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FRACTIONAL NONLINEAR NONLOCAL DELAY EVOLUTION EQUATIONS WITH NEW DELAY RESOLVENT FAMILY

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Abstract. In this note, we establish the existence and uniqueness of mild and classical solutions of a class of some nonlinear nonlocal delay evolution equations of fractional orders in Banach spaces. By introducing a new approach in the semigroup theory named delay resolvent family, the solution representation is formulated. Sufficient conditions set, fractional calculus and fixed point theorem will be needed. An example that provides the abstract results is also given.

Keywords: Fractional delay evolution equation, nonlocal condition, mild and classical solutions, resolvent family, fixed point theorem.

2010 Mathematics Subject Classification: 34K05, 34G20, 26A33, 35A05.

1 Introduction

The aim of this paper is to study the nonlinear nonlocal delay fractional evolution system of the form

$$ {}^{C}D_t^{\alpha}u(t) + A(t, u(\sigma_0(t)))\, u(t) = F\big(t, u(\sigma_1(t)), ..., u(\sigma_n(t)), \int_0^t G(t, s, u(\sigma_{n+1}(s)))\, \mathrm{d}s\big), \tag{1.1} $$

$$ A(0, u)[u(0) - u_0] = H(u), \tag{1.2} $$

in a Banach space X, where D^{α} is the Caputo fractional derivative of order $0 < \alpha \le 1, t \in J,\ u$ is an X-valued function on J and $u_0 \in X$. We assume that $-A(t, .)$ is a closed linear operator defined

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on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. It is also assumed that $-A(t,.)$ generates an evolution operator in the Banach space X. The nonlinear operators $F : J \times X^{n+1} \to X, G : \Lambda \times X \to X$ and $H : C(J,X) \to X$ are given abstract functions, $\sigma_{i,\ i=\overline{0,n+1}} : J \to J'$ are delay arguments. Here $J = [0,a], J' = [0,t]$ and $\Lambda = \{(t,s) : 0 \leq s \leq t \leq a\}$.

The theory of evolution equations is an important branch in abstract differential equations, see Zaidman [24, 25]. In this case, since the terms of such equation are probabilistic, it can be taken many modeling senses of applications.

The existence result to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [3, 4]. Deng [9] indicated that, using the nonlocal condition $u(0)+h(u) = u_0$ to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem $u(0) = u_0$. According to Deng's papers, the function h is considered of the form

$$h(u) = \sum_{k=1}^{p} c_k u(t_k), \tag{1.3}$$

where $c_k, k = 1, 2, ..., p$ are given constants and $0 \leq t_1 < ... < t_p \leq a$, see also Benchohra *et al.* [2], Dong *et al.* [10] and N'Guérékata [19].

In the last few decades, fractional differential equations have attracted the attention of many scientists in several topics, see for example, Kilbas *et al.* [15, 22] and Podlubny [21]. Recently, the junction between the mentioned fields, well known by (nonlocal) fractional evolution equations, has been considered by many authors, see for instance [12, 16, 18, 23, 26].

In this paper, motivated by our work [6], Araya et al. [1] and Mophou et al. [17], our concern is: What evolution operator should be used when the closed operator depends on the delay argument? For this reason, we will introduce a new concept in the theory of semigroups called delay resolvent family as possible answering. A new form of nonlocal condition has been presented with the help of Hille-Phillips principles [14].

2 Preliminary results

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach space Z, the norm of Z is denoted by $\|.\|_Z$. The space of all bounded linear operators endowed with the topology defined by the operator norm from X to Y is denoted by $B(X,Y)$ and $B(X,X)$ is written as $B(X)$.

2.1 Fractional integrals and derivatives

We recall some basic definitions in fractional calculus from [15, 21, 22].

Definition 2.1 *The fractional integral of order α with the lower limit zero for a function $f \in C([0,\infty))$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, \mathrm{d}s,\ t > 0,\ 0 < \alpha < 1,$$

provided the right side is point-wise defined on $[0, \infty)$*, where* Γ *is the gamma function.*

Definition 2.2 *Riemann-Liouville derivative of order* α *with the lower limit zero for a function* $f \in C([0, \infty))$ *can be written as*

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} \,\mathrm{d}s,\ t > 0,\ 0 < \alpha < 1.$$

Definition 2.3 *The Caputo derivative of order* α *for a function* $f \in C([0, \infty))$ *can be written as*

$$^{C}D^{\alpha} f(t) = {}^{L}D^{\alpha}(f(t) - f(0)),\ t > 0,\ 0 < \alpha < 1.$$

Remark 2.4 *(1) If* $f \in C^1([0, \infty))$*, then*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} \,\mathrm{d}s = I^{1-\alpha} f'(t),\ t > 0,\ 0 < \alpha < 1.$$

(2) The Caputo derivative of a constant is equal to zero.
(3) If f *is an abstract function with values in* E*, then integrals which appear in Definitions 2.5–2.6 are taken in Bochner's sense.*

2.2 Delay resolvent family

Definition 2.5 *A two parameter family of bounded linear operators* $Q(t, s), 0 \leq s \leq t \leq a$*, on* X *is called an evolution system if the following two conditions are satisfied*
(i) $Q(t, t) = I,\ Q(t, r)Q(r, s) = Q(t, s)$ *for* $0 \leq s \leq r \leq t \leq a$*,*
(ii) $(t, s) \mapsto Q(t, s)$ *is strongly continuous for* $0 \leq s \leq t \leq a$*.*

More details about evolution systems can be found in Pazy [20, Chapter 5 and Section 6.4 respectively].

Let E be the Banach space formed from $D(A)$ with the graph norm. Since $-A(t)$ is a closed operator, it follows that $-A(t)$ is in the set of bounded operators from E to X.

Definition 2.6 *Let* $A(t, u(\sigma(t))), \sigma(t) \leq t$*, be a closed linear delay operator with domain* $D(A)$ *defined on a Banach space* X *and* $\alpha > 0$*. Let* $\rho[A(t, .)]$ *be the resolvent set of* $A(t, .)$*. We call* $A(t, .)$ *the generator of an* (α, u)*-delay resolvent family if there exists* $\omega \geq 0$ *and a strongly continuous function* $R_{(\alpha, u\circ\sigma)} : \mathbb{R}_+^2 \to L(X)$ *such that* $\{\lambda^{\alpha} : \Re e(\lambda) > \omega\} \subset \rho[A(t, .)]$ *and for* $0 \leq s \leq t < \infty$*,*

$$(\lambda^{\alpha} I - A(s, u \circ \sigma(s)))^{-1} v = \int_0^{\infty} e^{-\lambda(t-s)} R_{(\alpha, u\circ\sigma)}(t, s)\, v \ \mathrm{d}t,\ \Re e(\lambda) > \omega,\ (u, v) \in X^2. \quad (2.1)$$

In this case, $R_{(\alpha, u\circ\sigma)}(t, s)$ *is called the* (α, u)*-delay resolvent family generated by* $A(t, u(\sigma(t)))$*, compare with [1, 6, 17].*

Remark 2.7

1. *If* $\sigma(t) = t$*, then (2.1) will be reduced to the introduced concept in [6].*
2. *We can deduce that (1.1)–(1.2) is well posed if and only if,* $-A(t, u(\sigma_0(t)))$ *is the generator of an* (α, u)*-delay resolvent family.*

2.3 Sufficient conditions

Let Ω be a subset of X. We assume the following conditions:
(H$_1$) The operator $[A(t,.)+\lambda^\alpha I]^{-1}$ exists in $B(X)$ for any λ with $\Re e\lambda \leq 0$ and

$$\|[A(t,.)+\lambda^\alpha I]^{-1}\| \leq \frac{C_\alpha}{|\lambda|+1},\ t\in J$$

where C_α is a positive constant independent of both t and λ, see [14].
(H$_2$) $H : C(J : \Omega) \to Y$ is Lipschitz continuous in X and bounded in Y, that is, there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|H(u)\|_Y \leq k_1,$$

$$\|H(u)-H(v)\|_Y \leq k_2 \max_{t\in J}\|u-v\|_{PC},\ u,v\in C(J:X).$$

For the conditions (H$_3$) and (H$_4$) let Z be taken as both X and Y.
(H$_3$) $G : \Lambda\times Z \to Z$ is continuous and there exist constants $k_3 > 0$ and $k_4 > 0$ such that

$$\int_0^t \|G(t,s,u)-G(t,s,v)\|_Z\,\mathrm{d}s \leq k_3\|u-v\|_Z,\ u,v\in X,$$

$$k_4 = \max\{\int_0^t \|G(t,s,0)\|_Z\,\mathrm{d}s :\ (t,s)\in\Lambda\}.$$

(H$_4$) $F : J\times Z^{n+1} \to Z$ is continuous and there exist constants $k_5 > 0$ and $k_6 > 0$ such that

$$\|F(t,u_1,...,u_{n+1})-F(t,v_1,...,v_{n+1})\|_Z \leq k_5\sum_{i=1}^{n+1}\|u_i-v_i\|_Z,\ u_i,v_i\in X,$$

$$k_6 = \max_{t\in J}\|F(t,0,...,0)\|_Z.$$

(H$_5$) The delay arguments $\sigma_i : J \to J'$ are absolutely continuous and there exist constants $c_i > 0$ such that $\sigma_i'(t) \geq c_i$, for $t\in J$ and $i = 1,...,n+1$.
Let us take $M_0 = \max\|R_{(\alpha,u_\sigma)}(t,s)\|_{B(Z)},\ 0\leq s\leq t\leq a, u\in\Omega$.
(H$_6$) There exist positive constants $r > 0$ and $\lambda_1,\lambda_2,\lambda_3 \in (0,\frac{1}{3})$ such that

$$M_0\{\|u_0\| + C_\alpha k_1 + a\{k_5[r(1/c_1+...+1/c_n+k_3/c_{n+1})+k_4]+k_6\}\} \leq r,$$

$$\lambda_1 = Ka\|u_0\|_Y + k_1Ka + C_\alpha M_0 k_2,$$

$$\lambda_2 = a^2K\{k_5[r(1/c_1+...+1/c_n+k_3/c_{n+1})+k_4]+k_6\},$$

$$\lambda_3 = aM_0k_5[1/c_1+...+1/c_n+k_3/c_{n+1}].$$

Definition 2.8 *By a mild solution of (1.1)–(1.2) we mean a function* $u\in C(J:X)$ *with values in* Ω *and* $u_0\in X$ *satisfying the integral equation*

$$\begin{aligned} u(t) &= R_{(\alpha,u\circ\sigma_0)}(t,0)\,u_0 + A^{-1}(0,u)\,R_{(\alpha,u\circ\sigma_0)}(t,0)\,H(u) \\ &\quad + \int_0^t R_{(\alpha,u\circ\sigma_0)}(t,s)\,F\big(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta\big)\,\mathrm{d}s. \end{aligned}$$

Definition 2.9 *By a classical solution of (1.1)–(1.2) on J, we mean a function u with values in X such that:*
1) u is a continuous function on J and $u(t) \in D(A)$,
2) $\dfrac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}$ exists and is continuous on $(0,a]$, $0<\alpha<1$,
3) u satisfies (1.1) on $(0,a]$ and the nonlocal condition (1.2), see [8, 11, 13].
(H_7) Further there exists a constant $K>0$ such that for every $u, v \in C(J:X)$ with values in Ω and every $\omega \in Y$ we have

$$\|A^{-1}(t,u)\,R_{(\alpha,u\circ\sigma_0)}(t,s)\,\omega - A^{-1}(t,v)\,R_{(\alpha,v\circ\sigma_0)}(t,s)\,\omega\| \le K\|\omega\|_Y \int_s^t \|u(\tau)-v(\tau)\|\,\mathrm{d}\tau.$$

Clearly, the last inequality is still verified when $A^{-1}(t,u)$ is the identity.

3 Main results

Now we are in position to state and prove our main results of this work.

Theorem 3.1 *Let $u_0 \in Y$ and $\Omega = \{u \in X : \|u\|_Y \le r\}, r>0$. If $-A(t,u(\sigma_0(t)))$ is the generator of an (α,u)-delay resolvent family and the assumptions (H_1)–(H_7) are satisfied, then (1.1)–(1.2) has a unique mild solution on J.*

Proof. Let S be a nonempty closed subset of $PC(J:X)$ defined by

$$S=\{u:\ u\in PC(J:X),\ \|u\|_Y\le r\},\ t\in J.$$

Consider a mapping P on S defined by

$$\begin{aligned}(Pu)(t) &= R_{(\alpha,u\circ\sigma_0)}(t,0)u_0 + A^{-1}(0,u)R_{(\alpha,u\circ\sigma_0)}(t,0)H(u)\\ &\quad + \int_0^t R_{(\alpha,u\circ\sigma_0)}(t,s)\,F\Big(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta\Big)\,\mathrm{d}s.\end{aligned}$$

For $u\in S$, we have

$$\begin{aligned}\|Pu(t)\|_Y &\le \|R_{(\alpha,u\circ\sigma_0)}(t,0)u_0\| + \|A^{-1}(0,u)R_{(\alpha,u\circ\sigma_0)}(t,0)H(u)\|\\ &\quad + \int_0^t \|R_{(\alpha,u\circ\sigma_0)}(t,s)\|\{\|F(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta)\\ &\qquad - F(s,0,...,0)\| + \|F(s,0,...,0)\|\}\mathrm{d}s.\end{aligned}$$

Using assumptions (H_1)–(H_5), we get

$$\begin{aligned}\|Pu(t)\|_Y &\le M_0\|u_0\| + C_\alpha M_0 k_1 + M_0\int_0^t \Big\{k_5[\|u(\sigma_1(s))\| + \cdots + \|u(\sigma_n(s))\|\\ &\quad + \int_0^s \|G(s,\eta,u(\sigma_{n+1}(\eta))) - G(s,\eta,0)\|\mathrm{d}\eta + \int_0^s \|G(s,\eta,0)\|\mathrm{d}\eta] + k_6\Big\}\mathrm{d}s\\ &\le M_0\|u_0\| + C_\alpha M_0 k_1 + M_0\int_0^t \Big\{k_5[\|u(\sigma_1(s))\|(\sigma_1'(s)/c_1) + \cdots +\\ &\quad + \|u(\sigma_n(s))\|(\sigma_n'(s)/c_n) + k_3\|u(\sigma_{n+1}(s))\|(\sigma_{n+1}'(s)/c_{n+1}) + k_4] + k_6\Big\}\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\leq M_0\|u_0\| + C_\alpha M_0 k_1 + M_0 k_5\Big\{\frac{1}{c_1}\int_{\sigma_1(0)}^{\sigma_1(t)} \|u(\tau_1)\|\mathrm{d}\tau_1 + \cdots \\
&\quad + \frac{1}{c_n}\int_{\sigma_n(0)}^{\sigma_n(t)} \|u(\tau_n)\|\mathrm{d}\tau_n + \frac{k_3}{c_{n+1}}\int_{\sigma_{n+1}(0)}^{\sigma_{n+1}(s)} \|u(\tau_{n+1})\|\mathrm{d}\tau_{n+1}\Big\} + aM_0(k_4k_5 + k_6) \\
&\leq M_0\|u_0\| + C_\alpha M_0 k_1 + aM_0\{k_5[r(1/c_1 + \cdots + 1/c_n + k_3/c_{n+1}) + k_4] + k_6\}.
\end{aligned}$$

From assumption (H_6), one gets $\|Pu(t)\|_Y \leq r$. Thus P maps S into itself. Now we shall show that P is a strict contraction on S which will ensure the existence of a unique continuous function satisfying (2.2) on J.
If $u, v \in S$, then

$$\begin{aligned}
\|Pu(t) - Pv(t)\| &\leq \|R_{(\alpha,u\circ\sigma_0)}(t,0)u_0 - R_{(\alpha,v\circ\sigma_0)}(t,0)u_0\| \\
&\quad + \|A^{-1}(0,u)R_{(\alpha,u\circ\sigma_0)}(t,0)H(u) - A^{-1}(0,v)R_{(\alpha,v\circ\sigma_0)}(t,0)H(v)\| \\
&+ \int_0^t \Big\|R_{(\alpha,u\circ\sigma_0)}(t,s)[F(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta)] \\
&\quad - R_{(\alpha,v\circ\sigma_0)}(t,s)[F(s,v(\sigma_1(s)),...,v(\sigma_n(s)),\int_0^s G(s,\eta,v(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta)]\Big\|\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\leq \|R_{(\alpha,u\circ\sigma_0)}(t,0)u_0 - R_{(\alpha,v\circ\sigma_0)}(t,0)u_0\| \\
&\quad + \|A^{-1}(0,u)R_{(\alpha,u\circ\sigma_0)}(t,0)H(u) - A^{-1}(0,v)R_{(\alpha,v\circ\sigma_0)}(t,0)H(u)\| \\
&\quad + \|A^{-1}(0,v)R_{(\alpha,v\circ\sigma_0)}(t,0)H(u) - A^{-1}(0,v)R_{(\alpha,v\circ\sigma_0)}(t,0)H(v)\| \\
&\quad + \int_0^t \Big\{\Big\|R_{(\alpha,u\circ\sigma_0)}(t,s)\,F\big(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta\big) \\
&\qquad - R_{(\alpha,v\circ\sigma_0)}(t,s)\,F\big(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta\big)\Big\| \\
&\qquad + \Big\|R_{(\alpha,v\circ\sigma_0)}(t,s)\,F\big(s,u(\sigma_1(s)),...,u(\sigma_n(s)),\int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta\big) \\
&\qquad - R_{(\alpha,v\circ\sigma_0)}(t,s)\,F\big(s,v(\sigma_1(s)),...,v(\sigma_n(s)),\int_0^s G(s,\eta,v(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta\big)\Big\|\Big\}\mathrm{d}s.
\end{aligned}$$

Using assumptions (H_1)–(H_7), we get

$$\begin{aligned}
\|Pu(t) - Pv(t)\| &\leq Ka\|u_0\|_Y \max_{\tau\in J}\|u(\tau) - v(\tau)\| \\
&\quad + k_1Ka\max_{\tau\in J}\|u(\tau) - v(\tau)\| + C_\alpha M_0 k_2 \max_{\tau\in J}\|u(\tau) - v(\tau)\| \\
&\quad + aK\int_0^t \Big\|F(s,u(\sigma_1(s)),...,u(\sigma_n(s)), \\
&\qquad\qquad \int_0^s G(s,\eta,u(\sigma_{n+1}(\eta)))\,\mathrm{d}\eta)\Big\|_Y \mathrm{d}s \times \max_{\tau\in J}\|u(\tau) - v(\tau)\| \\
&\quad + M_0k_5\int_0^t \Big[\|u(\sigma_1(s)) - v(\sigma_1(s))\| + \cdots + \|u(\sigma_n(s)) - v(\sigma_n(s))\| \\
&\qquad\qquad + \int_0^s \|G(s,\eta,u(\sigma_{n+1}(\eta))) - G(s,\eta,v(\sigma_{n+1}(\eta)))\|\,\mathrm{d}\eta\Big]\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(Ka\|u_0\|_Y + k_1Ka + C_\alpha M_0 k_2 + a^2K\{k_5[r(1/c_1 + \cdots \\
&\qquad + 1/c_n + k_3/c_{n+1}) + k_4] + k_6\}\Big) \max_{\tau\in J} \|u(\tau) - v(\tau)\| \\
&\quad + M_0k_5 \int_0^t \Big[\|u(\sigma_1(s)) - v(\sigma_1(s))\|(\sigma_1'(s)/c_1) + \cdots \\
&\qquad + \|u(\sigma_n(s)) - v(\sigma_n(s))\|(\sigma_n'(s)/c_n) \\
&\qquad + k_3\|u(\sigma_{n+1}(s)) - v(\sigma_{n+1}(s))\|(\sigma_{n+1}'(s)/c_{n+1})\Big]\,\mathrm{d}s \\
&\leq \Big(Ka\|u_0\|_Y + k_1Ka + C_\alpha M_0 k_2 + a^2K\{k_5[r(1/c_1 + \cdots + \\
&\qquad + 1/c_n + k_3/c_{n+1}) + k_4] + k_6\}\Big) \max_{\tau\in J} \|u(\tau) - v(\tau)\| \\
&\quad + aM_0k_5[1/c_1 + \cdots + 1/c_n + k_3/c_{n+1}] \max_{\tau\in J} \|u(\tau) - v(\tau)\| \\
&\leq (\lambda_1 + \lambda_2 + \lambda_3) \max_{\tau\in J} \|u(\tau) - v(\tau)\|
\end{aligned}
$$

Thus

$$\|Pu(t) - Pv(t)\| \leq \lambda \max_{\tau\in J} \|u(\tau) - v(\tau)\|,$$

where $0 < \lambda < 1$, which means that P is a strict contraction map from S into S and therefore by the Banach contraction principle there exists a unique fixed point $u \in S$ such that $Pu = u$. Hence u is a unique mild solution of (1.1)–(1.2) on J. □

Theorem 3.2 *Assume that*

(i) Conditions (H_1)–(H_7) *hold,*

(ii) Y *is a reflexive Banach space with norm* $\|.\|$*,*

(iii) The functions f *and* g *are uniformly Hölder continuous in* $t \in J$.

Then the problem (1.1)–(1.2) *has a unique classical solution on* J.

Proof. From (i), applying Theorem 3.1, the problem (1.1)–(1.2) has a unique mild solution $u \in S$. Set

$$\omega(t) = F(t, u(\sigma_1(t)), ..., u(\sigma_n(t)), \int_0^t G(t, s, u(\sigma_{n+1}(s)))\,\mathrm{d}s).$$

In order to prove the regularity of the mild solution, we use the further assumptions, it is easy to conclude that the function $\omega(t)$ is also uniformly Hölder continuous in $t \in J$.

Consider the following nonlocal delay fractional problem

$$\frac{\mathrm{d}^\alpha v(t)}{\mathrm{d}t^\alpha} + A(t, u(\sigma_0(t)))u(t) = \omega(t), \tag{3.1}$$

$$A(0, u)[u(0) - u_0] = H(u). \tag{3.2}$$

According to Pazy [20], (3.1), (3.2) has a unique solution v on J into X given by

$$v(t) = R_{(\alpha, u\circ\sigma_0)}(t, 0)u_0 + A^{-1}(0, u)R_{(\alpha, u\circ\sigma_0)}(t, 0)H(u) + \int_0^t R_{(\alpha, u\circ\sigma_0)}(t, s)\omega(s)\,\mathrm{d}s. \tag{3.3}$$

Noting that, each term on the right hand side of (3.3) belongs to $D(A)$, using the uniqueness of $v(t)$, we have that $u(t) \in D(A)$. It follows that u is a unique classical solution of (1.1)–(1.2) on J, see [5, 7]. $\square$

4 Application

Consider the fractional nonlocal delay integro-partial differential system of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x,t,u(x,\sin t))\frac{\partial^2 u(x,t)}{\partial x^2} = \sin u(x,t) + a_1(t)u(x,\sin t) + \int_0^t a_2(t,s)u(x,\sin s)\,\mathrm{d}s, \tag{4.1}$$

$$a(x,0,u(x,0))[u(x,0)-u_0(x)]'' = \sum_{k=1}^m c_k u(x,t_k),\ x\in[0,\pi], \tag{4.2}$$

$$u(0,t) = u(\pi,t) = 0,\ t \in J, \tag{4.3}$$

where $0 < \alpha \leq 1$, $0 < t_1 < ... < t_m < a$ and c_k are positive constants, $k = 1, ..., m$.
We define $A(t,.) : X \to X$ by $(A(t,.)w)(x) = a(x,t,.)w''$ with domain

$$D(A) = \{w \in X : w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = w(\pi) = 0\},$$

the functions $a(x,t,.), a_1(t)$ and $a_2(t,s)$ are continuous on Λ.

Let us take

$$X = L^2[0,\pi],\ C = C(J, S_r), S_r = \{y \in L^2[0,\pi] : \|y\| \leq r\}.$$

Put $F(t, u(.), ..., u(.), \int_0^t G(t,s,u(.))\,\mathrm{d}s) = \sin u(x,t) + a_1(t)u(x,.) + \int_0^t a_2(t,s)u(x,.)\,\mathrm{d}s$,

$$G(s,t,u(\sigma_{n+1}(t))) = a_2(t,s)u(x,\sin t), \sigma_i(t) = \sin t, i = 0,1,...,n+1$$

and $H(u(.,t)) = \sum_{k=1}^m c_k u(.,t_k)$.
Assume that, there exist constants $\eta \in (0,1]$ and C_α such that

$$\|[A(t_1,.) - A(t_2,.)]A^{-1}(s,.)\| \leq C_\alpha |t_1 - t_2|^\eta,\ t_1, t_2, s \in J.$$

Under these conditions each operator $-A(s,.)$, $s \in J$ generates an evolution operator $\exp(-t^\alpha A(s,.)), t > 0$ (which is compact, analytic and self-adjoint) and there exists a constant C_α such that

$$\|A^n(s,.)\exp(-t^\alpha A(s,.))\| \leq \frac{C_\alpha}{t^n},$$

where $n = 0, 1, t > 0, s \in J$, compare with [14]. It is clear that $-A(t,u(.))$ depends on $\sin t$, $(\sin t \leq t)$, which means that this evolution operator is a delay resolvent family. Thus we can deduce that the system (4.1)–(4.3) is an abstract formulation of (1.1)–(1.2). Further, all assumptions (H_1)–(H_7) are satisfied and it is possible to choose our constants in (H_6). Hence by Theorem 3.1, the system (4.1)–(4.3) has a unique mild solution on J. In addition, if the function

$\sin u(x,t)+a_1(t)u(x,.)+\displaystyle\int_0^t a_2(t,s)u(x,.)\,\mathrm{d}s$ is uniformly Hölder continuous in $t \in J$, then from Theorem 3.2, this mild solution in fact is a classical solution.

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