_2)\in(0,1]\times(0,1]$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t\partial x}$, the mixed second order partial derivative.

Definition 2.3 *[21] Let $r\in(0,1]\times(0,1]$ and $u\in L^1([0,a]\times[0,b])$. The Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(t,x) = (I_\theta^{1-r}D_{tx}^2 u)(t,x)$.*

The case $\sigma=(1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x),\ \text{for almost all}\ (t,x)\in[0,a]\times[0,b].$$

Example 2.4 *Let $\lambda,\omega\in(-1,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then*

$${}^cD_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)}t^{\lambda-r_1}x^{\omega-r_2},$$

for almost all $(t,x)\in[0,a]\times[0,b]$.

Let $\emptyset\neq\Omega\subset BC$, and let $G:\Omega\to\Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{2.1}$$

We introduce the following concept of attractivity of solutions for equation (2.1).

Definition 2.5 *Solutions of equation (2.1) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equations (2.1) belonging to $B(u_0, \eta) \cap \Omega$, we have that, for each $x \in [0,b]$,*

$$\lim_{t\to\infty} (v(t,x) - w(t,x)) = 0. \tag{2.2}$$

When the limit (2.2) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (2.1) are said to be uniformly locally attractive (or equivalently that solutions of (2.1) are locally asymptotically stable).

Lemma 2.6 *([7], p. 62) Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*
(a) D is uniformly bounded in BC,
(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+ \times [0,b]$,
i.e. equicontinuous on every compact of $\mathbb{R}_+ \times [0,b]$,
(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$, $x \in [0,b]$ there corresponds $T(\epsilon, x) > 0$ such that $|u(t,x) - \lim_{t\to\infty} u(t,x)| < \epsilon$ for any $t \geq T(\epsilon, x)$ and $u \in D$.

3 Main Results

Let us start by defining what we mean by a solution of of the problem (1.2)-(1.4).

Definition 3.1 *A function $u \in BC$ is said to be a solution of (1.2)-(1.4) if u satisfies equation (1.2) on J, equation (1.3) on $\tilde{J}$ and condition (1.4) is satisfied.*

Lemma 3.2 *([1]) Let $f \in L^1([0,a] \times [0,b])$; $a, b > 0$. A function $u \in AC([0,a] \times [0,b])$ is a solution of problem*

$$\begin{cases} ({}^cD_\theta^r u)(t,x) = f(t,x); \ (t,x) \in [0,a] \times [0,b], \\ u(t,0) = \varphi(t); \quad t \in [0,a], \\ u(0,x) = \psi(x); \quad x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(t,x)$ satisfies

$$u(t,x) = \mu(t,x) + (I_\theta^r f)(t,x); \ (t,x) \in [0,a] \times [0,b],$$

where

$$\mu(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

The following hypotheses will be used in the sequel:

(H_1) The function Φ is in BC and p is continuous and bounded on J.

(H_2) There exist a nondecreasing function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous function $d : \mathbb{R}_+ \times [0,b] \to \mathbb{R}_+$ such that

$$|f(t,x,u)| \leq d(t,x)\Lambda(|u|), \text{ for } (t,x) \in J \text{ and } u \in \mathbb{R}.$$

Moreover, assume that

$$\lim_{t\to\infty} I_\theta^r d(t,x) = 0;\ x \in [0,b],$$

(H_3) There exist continuous functions $q_i : \mathbb{R}_+ \times [0,b] \to \mathbb{R}_+$ such that

$$\left(1 + \sum_{i=1}^{m} |u_i|\right) |g(t,x,u_1,u_2,...,u_m)| \leq \sum_{i=1}^{m} |u_i| q_i(t,x);$$

for $(t,x) \in \mathbb{R}_+ \times [0,b]$ and for $u_i \in \mathbb{R};\ i = 1 \ldots m$. Moreover, assume that

$$\lim_{t\to\infty} I_\theta^r q_i(t,x) = 0;\ x \in [0,b];\ i = 1 \ldots m.$$

Remark 3.3 *Set*

$$\Phi^* := \sup_{(t,x)\in\tilde{J}} \Phi(t,x),\ \varphi^* := \sup_{t\in\mathbb{R}_+} \varphi(t),\ p^* := \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} p(t,x),$$

$$d^* := \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} I_\theta^r d(t,x) \ \textit{and}\ q_i^* := \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} I_\theta^r q_i(t,x);\ i = 1 \ldots m.$$

From the hypotheses, we infer that $\Phi^*,\ \varphi^*,\ p^*,\ d^*$ *and* $q_i^*;\ i = 1 \ldots m$ *are finite.*

Now, we shall prove the following theorem concerning the existence and the stability of a solution of problem (1.2)-(1.4).

Theorem 3.4 *Assume that* $(H_1) - (H_3)$ *and the following hypothesis hold*

(H_4) *There exists a constant* $\eta > 0$, *such that*

$$\max\left\{\Phi^*, \varphi^* + p^*\eta\left(d^*\Lambda(\eta) + \sum_{i=1}^{m} q_i^*\right)\right\} \leq \eta.$$

Then the problem (1.2)-(1.4) has at least one solution in the space BC. *Moreover, if there exists a constant* $\eta^* > 0$, *such that*

$$p^* d^* \Lambda(\eta + \eta^*) + p^* \sum_{i=1}^{m} q_i^* < 1,$$

and

$$2p^*\eta\left(d^*\Lambda(\eta+\eta^*) + \sum_{i=1}^{m} q_i^*\right) \leq \eta^*\left(1 - p^* d^* \Lambda(\eta+\eta^*) - p^* \sum_{i=1}^{m} q_i^*\right),$$

then solutions of problem (1.2)-(1.4) are locally asymptotically stable.

Proof. Let us define the operator N such that, for any $u \in BC$,

$$(Nu)(t,x) = \begin{cases} \Phi(t,x); & (t,x) \in \tilde{J}, \\ \varphi(t) + I_\theta^r \left[p(t,x)u(t,x)\left(f(t,x,u(t,x))\right.\right. & \\ \left.\left.-g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m))\right)\right]; & (t,x) \in J. \end{cases} \tag{3.1}$$

The operator N maps BC into BC. Indeed the map $N(u)$ is continuous on $[-T,\infty) \times [-\xi, b]$ for any $u \in BC$, and for each $(t,x) \in J$ we have

$$\begin{aligned} |(Nu)(t,x)| &\leq |\varphi(t)| + I_\theta^r |p(t,x)u(t,x)(f(t,x,u(t,x)) \\ &\quad - g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m)))| \\ &\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_1} \\ &\qquad \times |p(t,x)|\,|u(t,x)| \Big[|d(t,x)|\,\Lambda(|u(t,x)|) \\ &\quad + \left(\sum_{i=1}^m |u(\tau-\tau_i, s-\xi_i)|\, q_i(\tau,s)\right) \\ &\qquad \times \left(1 + \sum_{i=1}^m |u(\tau-\tau_i,s-\xi_i)|\right)^{-1}\Big] \mathrm{d}s\,\mathrm{d}\tau \\ &\leq \varphi^* + p^* \|u\|_{BC} \left(d^* \Lambda(\|u\|_{BC}) + \sum_{i=1}^m q_i^*\right), \end{aligned}$$

and for $(t,x) \in \tilde{J}$, we have

$$|(Nu)(t,x)| = |\Phi(t,x)| \leq \Phi^*.$$

Thus,

$$\|N(u)\|_{BC} \leq \max\left\{\Phi^*, \varphi^* + p^*\|u\|_{BC}\left(d^*\Lambda(\|u\|_{BC}) + \sum_{i=1}^m q_i^*\right)\right\}. \tag{3.2}$$

Hence, $N(u) \in BC$. This proves that the operator N maps BC into itself.
By Lemma 3.2, the problem of finding the solutions of the problem (1.2)-(1.4) is reduced to finding the solutions of the operator equation $N(u) = u$. Equation (3.2) and hypothesis (H_4) implies that N transforms the ball $B_\eta := B(0,\eta)$ into itself. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [10]. The proof will be given in several steps.

Step 1: *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in [-T,\infty) \times [-\xi, b]$, we have

$$\begin{aligned} &|(Nu_n)(t,x) - (Nu)(t,x)| \\ &\quad \leq I_\theta^r \Big[|p(t,x)|\, \Big|u_n(t,x)\big(f(t,x,u_n(t,x)) \\ &\qquad\quad - g(t,x,u_n(t-\tau_1,x-\xi_1),...,u_n(t-\tau_m,x-\xi_m))\big) \\ &\qquad - u(t,x)\Big(f(t,x,u(t,x)) - g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m))\Big)\Big|\Big] \\ &\quad \leq I_\theta^r \big[|p(t,x)|\,|u_n(t,x) - u(t,x)|\,|f(t,x,u_n(t,x)) \\ &\qquad - g(t,x,u_n(t-\tau_1,x-\xi_1),...,u_n(t-\tau_m,x-\xi_m))|\big] \end{aligned}$$

$$
\begin{aligned}
& + I_\theta^r \big[|p(t,x)\, u(t,x)| \, \big| f(t,x,u_n(t,x)) - f(t,x,u(t,x)) \\
& - g(t,x,u_n(t-\tau_1,x-\xi_1),...,u_n(t-\tau_m,x-\xi_m)) \\
& + g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m))\big|\big] \\
\leq\ & p^* I_\theta^r \big[|u_n(t,x) - u(t,x)| \, |f(t,x,u_n(t,x)) \\
& - g(t,x,u_n(t-\tau_1,x-\xi_1),...,u_n(t-\tau_m,x-\xi_m))|\big] \\
& + p^*\eta\, I_\theta^r |f(t,x,u_n(t,x)) - f(t,x,u(t,x))| \\
& + p^*\eta\, I_\theta^r |g(t,x,u_n(t-\tau_1,x-\xi_1),...,u_n(t-\tau_m,x-\xi_m)) \\
& - g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m))| \\
\leq\ & p^* \left(d^*\Lambda(\eta) + \sum_{i=1}^m q_i^* \right) \|u_n - u\|_{BC} \\
& + p^*\, \eta\, I_\theta^r \, |f(t,x,u_n(t,x)) - f(t,x,u(t,x))| \\
& + p^*\, \eta\, I_\theta^r \, |g(t,x,u_n(t-\tau_1,x-\xi_1),...,u_n(t-\tau_m,x-\xi_m)) \\
& \qquad -g(t,x,u(t-\tau_1,x-\xi_1),...,u(t-\tau_m,x-\xi_m))| \,. \qquad (3.3)
\end{aligned}
$$

Case 1. If $(t,x) \in \tilde{J} \cup ([0,a] \times [0,b]);\ a > 0$, then, since $u_n \to u$ as $n \to \infty$ and f,g are continuous, (3.3) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

Case 2. If $(t,x) \in (a,\infty) \times [0,b];\ a > 0$, then from our hypotheses and (3.3), we get

$$
\begin{aligned}
|(Nu_n)(t,x) - (Nu)(t,x)| &\leq p^* \left(d^*\Lambda(\eta) + \sum_{i=1}^m q_i^* \right) \|u_n - u\|_{BC} \\
& +2p^*\eta\Lambda(\eta) I_\theta^r d(t,x) + 2p^*\eta \sum_{i=1}^m (I_\theta^r q_i(t,x)). \qquad (3.4)
\end{aligned}
$$

Since $u_n \to u$ as $n \to \infty$ and $t \to \infty$, then (3.4) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

Step 2: *$N(B_\eta)$ is uniformly bounded.*
This is clear since $N(B_\eta) \subset B_\eta$ and B_η is bounded.

Step 3: *$N(B_\eta)$ is equicontinuous on every compact subset $[-T,a] \times [-\xi,b]$ of $[-T,\infty) \times [-\xi,b]$, $a > 0$.*
Let $(t_1,x_1),(t_2,x_2) \in [0,a] \times [0,b]$, $t_1 < t_2,\ x_1 < x_2$ and let $u \in B_\eta$. Thus we have

$$
\begin{aligned}
& |(Nu)(t_2,x_2) - (Nu)(t_1,x_1)| \\
& \leq \ |\varphi(t_2) - \varphi(t_1)| + I_\theta^r \Big[(p(t_2,x_2)|u(t_2,x_2)| - p(t_1,x_1)|u(t_1,x_1)|) \Big(f(t_2,x_2,u(t_2,x_2)) \\
& \quad - g(t_2,x_2,u(t_2-\tau_1,x_2-\xi_1),...,u(t_2-\tau_m,x_2-\xi_m)) \Big) \Big] \\
& \quad + I_\theta^r \left[p(t_1,x_1)|u(t_1,x_1)| |f(t_2,x_2,u(t_2,x_2)) - f(t_1,x_1,u(t_1,x_1))| \right. \\
& \quad + |g(t_2,x_2,u(t_2-\tau_1,x_2-\xi_1),...,u(t_2-\tau_m,x_2-\xi_m))
\end{aligned}
$$

$$- \left. g(t_1, x_1, u(t_1 - \tau_1, x_1 - \xi_1), ..., u(t_1 - \tau_m, x_1 - \xi_m))| \Big) \right] .$$

Thus

$$\begin{aligned}
&|(Nu)(t_2, x_2) - (Nu)(t_1, x_1)| \\
&\leq \quad |\varphi(t_2) - \varphi(t_1)| + I_\theta^r \left[\eta \, |p(t_2, x_2) - p(t_1, x_1)| \left(d^* \Lambda(\eta) + \sum_{i=1}^m q_i(t_2, x_2) \right) \right] \\
&\quad + p^* \eta I_\theta^r \Big[|f(t_2, x_2, u(t_2, x_2)) - f(t_1, x_1, u(t_1, x_1))| \\
&\qquad + |g(t_2, x_2, u(t_2 - \tau_1, x_2 - \xi_1), ..., u(t_2 - \tau_m, x_2 - \xi_m)) \\
&\qquad\quad - g(t_1, x_1, u(t_1 - \tau_1, x_1 - \xi_1), ..., u(t_1 - \tau_m, x_1 - \xi_m))| \Big].
\end{aligned}$$

From continuity of φ, p, f, g and as $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2$, $x_1 \leq 0 \leq x_2$ is obvious.

Step 4: $N(B_\eta)$ *is equiconvergent.*
Let $(t, x) \in \mathbb{R}_+ \times [0, b]$ and $u \in B_\eta$, then we have

$$\begin{aligned}
|(Nu)(t, x)| \quad &\leq \quad |\varphi(t)| + \frac{p^* \eta \Lambda(\eta)}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t - \tau)^{r_1 - 1} (x - s)^{r_2 - 1} d(\tau, s) \, \mathrm{d}s \, \mathrm{d}\tau \\
&\quad + \frac{p^* \eta}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=0}^m \left(\int_0^t \int_0^x (t - \tau)^{r_1 - 1} (x - s)^{r_2 - 1} q_i(\tau, s) \, \mathrm{d}s \, \mathrm{d}\tau \right) \\
&\leq \quad |\varphi(t)| + p^* \eta \Lambda(\eta) I_\theta^r d(t, x) + p^* \eta \sum_{i=0}^m I_\theta^r p_i(t, x).
\end{aligned}$$

Thus, for each $x \in [0, b]$, we get

$$|(Nu)(t, x)| \to 0, \ \text{ as } \ t \to +\infty.$$

Hence,

$$|(Nu)(t, x) - (Nu)(+\infty, x)| \to 0, \ \text{ as } \ t \to +\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.6, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [10], we deduce that N has a fixed point u which is a solution of the problem (1.2)–(1.4).

Step 5: *The uniform local attractivity for solutions.*
Let us assume that u_0 is a solution of problem (1.2)-(1.4) with the conditions of this theorem. Taking $u \in B(u_0, \eta^*)$, we have

$$\begin{aligned}
|(Nu)(t, x) - u_0(t, x)| \quad &= \quad |(Nu)(t, x) - (Nu_0)(t, x)| \\
&\leq \quad p^* I_\theta^r \big[|u(t, x) - u_0(t, x)| \, |f(t, x, u(t, x)) \\
&\qquad - g(t, x, u(t - \tau_1, x - \xi_1), ..., u(t - \tau_m, x - \xi_m))| \big] \\
&\quad + p^* \eta \, I_\theta^r \, |f(t, x, u(t, x)) - f(t, x, u_0(t, x))| \\
&\quad + p^* \eta \, I_\theta^r \, |g(t, x, u(t - \tau_1, x - \xi_1), ..., u(t - \tau_m, x - \xi_m)) \\
&\qquad - g(t, x, u_0(t - \tau_1, x - \xi_1), ..., u_0(t - \tau_m, x - \xi_m))|
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad p^*(2\eta+\eta^*)\left(d^*\Lambda(\eta+\eta^*)+\sum_{i=1}^m q_i^*\right)\\
&\leq \quad \eta^*.
\end{aligned}
$$

Thus we observe that N is a continuous function such that

$$N(B(u_0,\eta^*))\subset B(u_0,\eta^*).$$

Moreover, if u is a solution of problem (1.2)-(1.4), then

$$
\begin{aligned}
|u(t,x)-u_0(t,x)| &= |(Nu)(t,x)-(Nu_0)(t,x)|\\
&\leq p^*\left(\Lambda(\eta+\eta^*)I_\theta^r d(t,x)+\sum_{i=1}^m (I_\theta^r q_i(t,x))\right)\|u-u_0\|_{BC}\\
&\quad + 2p^*\eta\Lambda(\eta+\eta^*)I_\theta^r d(t,x)+2p^*\eta\sum_{i=1}^m (I_\theta^r q_i(t,x)).
\end{aligned}
$$

Thus

$$|u(t,x)-u_0(t,x)|\leq p^*(2\eta+\eta^*)\left(\Lambda(\eta+\eta^*)I_\theta^r d(t,x)+\sum_{i=1}^m (I_\theta^r q_i(t,x))\right). \tag{3.5}$$

By using (3.5) and the fact that $\lim_{n\to\infty} I_\theta^r d(t,x)=\lim_{n\to\infty} I_\theta^r q_i(t,x)=0;\ i=0\dots m$ we deduce that

$$\lim_{t\to\infty}|u(t,x)-u_0(t,x)|=0.$$

Consequently, all solutions of problem (1.2)-(1.4) are locally asymptotically stable.

4 An Example

As an application of our results we consider the following system of Lotka-Volterra type model of nonlinear delay differential equations of fractional order of the form

$$
\begin{aligned}
{}^cD_\theta^r u(t,x) &= p(t,x)u(t,x)\left[f(t,x,u(t,x))-g\left(t,x,u\left(t-1,x-\frac{1}{4}\right),\right.\right.\\
&\qquad \left.\left. u\left(t-\frac{2}{3},x-\frac{1}{5}\right)\right)\right];\quad \text{for } (t,x)\in J:=\mathbb{R}_+\times[0,1], \qquad (4.1)\\
u(t,x) &= \frac{1}{1+t^2};\quad \text{for } (t,x)\in\tilde{J}:=[-1,\infty)\times[-\frac{1}{4},1]\backslash(0,\infty)\times(0,1], \qquad (4.2)
\end{aligned}
$$

$$
\begin{cases}
u(t,0)=\dfrac{1}{1+t^2}; & t\in[0,\infty),\\
u(0,x)=1; & x\in[0,1],
\end{cases} \tag{4.3}
$$

where $(r_1,r_2)=(\frac{1}{4},\frac{1}{2}),\ p(t,x)=\frac{1}{1+t^2+x^2}$,

$$
\begin{cases}
f(t,x,u)=\dfrac{3cxt^{\frac{-1}{4}}|u|\sin t}{64(1+\sqrt{t})}; & (t,x)\in(0,\infty)\times[0,1],\ u\in\mathbb{R},\\
f(0,x,u)=0; & x\in[0,1],\ u\in\mathbb{R},
\end{cases}
$$

$c = \frac{3\sqrt{\pi}}{16}$, and

$$\begin{cases} g(t,x,u,v) = \dfrac{cxt^{\frac{-1}{4}}\left(|u|\sin t + |v|e^{-\frac{1}{t}}\right)}{4(1+\sqrt{t})(2+|u|+|v|)}; & (t,x) \in (0,\infty)\times[0,1],\ u,v \in \mathbb{R}, \\ \qquad g(0,x,u,v) = 0; & x \in [0,1] \text{ and } u,v \in \mathbb{R}. \end{cases}$$

First, we can see that the hypothesis (H_1) is satisfied with $\Phi^* = \varphi^* = p^* = 1$. The function f is continuous and satisfies assumption (H_2), witch $\Lambda(w) = w$ and

$$\begin{cases} d(t,x) = \dfrac{3cxt^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})}; & (t,x) \in (0,\infty)\times[0,1], \\ \quad d(0,x) = 0; & x \in [0,1]. \end{cases}$$

The function g satisfies assumption (H_3), with $q_1(t,x) = \frac{16}{3}d(t,x)$, and

$$\begin{cases} q_2(t,x) = \dfrac{cxt^{\frac{-1}{4}}e^{-\frac{1}{t}}}{4(1+\sqrt{t})}; & (t,x) \in (0,\infty)\times[0,1], \\ \quad q_2(0,x) = 0; & x \in [0,1]. \end{cases}$$

Also, for each $x \in [0,1]$, we get

$$\begin{aligned} I_\theta^r q_1(t,x) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}q_1(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &\leq \frac{c}{4(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}|\sin\tau|\,\mathrm{d}s\,\mathrm{d}\tau \\ &\leq \frac{c}{4(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}\,\mathrm{d}s\,\mathrm{d}\tau \\ &= \frac{1}{4}\frac{t^{\frac{-1}{2}}x^{\frac{3}{2}}}{(1+t^{-\frac{1}{2}})} \\ &\leq \frac{1}{4}t^{\frac{-1}{2}}x^{\frac{3}{2}} \longrightarrow 0 \text{ as } t\to\infty, \end{aligned}$$

and

$$\begin{aligned} I_\theta^r q_2(t,x) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}q_2(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &\leq \frac{c}{4(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}\int_0^t\int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}}s\tau^{\frac{-3}{4}}e^{-\frac{1}{\tau}}\,\mathrm{d}s\,\mathrm{d}\tau \\ &\leq \frac{16c}{12\sqrt{\pi}}\frac{t^{\frac{-1}{2}}x^{\frac{3}{2}}}{(1+t^{-\frac{1}{2}})} \\ &\leq \frac{1}{4}t^{\frac{-1}{2}}x^{\frac{3}{2}} \longrightarrow 0 \text{ as } t\to\infty. \end{aligned}$$

Thus

$$\lim_{t\to\infty} I_\theta^r d(t,x) = \lim_{t\to\infty} I_\theta^r q_i(t,x) = 0;\ x\in[0,1];\ i = 0,1,2,$$

and $d^* = \frac{3}{64},\ q_1^* = q_2^* = \frac{1}{4}$. Finally, we can see that the hypothesis (H_4) is satisfied. Indeed,

$$\varphi^* + p^*\eta\left(d^*\Lambda(\eta) + \sum_{i=1}^{m} q_i^*\right) = 1 + \frac{1}{2}\eta + \frac{3}{64}\eta^2.$$

So,

$$\varphi^* + p^*\eta\left(d^*\Lambda(\eta) + \sum_{i=1}^{m} q_i^*\right) \le \eta,$$

implies that

$$1 - \frac{1}{2}\eta + \frac{3}{64}\eta^2 \le 0,$$

which is satisfies for $\eta = 3$. Thus,

$$\max\left\{\Phi^*, \varphi^* + p^*\eta\left(d^*\Lambda(\eta) + \sum_{i=1}^{m} q_i^*\right)\right\} \le \eta.$$

Also,

$$p^*d^*\Lambda(\eta+\eta^*) + p^*\sum_{i=1}^{m} q_i^* < 1,$$

implies that $\eta^* < \frac{23}{3}$. We can take $\eta^* = 5$, because the inequality

$$2p^*\eta\left(d^*\Lambda(\eta+\eta^*) + \sum_{i=1}^{m} q_i^*\right) \le \eta^*\left(1 - p^*d^*\Lambda(\eta+\eta^*) - p^*\sum_{i=1}^{m} q_i^*\right),$$

is satisfies with $\eta = 3$ and $\eta^* = 5$. This inequality gives $\frac{21}{4} \le \frac{45}{8}$. Hence by Theorem 3.4, the problem (4.1)-(4.3) has a solution defined on $[-1,\infty) \times [-\frac{1}{4}, 1]$ and solutions of this problem are locally asymptotically stable.

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