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GALERKIN METHOD FOR THE BOUSSINESQ EQUATION WITH INTEGRAL CONDITION

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Abstract. In this article, the Galerkin method is proposed for solving a Boussinesq type equation with an integral condition. We construct a discrete numerical solution of the approximate problem. Then the convergence of the method and the well posedness of the problem under study are established.

Keywords: Boussinesq equation, Integral conditions, Approximate solution, Galerkin method.

2010 Mathematics Subject Classification: 35L05, 35L20, 35L99.

1 Introduction

The aim of this paper is the investigation of a non-local problem generated by a Boussinesq equation and an integral condition. Boussinesq equation is a nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata. Recent developments in numerical schemes for solving Boussinesq-type equation have placed immense interest in nonlinear dispersive wave models. Various Boussinesq type equations can describe varying degrees of accuracy in representing nonlinearity and dispersion. Boussinesq type equations are conventionally associated with relatively shallow water. The present work deals with the application of the Galerkin method to determine a function $u = u(x,t)$, that satisfies the Boussinesq equation for all $(x,t) \in Q = \Omega \times I$

$$lu = u_{tt} + (b(x,t)u_x)_x - \beta \Delta u_{tt} = f(x,t), \tag{1.1}$$

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subject to the initial conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \tag{1.2}$$

and the integral condition of second kind

$$\forall x \in \partial\Omega : \frac{\partial u}{\partial \eta}|_{(x,t)\in\partial\Omega\times I} + \int_\Omega k(x,\xi)u(\xi,t)\,\mathrm{d}\xi = 0. \tag{1.3}$$

where $x \in \Omega$ a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $t \in I = (0,T)$, $\varphi(x)$, $\psi(x)$, $k(x,\xi)$ are given functions and $b(x,t)$ is a nonnegative continuous function such that $|b_t(x,t)| < b_2$, $b_1 < b(x,t) < b_0$, for all $(x,t) \in Q$.

The coefficients in (1.1) are real-valued and are physically meaningful. The Boussinesq equation (1.1) describe flow of shallow-water waves having small amplitudes. If β is negative then (1.1) describes the irrotational flow of an inviscid liquid in a uniform rectangular channel.

Boussinesq equation (1.1) jointly with integral condition (1.3) is a new posed problem. Under some assumptions on the kernel K and the function b, existence and uniqueness of the generalized solution is established by using Galerkin method. Many physical phenomena can be modeled by non-classical boundary value problems with nonlocal conditions. When the integrals appearing in boundary conditions, we speak about integrals conditions. If the integrals appearing in the equation itself we arrive at the integro-differential equations. The study of these problems is typical this is due to the importance of non local conditions appearing in the mathematical modeling of various phenomena of physics, ecology, biology,.... Non local conditions come up when values of the function on the boundary is connected to values inside the domain or when direct measurements on the boundary are not possible. It is found that problems with nonlocal conditions have many applications in many problems such as population dynamics, the process of heat conduction, control theory, etc.. In particular, the introduction of non-local conditions can improve the qualitative and quantitative characteristics of the problem which lead to good results concerning existence, uniqueness and regularity of the solution.

The presence of an integral term in a boundary condition complicates greatly the application of standard functional or numerical methods. Various type of nonlocal problem with integral conditions were studied by many authors using different methods. Guezane-Lakoud et al [12] have applied the Galerkin method to a telegraph equation with an integral boundary condition and established the existence, uniqueness of a weak solution. Bahuguna et al in [5] have studied a neutral functional differential equation with a nonlocal initial condition via the Galerkin approximation. Dabas et al in [8] have used Rothe method to establish the existence and uniqueness of a weak solution. For more results on nonlocal problems we refer to [1–3, 6, 7, 9–11, 13–21].

The paper is organized as follows. In Section 2, we define the function spaces, state some inequalities and precise sense of the desired solution. In Section 3, we established the uniqueness of the solution. Finally, Section 4, is devoted to the construction of the approximate solution and its existence via the Galerkin Method.

2 Function spaces

Let $L^2(Q)$ be the usual space of Lebesgue square integrable real functions on Q whose inner product and norm will be denoted by $(\, , \,)$ and $\| \, . \|$ respectively.

$W^{1,2}(Q)$ is the Sobolev space consisting of functions such that all derivatives lower than one belong to $L^2(Q)$ equipped with the norm

$$\|u\|^2_{W^{1,2}(Q)} = \|u\|^2 + \|\nabla u\|^2 + \|u_t\|^2.$$

We define the space $W^{1,2}_T(Q) = \{v(x,t) \in W^{1,2}(Q),\ v(x,T) = 0\}$.

Now we define the sense of generalized solution. Multiplying the equation (1.1) by the function $v \in W^{1,2}_T(Q)$, and integrating by parts the resultant equality over Q, yields

$$\begin{aligned}
&(b\nabla u, \nabla v)_{L^2(Q)} + \beta(\nabla u_t, \nabla v_t)_{L^2(Q)} + (u_t, v_t)_{L^2(Q)} \\
= \quad & -(f,v)_{L^2(Q)} - \int_0^T \int_{\partial\Omega} b(s,t)\, v(s,t) \int_\Omega k(x,\xi)\, u(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\
& -\beta \int_0^T \int_{\partial\Omega} v_t (\int_\Omega k(x,\xi)\, u_t(\xi,t)\,\mathrm{d}\xi)\,\mathrm{d}s\,\mathrm{d}t \\
& - (v(x,0), \psi(x))_{L^2(\Omega)} + \beta(v(x,0), \Delta\psi(x))_{L^2(\Omega)}
\end{aligned} \tag{2.1}$$

Definition 2.1 *By a generalized solution of problem (1.1)–(1.3) we mean a function $u \in W^{1,2}(Q)$ such that identity (2.1) holds for all $v \in W^{1,2}_T(Q)$ and $u(x,0) = \varphi(x)$.*

We give some useful inequalities:

- Gronwall inequality. Let $h(t)$ and $y(t)$ be two nonnegative integrable functions on the interval I with $h(t)$ non decreasing. If for any $t \in I$, we have

$$y(t) \le h(t) + c\int_0^t y(\tau)\,\mathrm{d}\tau$$

  where c is a positive constant, then

$$y(t) \le h(t)e^{ct}.$$

- Cauchy–Schwarz inequality. If $L^2(Q) f, g \in L^2(I)$, then

$$(\int_I f(t)\, g(t)\,\mathrm{d}t)^2 \le (\int_I |f(t)|^2\,\mathrm{d}t)(\int_I |g(t)|^2\,\mathrm{d}t).$$

- ε–Cauchy inequality. For all $\alpha, \beta \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^*_+$, we have

$$|\alpha\beta| \le \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2.$$

- Trace inequality. If $v \in W^{1,2}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, then

$$\int_{\partial\Omega} |v|^2\,\mathrm{d}s \le \int_\Omega (\varepsilon|\nabla v|^2 + c(\varepsilon)|v|^2)\,\mathrm{d}x,$$

  where $c(\varepsilon)$ is a positive constant that depends only on ε and on the domain Ω.

3 Uniqueness of the generalized solution

Now we shall show that the generalized solution of problem (1.1)–(1.3), if it exists, is unique.

Theorem 3.1 *Assume that $\varphi, \psi \in W^{1,2}(\Omega)$, $f \in L^2(Q)$, $K \in C(\Omega \times \Omega)$ such that $\max\limits_{\overline{Q}} |K| \leq k_0$ and the derivatives $\dfrac{\partial K}{\partial \xi_i}$ exist. Then the generalized solution of problem* (1.1)–(1.3), *if it exists, is unique.*

Proof. Suppose that there exist two different generalized solutions u_1 and u_2 for the problem (1.1)–(1.3). Then the difference $U = u_1 - u_2$ is a generalized solution of the problem (1.1)–(1.3) with homogeneous equation and homogeneous conditions, that is $f = \varphi = \psi = 0$. We shall prove that $U = 0$ in Q. Let $v \in W_T^{1,2}(Q)$ and denote $Q_\tau = \{(x,t); 0 < x < 1,\ 0 < t \leq \tau \leq T\}$. Consider the function

$$v(x,t) = \begin{cases} \int_t^\tau U(x,s)\,\mathrm{d}s & 0 \leq t \leq \tau, \\ 0 & \tau \leq t \leq T. \end{cases}$$

The identity (2.1) becomes

$$\begin{aligned} &(b\nabla U, \nabla v)_{L^2(Q_T)} + \beta(\nabla U_t, \nabla v_t)_{L^2(Q_T)} + (U_t, v_t)_{L^2(Q_T)} \\ = \quad & -\int_0^T \int_{\partial\Omega} b(s,t) v(s,t) \int_\Omega k(x,\xi)\, U(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\ & -\beta \int_0^T \int_\Omega v_t(\int_\Omega k(x,\xi)\, U_t(\xi,t)\,\mathrm{d}\xi)\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{3.1}$$

Substituting v into (3.1), integrating by parts, then using the fact that $v_t(x,t) = -U(x,t)$, it follows

$$\begin{aligned} &-\int_\Omega b(x,0)(\nabla v)^2(x,0)\,\mathrm{d}x + \beta \int_\Omega (\nabla U)^2(x,\tau)\,\mathrm{d}x + \int_\Omega (U)^2(x,\tau)\,\mathrm{d}x \\ = \quad & \int_0^\tau \int_\Omega b_t(\nabla v)^2\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^\tau \int_{\partial\Omega} b(s,t)\, v(s,t) \int_\Omega k(x,\xi) U(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\ & + 2\beta \int_0^\tau \int_{\partial\Omega} v_t(s,t)(\int_\Omega k(x,\xi)\, U_t(\xi,t)\,\mathrm{d}\xi)\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{3.2}$$

Using the assumption on the functions K and b we get

$$\begin{aligned} &-b_0 \int_\Omega (\nabla v)^2(x,0)\,\mathrm{d}x + \beta \int_\Omega (\nabla U)^2(x,\tau)\,\mathrm{d}x + \int_\Omega (U)^2(x,\tau)\,\mathrm{d}x \\ \leq \quad & b_2 \|\nabla v\|^2_{L^2(Q)} + 2b_0 k_0 \int_0^\tau \int_{\partial\Omega} |v(s,t)| \int_\Omega |U(\xi,t|)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\ & + 2\beta k_0 \int_0^\tau \int_{\partial\Omega} |v_t(s,t)| (\int_\Omega |U_t(\xi,t)|\,\mathrm{d}\xi) ds\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

Now, apply the Cauchy–Schwarz inequality to the two last terms in the right hand side of (3.3), use the ε–Cauchy inequality with $\varepsilon = 1$, and the trace inequality. Remarking that $\|v\|^2_{L^2(Q_\tau)} \leq$

$T^2\|U\|^2_{L^2(Q_\tau)}$, we obtain

$$\begin{aligned} -b_0\int_\Omega(\nabla v)^2(x,0)\,\mathrm{d}x+\beta\int_\Omega(\nabla U)^2(x,\tau)\,\mathrm{d}x+\int_\Omega(U)^2(x,\tau)\,\mathrm{d}x \\ \le\ (b_2+b_0k_0\varepsilon)\|\nabla v(x,t)\|^2_{L^2(Q_\tau)}+\beta k_0\varepsilon\|\nabla U\|^2_{L^2(Q_\tau)}+\beta k_0\|U_t\|^2_{L^2(Q_\tau)} \\ +(b_0k_0+b_0k_0c(\varepsilon)T^2+\beta k_0c(\varepsilon))\|U\|^2_{L^2(Q_\tau)} \end{aligned} \tag{3.4}$$

Let $\varkappa\in W^{1,2}_T(Q)$ such that

$$\varkappa(x,t)=\begin{cases}U(x,t) & 0\le t\le\tau\ ,\\ 0 & \tau\le t\le T\ .\end{cases}$$

Substituting $\varkappa$ into (3.1) then integrating by parts the resultant equality to get

$$\begin{aligned} &b_1\|\nabla U\|^2_{L^2(Q_\tau)}+\beta\|\nabla U_t\|^2_{L^2(Q_\tau)}+\|U_t\|^2_{L^2(Q_\tau)} \\ &=\ -\int_0^\tau\int_{\partial\Omega}b(s,t)\,U(s,t)\int_\Omega k(x,\xi)\,U(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\ &\quad-\beta\int_0^\tau\int_{\partial\Omega}U_t(\int_\Omega k(x,\xi)U_t(\xi,t)\,\mathrm{d}\xi)\,\mathrm{d}s\,\mathrm{d}t\ . \end{aligned}$$

Using the same argument as previously we obtain

$$\begin{aligned} b_1\|\nabla U\|^2_{L^2(Q_\tau)}+\beta\|\nabla U_t\|^2_{L^2(Q_\tau)}+\|U_t\|^2_{L^2(Q_\tau)} \\ \le\ b_0k_0\varepsilon\|\nabla U\|^2_{L^2(Q_\tau)}+b_0k_0(c(\varepsilon)+1)\|U\|^2_{L^2(Q_\tau)}+\varepsilon\beta k_0\|\nabla U_t\|^2_{L^2(Q_\tau)} \\ +\beta k_0(c(\varepsilon)+1)\|U_t\|^2_{L^2(Q_\tau)} \end{aligned} \tag{3.5}$$

Summing (3.4) and (3.5) yields

$$\begin{aligned} \int_\Omega[-b_0\nabla v^2(x,0)+\beta\nabla U^2(x,\tau)+U^2(x,\tau)]\,\mathrm{d}x \\ \le\ (b_2+b_0k_0\varepsilon)\|\nabla v(x,t)\|^2_{L^2(Q_\tau)}+(2\beta k_0\varepsilon-b_1)\|\nabla U\|^2_{L^2(Q_\tau)} \\ +(2b_0k_0+b_0k_0c(\varepsilon)T^2+b_0k_0c(\varepsilon)+\beta k_0c(\varepsilon))\|U\|^2_{L^2(Q_\tau)} \\ +(2\beta k_0-1+\beta k_0c(\varepsilon))\|U_t\|^2_{L^2(Q_\tau)}+(\varepsilon\beta k_0-\beta)\|\nabla U_t\|^2_{L^2(Q_\tau)} \end{aligned} \tag{3.6}$$

Now choosing ε such that $\varepsilon k_0\le 1$ and $\beta k_0(2+c(\varepsilon))\le 1$, inequality (3.6) becomes

$$\begin{aligned} \int_\Omega(-b_0\nabla v^2(x,0)+\beta\nabla U^2(x,\tau)+U^2(x,\tau))\,\mathrm{d}x \\ \le\ (b_2+b_0k_0\varepsilon)\|\nabla v(x,t)\|^2_{L^2(Q_\tau)}+(2\beta k_0\varepsilon-b_1)\|\nabla U\|^2_{L^2(Q_\tau)} \\ +(2b_0k_0+b_0k_0c(\varepsilon)T^2+b_0k_0c(\varepsilon)+\beta k_0c(\varepsilon))\|U\|^2_{L^2(Q_\tau)}. \end{aligned} \tag{3.7}$$

Let us denote $C_1=\max(b_2+b_0k_0\varepsilon,2\beta k_0\varepsilon-b_1,2b_0k_0+b_0k_0c(\varepsilon)T^2+b_0k_0c(\varepsilon)+\beta k_0c(\varepsilon))$, then (3.8) gives

$$\begin{aligned} \int_\Omega(-b_0\nabla v^2(x,0)+\beta\nabla U^2(x,\tau)+U^2(x,\tau))\,\mathrm{d}x \\ \le\ C_1(\|\nabla v(x,t)\|^2_{L^2(Q_\tau)}+\|\nabla U\|^2_{L^2(Q_\tau)}+\|U\|^2_{L^2(Q_\tau)})\ . \end{aligned} \tag{3.8}$$

Consider the function

$$w(x,t)=\begin{cases}-\int_0^t u(x,s)\,\mathrm{d}s & 0\le t\le\tau\,,\\ 0 & \tau\le t\le T\,.\end{cases}$$

It is easy to see that $v(x,t) = w(x,t) - w(x,\tau)$, $\nabla w(x,\tau) = -\nabla v(x,0)$ and $\nabla v^2(x,t) \le 2\,\nabla w^2(x,\tau) + 2\,\nabla w^2(x,t)$, consequently, substituting w in (3.8), we get

$$\begin{aligned}&\int_\Omega ((b_0-2\tau C_1)\nabla w^2(x,\tau)+\beta\nabla U^2(x,\tau)+U^2(x,\tau))\,\mathrm{d}x\\ &\le\;C_1(2\|\nabla w^2(x,\tau)\|^2_{L^2(Q_\tau)}+\|\nabla U\|^2_{L^2(Q_\tau)}+\|U\|^2_{L^2(Q_\tau)})\,. \end{aligned}\tag{3.9}$$

Since τ is arbitrary chosen, let $1-2\tau C_1>0$, then (3.9) becomes

$$\begin{aligned}&\int_\Omega (\nabla w^2(x,\tau)+\nabla U^2(x,\tau)+U^2(x,\tau))\,\mathrm{d}x\\ &\le\;C_2\int_0^\tau\int_\Omega ((\nabla w)^2+\nabla U^2+U^2)\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

where $C_2=\dfrac{2C_1}{\min((b_0-2\tau C_1),\beta,1)}$. Applying Gronwall inequality, we get for all $\tau\in\left]0,\frac{1}{C_2}\right[$,

$$\int_\Omega (\nabla w^2(x,\tau)+\nabla U^2(x,\tau)+U^2(x,\tau))\,\mathrm{d}x\le 0,$$

we conclude that $U(x,\tau)=0$, for all $x\in\Omega$ and $\tau\in\left]0,\dfrac{1}{C_2}\right[$. If $T\le\dfrac{1}{C_2}$, then $U=0$ in Q. In the case where $T\ge\dfrac{1}{C_2}$, we see that $\left]0,T\right[\subset\cup_{n=1}^{n=n_0}\left]\dfrac{n-1}{C_2},\dfrac{n}{C_2}\right[$, where $n_0=[C_2T]+1$, $[C_2T]$ is the entire part of C_2T, then repeating the preceding reasoning for $\tau\in\left]\dfrac{n-1}{C_2},\dfrac{n}{C_2}\right[$, we get $U(x,\tau)=0$, for all $\tau\in\left]\dfrac{n-1}{C_2},\dfrac{n}{C_2}\right[$ and then $U(x,t)=0$ in Q. Thus, the uniqueness is proved. □

4 Existence of generalized solution

In this section, we shall prove the existence of a generalized solution of problem (1.1)–(1.3) by using Galerkin's method.

Theorem 4.1 *Assume that the assumptions of Theorem 3.1 hold, then problem (1.1)–(1.3) has a unique solution $u\in W^{1,2}(Q)$.*

Proof. Let $\{w_k(x)\}$ be a fundamental system in $W^{1,2}(\Omega)$, such that $(w_k,w_i)_{L_2(\Omega)}=\delta_{k,i}$. Now we will try to find an approximate solution of the problem (1.1)–(1.3) in the form

$$u^n(x)=\sum_{k=1}^n d_k(t)w_k(x).\tag{4.1}$$

The approximations of the functions $\varphi(x)$ and $\psi(x)$ are denoted respectively by

$$\varphi^{(n)}(x) = \sum_{k=1}^{n} \varphi_k w_k(x), \psi^{(n)}(x) = \sum_{k=1}^{n} \psi_k w_k(x) \tag{4.2}$$

$$d_k(0) = \varphi_k, d_k'(0) = \psi_k.$$

Substituting the approximate solution in equation (1.1), multiplying both sides by w_l, yields

$$\int_\Omega w_l u_{tt}^n \,\mathrm{d}x + \int_\Omega w_l (b u_x^n)_x \,\mathrm{d}x - \int_\Omega \beta \Delta u_{tt}^n w_l \,\mathrm{d}x = \int_\Omega f w_l \,\mathrm{d}x \,. \tag{4.3}$$

Integration by parts with respect to x over Ω yields

$$\begin{aligned}(u_{tt}^n, w_l)_{L^2(\Omega)} - (b\nabla u^n, \nabla w_l)_{L^2(\Omega)} + \beta(\nabla u_{tt}^n, \nabla w_l)_{L^2(\Omega)} \\ - \int_{\partial\Omega} b w_l(x) \int_\Omega k(x,\xi) u^n(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}s \\ + \beta \int_{\partial\Omega} w_l(x) \,(\int_\Omega k(x,\xi) u_{tt}^n(\xi,t)\,\mathrm{d}\xi)\,\mathrm{d}s \quad &= \quad (f, w_l)_{L^2(\Omega)}, \end{aligned} \tag{4.4}$$

substituting (4.1) in (4.4) gives for $l = \overline{1,n}$

$$\begin{aligned}(f, w_l)_{L^2(\Omega)} \quad &= \quad \sum_{k=1}^{n} d_k''(t)\,(w_k, w_l)_{L^2(\Omega)} - \sum_{k=1}^{n} d_k(t)((b\nabla w_k, \nabla w_l)_{L^2(\Omega)} \\ &\quad + \int_{\partial\Omega} b(s,t) w_l(s) \int_\Omega k(x,\xi) w_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}s) \\ &\quad + \beta \sum_{k=1}^{n} d_k''(t)\,((\nabla w_k, \nabla w_l)_{L^2(\Omega)} + \int_{\partial\Omega} w_l(s) \int_\Omega k(x,\xi) w_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}s) \,. \end{aligned} \tag{4.5}$$

Let

$$\begin{aligned}\gamma_{kl}(t) \quad &= \quad -\Big((b\nabla w_k, \nabla w_l)_{L^2(\Omega)} + \int_{\partial\Omega} b(s,t) w_l(s) \int_\Omega k(x,\xi) w_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\Big)\,, \\ \chi_{kl} \quad &= \quad (\nabla w_k, \nabla w_l)_{L^2(\Omega)} + \int_{\partial\Omega} w_l(s) \int_\Omega k(x,\xi) w_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\,, \\ f_l \quad &= \quad (f, w_l)\,. \end{aligned}$$

Then (4.5) can be written as

$$\sum_{k=1}^{n} d_k''(t)(\delta_{kl} + \beta\chi_{kl}) + d_k(t)\gamma_{kl}(t) = f_l(t)\,.$$

We obtain a system of differential equations of second order with respect to the variable t with smooth coefficients and the initial conditions $d_k(0) = \alpha_k$, $d_k'(0) = \beta_k$, consequently we get a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. So it has a unique solution $u^{(n)}$ satisfying (4.3). □

Lemma 4.2 *The sequence $u^{(n)}$is bounded.*

Proof. Multiplying each equation of (4.4) by the appropriate $d'_k(t)$, summing over k from 1 to n, then integrating the resultant equality with respect to t from 0 to τ, with $\tau \leq T$, yields

$$\begin{aligned}
&\frac{1}{2}\int_\Omega [(u_t^n(x,\tau))^2 + b(x,t)|\nabla u^n(x,\tau)|^2 + \beta|\nabla u_t^n(x,\tau)|^2]\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega [u_t^n(x,0)^2 + \beta|\nabla u_t^n(x,0)|^2 + b|\nabla u^n(x,0)|^2]\,\mathrm{d}x \qquad (4.6)\\
&\quad + \int_{\partial\Omega}\int_0^\tau b_t u^n(x,t)\int_\Omega k(x,\xi)u^n(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_{\partial\Omega}\int_0^\tau b u^n(x,t)\int_\Omega k(x,\xi)u_t^n(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_{\partial\Omega} b(x,0)u^n(x,0)\int_\Omega k(x,\xi)u^n(\xi,0)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \beta\int_{\partial\Omega} u_t^n(x,\tau)(\int_\Omega k(x,\xi)u_t^n(\xi,\tau)\,\mathrm{d}\xi)\,\mathrm{d}s \\
&\quad - \int_{\partial\Omega} b(x,\tau)u^n(x,\tau)\int_\Omega k(x,\xi)u^n(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad - \beta\int_{\partial\Omega} u_t^n(x,0)(\int_\Omega k(x,\xi)u_t^n(\xi,0)\,\mathrm{d}\xi)\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^\tau\int_\Omega b_t(\nabla u^n)^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega f u_t^n\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

With the help of Cauchy–Schwarz inequality, ε–Cauchy inequality, trace inequality and remarking that $\|u^n(x,\tau)\|^2_{L^2(\Omega)} = \|u_t^n(x,t)\|^2_{L^2(Q_\tau)} + \|u^n(x,0)\|^2_{L^2(\Omega)}$, we can estimate the last eight terms in the right hand side of (4.6) as follows:

$$\begin{aligned}
(1) &= \int_{\partial\Omega}\int_0^\tau (b_t u^n(x,t)\int_\Omega k(x,\xi)u^n(\xi,t)\,\mathrm{d}\xi)\,\mathrm{d}t\,\mathrm{d}s \\
&\leq b_2k_0\frac{\varepsilon}{2}\|\nabla u^n(x,t)\|^2_{L^2(Q_\tau)} + b_2k_0(\frac{c(\varepsilon)}{2}+\frac{1}{2})\|u^n(x,t)\|^2_{L^2(Q_\tau)}
\end{aligned}$$

$$\begin{aligned}
(2) &= \int_0^\tau\int_{\partial\Omega} b u^n(x,t)\int_\Omega k(x,\xi)u_t^n(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s \\
&\leq b_0k_0\frac{\varepsilon}{2}\|\nabla u^n(x,t)\|^2_{L^2(Q_\tau)} + b_0k_0\frac{c(\varepsilon)}{2}\|\nabla u^n(x,t)\|^2_{L^2(Q_\tau)} \\
&\quad + \frac{1}{2}\|u_t^n(x,t)\|^2_{L^2(Q_\tau)}
\end{aligned}$$

$$\begin{aligned}
(3) &= \int_{\partial\Omega} b(x,0)u^n(x,0)\int_\Omega k(x,\xi)u^n(\xi,0)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leq b_0k_0\frac{\varepsilon}{2}\|\nabla u^n(x,0)\|^2_{L^2(\Omega)} + b_0k_0(\frac{c(\varepsilon)}{2}+\frac{1}{2})\|u^n(x,0)\|^2_{L^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
(4) &= \beta\int_{\partial\Omega} u_t^n(x,\tau)\int_\Omega k(x,\xi)u_t^n(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leq \beta k_0\frac{\varepsilon}{2}\|\nabla u_t^n(x,\tau)\|^2_{L^2(\Omega)} + \beta k_0(\frac{c(\varepsilon)}{2}+\frac{1}{2})\|u_t^n(x,\tau)\|^2_{L^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
(5) &= -\int_{\partial\Omega} b(x,\tau)u^n(x,\tau)\int_{\Omega} k(x,\xi)u^n(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leq b_0k_0\frac{\varepsilon}{2}\|\nabla u^n(x,\tau)\|^2_{L^2(\Omega)} + b_0k_0(\frac{c(\varepsilon)}{2}+\frac{1}{2})\|u^n_t(x,t)\|^2_{L^2(Q_\tau)} \\
&\quad + b_0k_0(\frac{c(\varepsilon)}{2}+\frac{1}{2})\|u^n(x,0)\|^2_{L^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
(6) &= -\beta\int_{\partial\Omega} u^n_t(x,0)\int_{\Omega} k(x,\xi)u^n_r(\xi,0)\ \mathrm{d}\xi ds \\
&\leq \beta k_0\frac{\varepsilon}{2}\|\nabla u^n_t(x,0)\|^2_{L^2(\Omega)} + \beta k_0(\frac{c(\varepsilon)+1}{2})\|u^n_t(x,0)\|^2_{L^2(\Omega)} \\
(7) &= \int_0^\tau\int_\Omega f u^n_t\,\mathrm{d}x\,\mathrm{d}t \ \leq\ \frac{1}{2}\|f(x,t)\|^2_{L^2(Q_\tau)} + \frac{1}{2}\|u^n_t(x,t)\|^2_{L^2(Q_\tau)} \\
(8) &= \int_0^\tau\int_\Omega b_t(\nabla u^n)^2\,\mathrm{d}x\,\mathrm{d}t \ \leq\ b_2\|\nabla u^n(x,t)\|^2_{L^2(Q_\tau)}\ .
\end{aligned}$$

Substituting the eight integrals in (4.6) yields

$$\begin{aligned}
&(1-\beta k_0(c(\varepsilon)+1))\|u^n_t(x,\tau)\|^2_{L^2(\Omega)} + (\beta-\beta k_0\varepsilon)\|\nabla u^n_t(x,\tau)\|^2_{L^2(\Omega)} \\
&+(b_1-b_0k_0\varepsilon)\|\nabla u^n(x,\tau)\|^2_{L^2(\Omega)} \\
&\leq \|f(x,t)\|^2_{L^2(Q_\tau)} + (\beta+\beta k_0\varepsilon)\|\nabla u^n_t(x,0)\|^2_{L^2(\Omega)} \\
&\quad + (\beta+\beta k_0(c(\varepsilon)+1))\|u^n_t(x,0)\|^2_{L^2(\Omega)} + (\beta b_0+b_0k_0\varepsilon)\|\nabla u^n(x,0)\|^2_{L^2(\Omega)} \\
&\quad + 2\,b_0\,k_0(c(\varepsilon)+1)\,\|u^n(x,0)\|^2_{L^2(\Omega)} + (b_0+b_2)k_0\varepsilon\|\nabla u^n(x,t)\|^2_{L^2(Q_\tau)} \\
&\quad + (b_2k_0c(\varepsilon)+1)\|u^n(x,t)\|^2_{L^2(Q_\tau)} + (1+b_0k_0(c(\varepsilon)+1))\|u^n_t(x,t)\|^2_{L^2(Q_\tau)}\ .
\end{aligned}$$

Choosing ε such that $\beta k_0(c(\varepsilon)+1)\leq 1, k_0\varepsilon\leq 1$, setting $m = \min(\beta k_0(c(\varepsilon)+1),\ \beta(1-k_0\varepsilon),\ (b_1-b_0k_0\varepsilon))$, $M = \max(1+b_2k_0c(\varepsilon),\ k_0\varepsilon(b_0+b_2),$ $1+b_0k_0(c(\varepsilon)+1), 2b_0k_0(c(\varepsilon)+1), \beta b_0+b_0k_0\varepsilon, \beta+\beta k_0(c(\varepsilon)+1), \beta+\beta k_0\varepsilon)$ and $M_1=\dfrac{M}{m}$, then using elementary estimates, (4.7) becomes

$$\begin{aligned}
&\|u^n_t(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u^n(x,\tau)\|^2_{L^2(\Omega)} + \|u^n(x,\tau)\|^2_{L^2(\Omega)} \\
&\leq M_1(\|u^n_t(x,t)\|^2_{L^2(Q_\tau)} + \|\nabla u^n(x,t)\|^2_{L^2(Q_\tau)} \\
&\quad + \|u^n(x,t)\|^2_{L^2(Q_\tau)} + \|\nabla u^n_t(x,0)\|^2_{L^2(\Omega)} + \|u^n_t(x,0)\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla u^n(x,0)\|^2_{L^2(\Omega)} + \|u^n(x,0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q_\tau)}).
\end{aligned} \tag{4.7}$$

Applying the Gronwall inequality to (4.7) we obtain

$$\begin{aligned}
&\|u^n_t(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u^n(x,\tau)\|^2_{L^2(\Omega)}\|u^n(x,\tau)\|^2_{L^2(\Omega)} \\
&\leq e^{M_1T}(\|\psi^n\|^2_{L^2(\Omega)} + \|\nabla\psi^n\|^2_{L^2(\Omega)} + \|\varphi^n\|^2_{L^2(\Omega)} + \|\nabla\varphi^n\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q_\tau)})
\end{aligned}$$

Integrating (4.8) with respect to τ on $[0,T]$ yields

$$\|u^n\|^2_{W^{1,2}(Q)} \leq Te^{M_1T}(\|\varphi\|^2_{W^{1,2}(\Omega)} + \|\psi\|^2_{W^{1,2}(\Omega)} + \|f\|^2_{L^2(Q)}).$$

Consequently the sequence $\{u^n\}$ in $W^1_2(Q_\tau)$, therefore we can extract a subsequence which we denote by $\{u^{(n_k)}\}$ that is weakly convergent, then we prove that its limit is the desired solution of the problem (1.1)–(1.3). □

Lemma 4.3 *The limit of the subsequence* $\{u^{(n_k)}\}$ *is the solution of the problem* (1.1)–(1.3).

Proof. For this, we prove that the limit of the subsequence $\{u^{(n_k)}\}$ satisfies the identity (2.1) for any function $v = \sum_{i=1}^n v_i(t)w_i(x) \in W_T^{1,2}(Q)$. Since the set $S_n = \{v(x,t) = \sum_{k=1}^n v_k(t)w_k(x),\ v_k(t) \in C^2(0,T), v_k(T) = 0\}$ is such that $\overline{\cup_{n=1}^\infty S_n} = W_T^{1,2}(Q)$, it suffices to prove (2.1) for $v \in S_n$. Multiplying (4.4) by $v_i(t) \in W^{1,2}(0,T)$, $v_i(T) = 0$, then taking the sum from $i = 0$ to n, we obtain

$$\begin{aligned}
&\int_\Omega (u_{tt}^{(n_k)} v - b\nabla u^{(n_k)} \nabla v + \beta \nabla u_{tt}^{(n_k)} \nabla v)\, \mathrm{d}x \\
&\quad - \int_{\partial\Omega} bv(\int_\Omega K(x,\xi) u^{(n_k)}(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s \\
&\quad + \beta \int_{\partial\Omega} v(\int_\Omega k(x,\xi) u_{tt}^n(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s \quad = \quad \int_\Omega fv\, \mathrm{d}x\,. \qquad (4.8)
\end{aligned}$$

Integrating by parts (4.8) on $[0,T]$ we get

$$\begin{aligned}
&-(u_t^{(n_k)}, v_t)_{L^2(Q)} - (bu^{(n_k)}, \nabla v)_{L^2(Q)} - \beta(\nabla u_t^{(n_k)}, \nabla v_t)_{L^2(Q)} \\
&\quad -(\psi^{(n_k)}, v(x,0))_{L^2(\Omega)} - \beta(\nabla\psi^{(n_k)}, \nabla v(x,0))_{L^2(\Omega)} \\
&\quad - \int_0^T \int_{\partial\Omega} b(s,t)\, v(s,t)(\int_\Omega K(x,\xi) u^{(n_k)}(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t \\
&\quad -\beta \int_{\partial\Omega} v(s,0)(\int_\Omega k(x,\xi)\psi^{(n_k)}(\xi)\, \mathrm{d}\xi)\, \mathrm{d}s \\
&\quad -\beta \int_0^T \int_{\partial\Omega} v_t(s,t)(\int_\Omega k(x,\xi) u_t^n(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t \quad = \quad (f,v)_{L^2(Q)}\,. \qquad (4.9)
\end{aligned}$$

Using the condition (1.3) we see that (4.9) is equivalent to

$$\begin{aligned}
&-(u_t^{(n_k)}, v_t)_{L^2(Q)} - (bu^{(n_k)}, \nabla v)_{L^2(Q)} - \beta(\nabla u_t^{(n_k)}, \nabla v_t)_{L^2(Q)} \\
&\quad -(\psi^{(n_k)}, v(x,0))_{L^2(\Omega)} + \beta(\Delta\psi^{(n_k)}, v(x,0))_{L^2(\Omega)} \\
&\quad - \int_0^T \int_{\partial\Omega} b(s,t)\, v(s,t)(\int_\Omega K(x,\xi) u^{(n_k)}(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t \\
&\quad -\beta \int_0^T \int_{\partial\Omega} v_t(s,t)(\int_\Omega k(x,\xi) u_t^n(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t \quad = \quad (f,v)_{L^2(Q)}\,. \qquad (4.10)
\end{aligned}$$

Denote the weak limit of the subsequence $\{u^{(n_k)}\}$ by u. When k tends to infinity, we see that $\int_0^T \int_{\partial\Omega} bv(\int_\Omega K(x,\xi) u^{(n_k)}(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t$ tends to $\int_0^T \int_{\partial\Omega} bv(\int_\Omega K(x,\xi) u(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t$ and $\beta \int_0^T \int_{\partial\Omega} v_t(s,t)(\int_\Omega k(x,\xi) u_t^{n_k}(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t$ tends to $\beta \int_0^T \int_{\partial\Omega} v_t(s,t)(\int_\Omega k(x,\xi) u_t(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t$. Indeed, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega} bv(\int_\Omega K(x,\xi) u^{(n_k)}(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t - \int_0^T \int_{\partial\Omega} bv(\int_\Omega K(x,\xi) u(\xi,t)\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^T \int_{\partial\Omega} v(\int_\Omega K(x,\xi)(u^{(n_k)}(\xi,t) - u(\xi,t))\, \mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t \\
&\leq b_0 k_0 |\partial\Omega| (\int_0^T \int_{\partial\Omega} |v|^2\, \mathrm{d}s\, \mathrm{d}t)^{\frac{1}{2}} \times (\int_0^T \int_\Omega |u^{(n_k)}(\xi,t) - u(\xi,t)|^2\, \mathrm{d}\xi\, \mathrm{d}t)^{\frac{1}{2}} \to 0.
\end{aligned}$$

Doing the same reasoning for the second limit, then by passing to the limit in (4.10), we get that u satisfies (2.1). $\square$

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